International Journal of Pure and Applied Mathematics
Volume 63 No. 2 2010, 243-254

ON A CLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY A LINEAR OPERATOR

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Abstract: Let S_H denote the class of functions $f = h + \bar{g}$ which are harmonic, univalent and sense preserving in the unit disc Δ. We define a new subclass $SHL(\alpha, \beta)$ by using a linear operator of harmonic univalent functions. In this paper, coefficient bounds, distortion bounds and extreme points are obtained.

AMS Subject Classification: 30C45, 30C50, 31C05
Key Words: harmonic functions, harmonic starlike function, linear operator, distortion bounds

1. Introduction

A continuous functions $f = u + iv$ is a complex valued harmonic functions in complex plane $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $\mathcal{D} \subset \mathbb{C}$ we can write $f(z) = h(z) + \bar{g}(z)$, where $h(z)$ and $g(z)$ are analytic in $\mathcal{D}$. We call $h(z)$ the analytic part and $g(z)$ the co-analytic part of $f(z)$. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$, see [2].

S_H denote the class of functions $f = h + \bar{g}$ that are harmonic, univalent and sense-preserving in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. For $f = h + \bar{g} \in S_H$ we may express the analytic function h and g as

Received: July 20, 2010
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$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1, \quad z \in \Delta. \tag{1}$$

Observe that S_H reduces to S, the class of normalized univalent analytic functions, if the co-analytic part of f is zero.

A function $f(z) = h + \bar{g}$, where $h(z)$ and $g(z)$ defined in (1) is harmonic starlike for $|z| = r < 1$, if

$$\frac{\partial}{\partial \theta}(\arg f(re^{i\theta})) = \text{Re}\left\{ \frac{zh'(z) - \overline{zg'(z)}}{h(z) + g(z)} \right\} > 0, \quad z \in \Delta.$$

The class S_H^- as the subclass of S_H consisting of all functions $f = h + \bar{g}$, where h and g are given by

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = -\sum_{k=1}^{\infty} b_k z^k, \quad z \in \Delta. \tag{2}$$

In 1984, Clunie and Sheil-Small [2] investigated the class S_H as well as its subclasses and obtained some coefficient bounds. Since then, many authors have carried out the study on the class S_H and on its subclasses such that Silverman [9], Silverman and Silvia [10] and Jahangiri [5].

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the unit disc Δ. The Hadamard product of $f(z),\ g(z) \in \mathcal{A}$ denoted by $(f * g)(z)$, is defined by, $(f * g)(z) = \sum_{k=1}^{\infty} a_k b_k z^k, \quad z \in \Delta.$

Let the linear operator $\mathcal{L}(a, c)f(z)$ (see [1]) be defined by

$$\mathcal{L}(a, c)f(z) = z + \sum_{k=2}^{\infty} \frac{(a)_k}{(c)_k} z^k, \quad a \in \mathbb{R},\ c \neq \{0, -1, -2, \cdots\},\ z \in \Delta,$$

or equivalently, by

$$\mathcal{L}(a, c)f(z) = \phi(a, c; z) * f(z), \quad f \in \mathcal{A},$$

where the function $\Phi(a, c; z)$ defined by

$$\Phi(a, c; z) = z + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^k, \qquad a \in \mathbb{R},\ c \in \mathbb{R} - \mathbb{Z}_0^- = \{0, -1, -2, \cdots\},\ z \in \Delta,$$

where $(a)_k$ is the Pochhammer symbol or the shifted factorial given by

$$(a)_k \quad = \quad \frac{\Gamma(a + k)}{\Gamma(a)}$$

$$= \begin{cases} 1, & k = 0 \\ a(a+1)(a+2)\cdots(a+k-1), & k \in \mathbb{N}, \end{cases} \tag{3}$$

Here Γ is the usual Gamma function.

For $0 < \alpha \leq 1$, $0 \leq \beta < 1$ and $a,\ c > 0$, we define $SHL(\alpha, \beta)$, the class of all function of the form (1)

$$\mathrm{Re}\left\{(1-\alpha)\frac{\mathcal{L}(a,c)h(z) + \mathcal{L}(a,c)g(z)}{z} + \alpha[\mathcal{L}(a,c)h(z) + \mathcal{L}(a,c)g(z)]'\right\} > \beta,$$

where $\mathcal{L}(a,c)f(z) = \mathcal{L}(a,c)h(z) + \overline{\mathcal{L}(a,c)g(z)}$.

For $\alpha = 1$, $\beta = 0$ and $a = n+1, c = 1$ the class $SHL(\alpha, \beta) \equiv \mathcal{H}_0^n$ which is studied in [3] and for $a = 1, c = 1$ the class $SHL(\alpha, \beta) \equiv SHP_0(\alpha, \beta)$ which is studied in [8].

We further denote by $SHL^-(\alpha, \beta)$ the subclass of $SHL(\alpha, \beta)$ such that the functions are $h(z)$ and $g(z)$ of the form (2).

2. Main Results

Theorem 1. *Let $f = h + \bar{g}$ be given by (1). Let*

$$\sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)(|a_k| + |b_k|) \leq 2 - \beta, \tag{4}$$

where $|a_1| = 1$, $\alpha \geq 0$, $0 \leq \beta < 1$, and $a,\ c > 0$, $\frac{a(1+\alpha)}{c} \geq 2$.

Then f is harmonic univalent sense preserving in Δ and $f \in SHL(\alpha, \beta)$.

Proof. For $|z_1| \leq |z_2| < 1$, we have by (4),

$$\begin{aligned} |f(z_1) - f(z_2)| &\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\ &\geq |z_1 - z_2|\left(1 - \sum_{k=2}^{\infty} k|a_k||z_2|^{k-1} - \sum_{k=1}^{\infty} k|b_k||z_2|^{k-1}\right) \\ &\geq |z_1 - z_2|\left[1 - \left(\sum_{k=2}^{\infty} k(|a_k| + |b_k|) + |b_1|\right)\right] \quad (|z_2| < 1) \\ &\geq |z_1 - z_2|[1 - (\sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)(|a_k| + |b_k|) + |b_1|)] \\ &\geq |z_1 - z_2|[1 - (1 - \beta - |b_1| + |b_1|)] = \beta|z_1 - z_2|. \end{aligned}$$

Hence, f is univalent in Δ. We note that f is sense preserving in Δ. This is because

$$|h'(z)| \geq 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1} \geq 1 - \sum_{k=2}^{\infty} k|a_k| > 1 - \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)|a_k|$$
$$\geq \beta + \sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)|b_k| \geq \sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)|b_k|$$
$$> \sum_{k=2}^{\infty} k|b_k||z|^{k-1} = |g'(z)| \quad (z \in \Delta).$$

Now we show that for $f \in SHL(\alpha,\beta)$. Using the fact that Re $w > \beta$ if and only if $|1-\beta+w| \geq |1+\beta-w|$, it suffices to show that

$$|1-\beta+(1-\alpha)\frac{\mathcal{L}(a,c)h(z)+\mathcal{L}(a,c)g(z)}{z} + \alpha[\mathcal{L}(a,c)h(z)+\mathcal{L}(a,c)g(z)]'|$$
$$- |1+\beta+(1-\alpha)\frac{\mathcal{L}(a,c)h(z)+\mathcal{L}(a,c)g(z)}{z} - \alpha[\mathcal{L}(a,c)h(z)+\mathcal{L}(a,c)g(z)]'|$$
$$\geq 0. \quad (5)$$

Substituting for h and g in (5) yields

$$|1-\beta+(1-\alpha)\frac{\mathcal{L}(a,c)h(z)+\mathcal{L}(a,c)g(z)}{z} + \alpha[\mathcal{L}(a,c)h(z)+\mathcal{L}(a,c)g(z)]'|$$
$$- |1+\beta-(1-\alpha)\frac{\mathcal{L}(a,c)h(z)+\mathcal{L}(a,c)g(z)}{z} - \alpha[\mathcal{L}(a,c)h(z)+\mathcal{L}(a,c)g(z)]'|$$
$$= |2-\beta+\sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)a_k z^{k-1} + \sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)b_k z^{k-1}|$$
$$- |\beta - \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)a_k z^{k-1} - \sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)b_k z^{k-1}|$$
$$> 2[(1-\beta) - \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)|a_k| - \sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)|b_k|] > 0.$$

The harmonic mapping

$$f(z) = z + \sum_{k=2}^{\infty} \frac{1-\beta}{\frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)} x_k z^k + \sum_{k=1}^{\infty} \frac{1-\beta}{\frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)} \overline{y_k z^k}, \quad (6)$$

where $\sum_{k=2}^{\infty}|x_k| + \sum_{k=1}^{\infty}|y_k| = 1$ show that the coefficient bound given by (4) is sharp. The function of the form (6) are in $SHL(\alpha,\beta)$ because

$$\sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)(|a_k|+|b_k|) = 1 + (1-\beta)(\sum_{k=2}^{\infty}|x_k| + \sum_{k=1}^{\infty}|y_k|) = 2-\beta.$$

The restriction placed in Theorem 1 on the moduli of the coefficients of $f = h+\bar{g}$

enables to conclude, for an arbitrary rotation of the coefficients of f, that the resulting function would still be in $SHL(\alpha,\beta)$. $\square$

In particular, for $a=1$ and $c=1$ in the above which is same as Theorem 1 given in [8] for $\lambda=0$.

For $a=n+1$, $c=1$ and $\alpha=1$, $\beta=0$ in the above theorem we have:

Corollary 2. *Let $f=h+\bar{g}$ be given by (1). Let*

$$\sum_{k=1}^{\infty} kC(n,k)(|a_k|+|b_k|)\leq 2,$$

where

$$C(n,k)=\binom{k+n-1}{n}$$

and $a_1=1$. Then f is harmonic univalent sense preserving in Δ and $f\in\mathcal{H}_0^n=\{f\in\mathcal{A}:\mathrm{Re}(D^nf(z))>0,\quad z\in\Delta\}$, here $D^nf(z)$ is Ruscheyegh derivative (see [7]).

This corollary is proved in [3] for $\lambda=0$.

We next show that the condition (4) is also necessary for functions in $SHL^{-}(\alpha,\beta)$.

Theorem 3. *Let $f=h+\bar{g}$ be given by (2). Let*

$$\sum_{k=1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)(|a_k|+|b_k|)\leq 2-\beta, \tag{7}$$

where $|a_1|=1$, $\alpha\geq 0$, $0\leq\beta<1$, $a,\ c>0$ and $\frac{a(1+\alpha)}{c}\geq 2$.

Then f is harmonic univalent sense preserving in Δ and $f\in SHL^{-}(\alpha,\ \beta)$.

Proof. The if part follows from Theorem 1. For the only if part, we show that if $f\in SHL^{-}(\alpha,\ \beta)$ then (7) holds. Note that a necessary condition and sufficient condition for $f=h+\bar{g}$ given by (2) be in $SHL^{-}(\alpha,\ \beta)$, is that

$$\mathrm{Re}\left\{(1-\alpha)\frac{\mathcal{L}(a,c)h(z)+\mathcal{L}(a,c)g(z)}{z}+\alpha[\mathcal{L}(a,c)h(z)+\mathcal{L}(a,c)g(z)]'\right\}\geq\beta,$$

equivalently

$$\mathrm{Re}\{1-\sum_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)|a_k|z^{k-1}+\sum_{k=1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)|b_k|z^{k-1}\}>\beta,$$

$z\to 1^{-}$, then

$$1 - \sum_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}(1 - \alpha + k\alpha)|a_k| - \sum_{k=1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}(1 - \alpha + k\alpha)|b_k| > \beta. \quad \square$$

Our next theorem yields upper and lower bounds for $|f(z)|$.

Theorem 4. *If $f \in SHL^{-}(\alpha, \beta)$ where $\alpha \geq 0$, $0 \leq \beta < 1$, $a, c > 0$ and $\frac{a(1+\alpha)}{c} \geq 2$. Then*

$$|f(z)| \leq (1+|b_1|)r + \frac{c}{(1+\alpha)a}(1-|b_1|-\beta)r^2$$

and

$$|f(z)| \geq (1+|b_1|)r - \frac{c}{(1+\alpha)a}(1-|b_1|-\beta)r^2.$$

Proof.

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k, \\
&< (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2 \quad (|z| = r < 1) \\
&\leq (1+|b_1|)r + \frac{c}{(1+\alpha)a}\sum_{k=2}^{\infty}(1-\alpha+k\alpha)\frac{(a)_{k-1}}{(c)_{k-1}}(|a_k|+|b_k|)r^2 \\
&\leq (1+|b_1|)r + \frac{c}{(1+\alpha)a}(1-|b_1|-\beta)r^2
\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &\geq (1-|b_1|)r - \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k, \\
&> (1-|b_1|)r - \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2 \quad (|z| = r < 1) \\
&\geq (1-|b_1|)r - \frac{c}{(1+\alpha)a}\sum_{k=2}^{\infty}(1-\alpha+k\alpha)\frac{(a)_{k-1}}{(c)_{k-1}}(|a_k|+|b_k|)r^2 \\
&\geq (1-|b_1|)r - \frac{c}{(1+\alpha)a}(1-|b_1|-\beta)r^2.
\end{aligned}$$

The bounds given in this theorem for the functions $f = h + \bar{g}$ of the form (1) also holds for (2) if the coefficient bound (4) satisfied. The functions

$$f(z) = 1 + |b_1|\bar{z} - \frac{c}{(1+\alpha)a}(1-|b_1|-\beta)\bar{z}^2,$$

$$f(z) = 1 - |b_1|z - \frac{c}{(1+\alpha)a}(1-|b_1|-\beta)z^2$$

for $|b_1| \le 1-\beta$ show that the bounds given in Theorem 3 are sharp.

The following result follows from the left hand inequality in Theorem 3. □

Theorem 5. *$f \in SHL^-(\alpha,\beta)$ $(\alpha \ge 0,\ 0 \le \beta < 1)$ and $a,\ c > 0$, $\frac{a(1+\alpha)}{c} \ge 2$ if and only if f can be expressed as*

$$f(z) = \sum_{k=1}^{\infty} (\mu_k h_k + \eta_k g_k), \tag{8}$$

where $z \in \Delta$, $h_1(z) = z$ and

$$h_k(z) = z - \frac{(1-\beta)(c)_{k-1}}{(a)_{k-1}(1-\alpha+k\alpha)} z^k, \quad k = 2,3,\dots,$$

$$g_k(z) = z - \frac{(1-\beta)(c)_{k-1}}{(a)_{k-1}(1-\alpha+k\alpha)} \bar{z}^k, \quad k = 2,3,\cdot,$$

$$\sum_{k=1}^{\infty} (\mu_k + \eta_k) = 1 \quad (\mu_k, \eta_k \ge 0).$$

In particular the extreme points of $SHL^-(\alpha,\beta)$ are $\{g_k\}$ and $\{h_k\}$.

Proof. For functions f of the form (8) we have

$$f(z) = \sum_{k=1}^{\infty} (\mu_k h_k + \eta_k g_k) = \sum_{k=1}^{\infty} (\mu_k + \eta_k) z$$

$$- \sum_{k=2}^{\infty} \frac{(c)_{k-1}(1-\beta)}{(a)_{k-1}(1-\alpha+k\alpha)} \mu_k z^k - \sum_{k=1}^{\infty} \frac{(c)_{k-1}(1-\beta)}{(a)_{k-1}(1-\alpha+k\alpha)} \eta_k \bar{z}^k.$$

Then

$$\sum_{k=2}^{\infty} \frac{(a)_{k-1}(1-\alpha+k\alpha)}{(c)_{k-1}(1-\beta)} \frac{(c)_{k-1}(1-\beta)}{(a)_{k-1}(1-\alpha+k\alpha)} \mu_k$$

$$- \sum_{k=1}^{\infty} \frac{(a)_{k-1}(1-\alpha+k\alpha)}{(c)_{k-1}(1-\beta)} \frac{(c)_{k-1}(1-\beta)}{(a)_{k-1}(1-\alpha+k\alpha)} \eta_k = \sum_{k=2}^{\infty} \mu_k + \sum_{k=2}^{\infty} \eta_k = 1 - \mu_1 \le 1.$$

Conversely, suppose that $f \in SHL^-(\alpha,\beta)$. Then

$$\sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} (1-\alpha+k\alpha)(|a_k| + |b_k|) \le 2-\beta.$$

Setting

$$\mu_k = \frac{(a)_{k-1}(1-\alpha+k\alpha)}{(c)_{k-1}(1-\beta)} |a_k|, \quad 0 \le \mu_k \le 1, \quad k = 2,3,\dots,$$

$$\eta_k = \frac{(a)_{k-1}(1-\alpha+k\alpha)}{(c)_{k-1}(1-\beta)}|b_k|, \quad 0 \le \eta_k \le 1, \quad k = 1,2,3,\dots,$$

and $\mu_1 = 1 - \eta_1 - \sum_{k=2}^{\infty}(\mu_k + \eta_k)$ we obtain $f(z) = \sum_{k=1}^{\infty}(\mu_k h_k + \eta_k g_k)$ as required. □

Theorem 6. *Each member of $SHL^{-}(\alpha,\beta)$ maps Δ onto a starlike domain where $\alpha \ge 0$, $0 \le \beta < 1$ and $a,\ c > 0$, $\frac{a(1+\alpha)}{c} \ge 2$.*

Proof. We need to show only if $f = h + \bar{g} \in SHL^{-}(\alpha,\beta)$, then

$$\mathrm{Re}\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + g(z)}\right\} > 0.$$

Using the fact that Re$w > 0$ if and only if $|1+w| > |1-w|$, it suffices to show that

$$\begin{aligned}
&|h(z) + \overline{g(z)} + zh'(z) - \overline{zg'(z)}| - |h(z) + \overline{g(z)} - zh'(z) + \overline{zg'(z)}| \\
&= \left|2z - \sum_{k=2}^{\infty}(k+1)|a_k|z^k + \sum_{k=1}^{\infty}(k-1)|b_k|\bar{z}^k\right| \\
&\quad - \left|\sum_{k=2}^{\infty}(k-1)|a_k|z^k - \sum_{k=1}^{\infty}(k+1)|b_k|\bar{z}^k\right| \\
&\ge 2|z| - \left|\sum_{k=2}^{\infty}(k+1)|a_k|z^k + \sum_{k=1}^{\infty}(k-1)|b_k|\bar{z}^k\right| \\
&\quad - \left|\sum_{k=2}^{\infty}(k-1)|a_k|z^k - \sum_{k=1}^{\infty}(k+1)|b_k|\bar{z}^k\right| \\
&\ge 2|z|\left[1 - \left(\sum_{k=2}^{\infty}k|a_k||z|^{k-1} + \sum_{k=1}^{\infty}k|b_k||\bar{z}|^{k-1}\right)\right] \\
&\ge 2|z|\left[1 - \left(\sum_{k=2}^{\infty}k|a_k||z|^{k-1} + \sum_{k=1}^{\infty}k|b_k||\bar{z}|^{k-1}\right)\right] \\
&> 2|z|[1 - (\sum_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)|a_k| + \sum_{k=1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)|b_k|)] \\
&> 2|z|[1-(1-\beta)] = 2|z|\beta \ge 0.
\end{aligned}$$

□

Theorem 7. *If $f(z) \in SHL^{-}(\alpha,\beta)$ $\alpha \ge 0$, $0 \le \beta < 1$, a, $c >$*

0 *and* $\frac{a(1+\alpha)}{c} \geq 2$, *then* f *is convex in the disc*

$$|z| < \min\left[\frac{1-\beta-|b_1|}{k}\right]^{\frac{1}{k-1}}, \quad k = 2, 3, \ldots, \quad (1-\beta) > |b_1|.$$

Proof. Let $f(z) \in SHL^-(\alpha, \beta)$ and let r be fixed such that $0 < r < 1$, then $\frac{1}{r} f(rz) \in SHL^-(\alpha, \beta)$ and have

$$\sum_{k=2}^{\infty} k^2(|a_k| + |b_k|) r^{k-1} = \sum_{k=2}^{\infty} k(|a_k| + |b_k|)(kr^{k-1})$$
$$\leq \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}(1 - \alpha + k\alpha)(|a_k| + |b_k|)(kr^{k-1}) \leq 1 - \beta - |b_1|,$$

provided $kr^{k-1} \leq 1 - \beta - |b_1|$. This holds if

$$r < \left[\frac{1-\beta-|b_1|}{k}\right]^{\frac{1}{k-1}}, \quad k = 2, 3, \ldots, \quad (1-\beta) > |b_1|. \qquad \square$$

For $f(z) \in S_H^-$ and $\delta \geq 0$ we define neighborhood $N_\delta(f)$ as follows, see [7]

$$N_\delta(f) = \{F(z)$$
$$= z - \sum_{k=2}^{\infty} A_k z^k - \sum_{k=1}^{\infty} |B_k| \bar{z}^k : \sum_{k=1}^{\infty} k(|a_k - A_k| + |b_k - B_k|) \leq \delta\}. \quad (9)$$

From (9), we obtain

$$\sum_{k=1}^{\infty} k(|a_k - A_k| + |b_k - B_k|) = |b_1 - B_1| + \sum_{k=2}^{\infty} k(|a_k - A_k| + |b_k - B_k|) \leq \delta.$$

The following lemma is required for the proof of Theorem 8.

Lemma. (see [9]) *For* $f(z)$ *of the form* (2), *then* f *is harmonic starlike function if and only if*

$$\sum_{k=2}^{\infty} k(a_k + b_k) \leq 1.$$

Theorem 8. *Let* $f(z) \in SHL^-(\alpha, \beta)$, $\alpha \geq 0$, $0 \leq \beta < 1$, a, $c > 0$, $\frac{a(1+\alpha)}{c} \geq 2$ *and* $\delta \leq \beta$. *If* $F \in N_\delta(f)$, *then* F *is a harmonic starlike function.*

Proof. Let $F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k - \sum_{k=1}^{\infty} |B_k| \bar{z}^k \in N_\delta(f)$. We have

$$\sum_{k=2}^{\infty} k(|A_k| + |B_k|) + |B_1|)$$

$$= \sum_{k=2}^{\infty} k(|a_k - A_k| + |b_k - B_k|) + \sum_{k=2}^{\infty} k(|a_k| + |b_k|) + |B_1 - b_1| + |b_1|$$
$$\leq \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)(|a_k - A_k| + |b_k - B_k|)$$
$$+ \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}(1-\alpha+k\alpha)(|a_k| + |b_k|) + |B_1 - b_1| + |b_1|$$
$$\leq \delta + |b_1| + (1-\beta-|b_1|) \leq 1.$$

Hence $F(z)$ is a harmonic starlike function. □

For harmonic functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=1}^{\infty} |b_k| \bar{z}^k, \tag{10}$$

$$F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k - \sum_{k=1}^{\infty} |B_k| \bar{z}^k. \tag{11}$$

The convolution of two harmonic functions (10) and (11) is given by

$$(f * F)(z) = z - \sum_{k=2}^{\infty} |a_k||A_k| z^k - \sum_{k=1}^{\infty} |b_k||B_k| \bar{z}^k. \tag{12}$$

To show the class $SHL^{-}(\alpha,\beta)$ is closed under convolution, we proved the following result

Theorem 9. *For $0 \leq \alpha_1 \leq \alpha_2 < 1$, $0 \leq \beta_1 \leq \beta_2 < 1$ and $\frac{a(1+\alpha)}{c} \geq 2$. Let $f \in SHL^{-}(\alpha_2,\beta_2)$ and $F \in SHL^{-}(\alpha_1,\beta_1)$. Then $(f * F) \in SHL^{-}(\alpha_2,\beta_2) \subset SHL^{-}(\alpha_1,\beta_1)$.*

Proof. Let

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=1}^{\infty} |b_k| \bar{z}^k$$

be in the class $SHL^{-}(\alpha_2,\beta_2)$ and $F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k - \sum_{k=1}^{\infty} |B_k| \bar{z}^k$ be in $SHL^{-}(\alpha_1,\beta_1)$. Then the convolution $f * F$ is given by (12). We wish to show that the coefficient of $f * F$ satisfy the required condition in Theorem 3. For $F \in SHL^{-}(\alpha_1,\beta_1)$ we note that $|A_k| < 1$ and $|B_k| < 1$. Now, for the convolution $f * F$, we obtain

$$\sum_{k=2}^{\infty} \frac{(a)_{k-1}(1-\alpha_1+k\alpha_1)}{(c)_{k-1}(1-\beta_1)} |a_k||A_k| + \sum_{k=1}^{\infty} \frac{(a)_{k-1}(1-\alpha_1+k\alpha_1)}{(c)_{k-1}(1-\beta_1)} |b_k||B_k|$$

$$\leq \sum_{k=2}^{\infty} \frac{(a)_{k-1}(1-\alpha_1+k\alpha_1)}{(c)_{k-1}(1-\beta_1)}|a_k| + \sum_{k=1}^{\infty} \frac{(a)_{k-1}(1-\alpha_1+k\alpha_1)}{(c)_{k-1}(1-\beta_1)}|b_k|$$

$$\leq \sum_{k=2}^{\infty} \frac{(a)_{k-1}(1-\alpha_2+k\alpha_2)}{(c)_{k-1}(1-\beta_2)}|a_k| + \sum_{k=1}^{\infty} \frac{(a)_{k-1}(1-\alpha_2+k\alpha_2)}{(c)_{k-1}(1-\beta_2)}|b_k| \leq 1.$$

Since $0 \leq \alpha_1 \leq \alpha_2$, $0 \leq \beta_1 \leq \beta_2 < 1$ and $f \in SHL^-(\alpha_2, \beta_2)$, therefore $(f * F) \in SHL^-(\alpha_2, \beta_2) \subset SHL^-(\alpha_1, \beta_1)$. □

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