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THE FACTORIZATION OF MONOTONE MORPHISM IN A TOPOS

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Abstract: In this paper, we investigate the factorization of a monotone morphism between two partially ordered objects in an arbitrary elementary topos. The factorization theorem in an arbitrary elementary topos is obtained.

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1. Introduction and Preliminaries

The development of topos theory resulted from the confluence of two streams of mathematical thought from the 20-th Sixties. The first of these is the development of an axiomatic treatment of sheaf theory by Grothendieck. This axiomatic development culminated in the discovery by Giraud that a category is equivalent to a category of sheaves for a Grothendieck topology if and only if it satisfies the conditions for being what is now called a Grothendieck topos. The main purpose of the axiomatic development is to be able to define sheaf cohomology. The second stream is Lawvere's continuing search for a natural way of founding mathematics (universal algebra, set theory, category theory, etc.) on the basic notions of morphism and composition of morphisms. All formal (and naive) presentations of set theory up to then had taken as primitives the notions of elements and sets with membership as the primitive relation. Now a topos can be considered both as a "generalized space" and as a "generalized

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universe of sets". Topos theory unifies this two seemingly wholly distinct mathematical aspects.

Recall a topos $\mathcal{E}$ is a category which has finite limits and every object of $\mathcal{E}$ has a power object. For a fixed object A of category $\mathcal{E}$, the power object of A is an object PA which represents $\mathrm{Sub}(-\times A)$, so that $\mathrm{Hom}_{\mathcal{E}}(-, PA) \simeq \mathrm{Sub}(-\times A)$ naturally. It says precisely that for any arrow $B' \xrightarrow{f} B$, the following diagram commutes, where φ is the natural isomorphism.

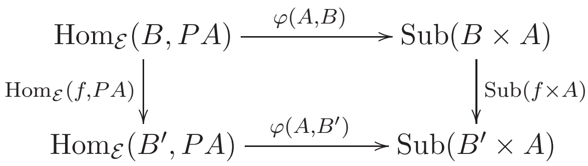

Figure 1:

As a matter of fact, the category of sheaves of sets on a topological space is a topos. In particular, the category of sets is a topos. For details of the treatment of toposes and sheaves please see Johnstone [6], Mac and Moerdijk [10], Joyal and Tierney [8], Johnstone and Joyal [7]. For a general background on category theory please refer to [2], [9]

In [10], Lattice and Heyting Algebra objects in a topos are well defined. In this paper we develop our study in the more general and more natural context of partially ordered object and the factorization theorem in categorical sense. More details about lattice and locale please see [3], [4], [1], [5].

2. Main Results

Definition 1. (see [10]) A subobject $\leq_L \rightarrowtail L \times L$ is called an internal partial order on L, provided that the following conditions are satisfied

1) Reflexivity: The diagonal $L \xrightarrow{\delta} L \times L$ factors through $\leq_L \xrightarrow{e_L} L \times L$, as in

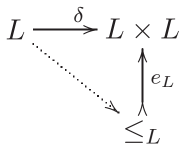

Figure 2: Reflexivity

2) Antisymmetry: The intersection $\leq_L \cap \geq_L$ is contained in the diagonal, as in the following pullback

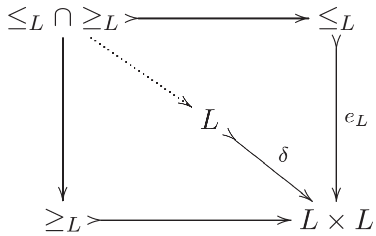

Figure 3: Antisymmetry

Here $\geq_L$ is defined as the composite $\leq_L \xrightarrow{e_L} L \times L \xrightarrow{\tau} L \times L$ with τ as the twist map interchanging the factors of the product.

3) Transitivity: The subobject $C \xrightarrow{\langle \pi_1 ev, \pi_2 eu \rangle} L \times L$ factors through $\leq_L \xrightarrow{e_L} L \times L$, as in

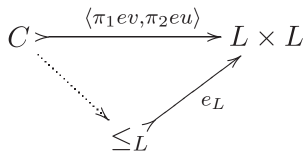

Figure 4: Transitivity

where C is the following pullback

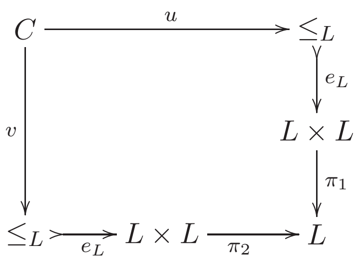

Figure 5: The definition of C

An object L endowed with an internal partial order $\leq_L$ is called a partially ordered object.

Let L and M be two partially ordered objects. We can define the product of partially ordered object $L \times M$ of L and M as the product object $L \times M$ endowed with the "pointwise order" $\leq_L \times \leq_M \rightarrowtail L \times L \times M \times M \simeq L \times M \times L \times M$. Also, a subobject B of a partially ordered object $(L, \leq_L)$ is again a partial order object endowed with the induced partial order $\leq_B$, as in the pullback

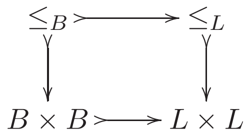

Figure 6: The induced partial order

We now turn to the discussion of morphisms between partial order objects.

In [10], for morphisms $L \overset{f}{\underset{g}{\rightrightarrows}} M$ between two objects in a topos, $f \leq g$ is defined to be $L \xrightarrow{\langle f,g \rangle} M \times M$ factors through $\leq_M \overset{e_M}{\rightarrowtail} M \times M$, as in

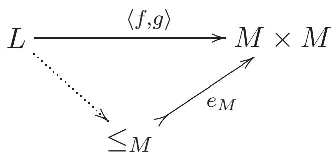

Figure 7: The first definition of $f \leq g$

Lemma 2. *Let L, M be two partially ordered objects with a pair of morphisms $L \overset{f}{\underset{g}{\rightrightarrows}} M$. Then $f \leq g$ if and only if $fr \leq gr$ for every morphism $A \xrightarrow{r} L$.*

Proof. $\Rightarrow$ Suppose $f \leq g$, then there exists a morphism $L \xrightarrow{k} \leq_M$ such that $\langle f, g \rangle = e_M k$. So $\langle fr, gr \rangle = \langle f, g \rangle r = e_M kr$, which means the outer triangle of Figure 8 below is commutative, i.e., $\langle fr, gr \rangle$ factors through $\leq_M \overset{e_M}{\rightarrowtail} M \times M$.

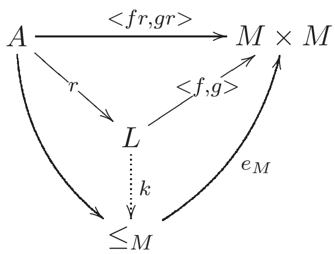

Figure 8: Equivelence of two definitions

$\Leftarrow$ Indeed, in order to verify this, we can take the fixed identity morphism

$L \xrightarrow{1_L} L$, then $f \leq g$ is obvious. □

Corollary 3. *Let L, M be two partially ordered objects and $L \xrightarrow{f} M$ be a morphism. Then $f \leq f$.*

Proof. Since $p_i\langle f, f\rangle = p_i\delta f$ with $p_i : M \times M \to M$ $(i = 1, 2)$ being projections, $\langle f, f\rangle = \delta f$. And by Definition 1, we know δ factors through $\leq_M \xrightarrow{e_M} M \times M$. It follows that the outer square is commutative as in the following Figure 9.

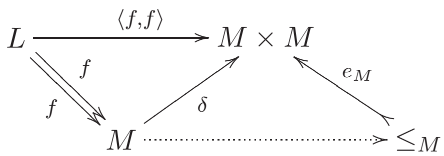

Figure 9: Reflexivity of f

So we have that $\langle f, f\rangle$ factors through $\leq_M \xrightarrow{e_M} M \times M$, thus $f \leq f$. □

Corollary 4. *Let L, M be two partially ordered objects and f, g, h morphisms between L and M. Then $f \leq g$ and $g \leq h$ imply $f \leq h$.*

Corollary 5. *Let L, M be two partially ordered objects and $f : L \to M, g : M \to L$ be morphisms. Then $f \leq g$ and $g \leq f$ imply $f = g$.*

Proof. $g \leq f$ implies that $\langle g, f\rangle : L \to M \times M$ can be factored through $\leq_M \rightarrowtail M \times M$, equivalently, $\langle f, g\rangle$ can be factored through $\geq_M \rightarrowtail M \times M$. Thus $\langle f, g\rangle : L \to M \times M$ can be factored through $\delta_M = \leq_M \cap \geq_M \rightarrowtail M \times M$. This shows $f = g$. □

The above argument shows that for two partially ordered objects L and M, the relation $\leq$ defined on the morphism set $\mathrm{Mor}(L, M)$ is a partial order relation.

Definition 6. (see [10]) Let L, M be two partially ordered objects in $\mathcal{E}$. A morphism $L \xrightarrow{f} M$ is called order-preserving or monotone if the composite

$\leq_L \xrightarrow{e_L} L \times L \xrightarrow{f\times f} M \times M$ factors through $\leq_M$, as in

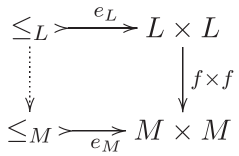

Figure 10: The definition of a monotone morphism

Lemma 7. *A morphism $L \xrightarrow{f} M$ between two partial ordered objects is order-preserving if and only if $r \leq s$ implies $fr \leq fs$ for every pair of parallel morphisms $A \overset{r}{\underset{s}{\rightrightarrows}} L$.*

Proof. $\Rightarrow$ We first show $\langle fr, fs\rangle = f \times f\langle r, s\rangle$. This may be pictured as in the following Figure 11, where p_1, p_2, π_1, π_2 are projections.

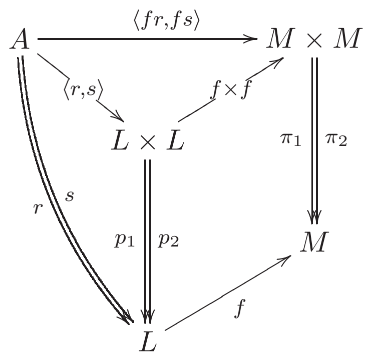

Figure 11: Universal property of product

By the universal property of $M \times M$, it follows that $fp_i = \pi_i f \times f$, $i = 1, 2$. Similarly, $r = p_1\langle r, s\rangle, s = p_2\langle r, s\rangle$. Then $fp_i\langle r, s\rangle = \pi_i f \times f\langle r, s\rangle$, so $fr = \pi_1 f \times f\langle r, s\rangle, fs = \pi_2 f \times f\langle r, s\rangle$. By the universal property of $M \times M$, we also have $fr = \pi_1\langle fr, fs\rangle, fs = \pi_2\langle fr, fs\rangle$. So, $\pi_1 < fr, fs >= \pi_1 f \times f\langle r, s\rangle, \pi_2\langle fr, fs\rangle = \pi_2 f \times f\langle r, s\rangle$, thus $\langle fr, fs\rangle = f \times f\langle r, s\rangle$.

Now suppose $r \leq s$, then there exists a morphism $A \xrightarrow{k} \leq_L$ with $\langle r, s\rangle = e_L k$. It follows that the left triangle of in Figure 12 is commutative. Since f is monotone, the right square of the Figure 12 is commutative, i.e., there exists $\leq_L \xrightarrow{m} \leq_M$ such that $f \times fe_L = e_M m$. So $\langle fr, fs\rangle = f \times f\langle r, s\rangle = f \times fe_L k = e_M mk$, which means the outer of the Figure 12 is commutative.

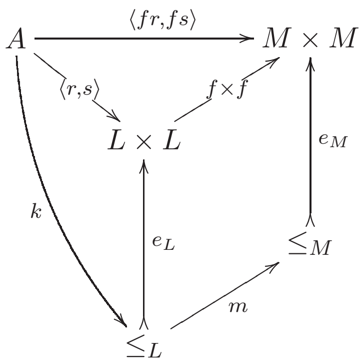

Figure 12: The relation between $< fr, fs >$ and e_M

Thus, $\langle fr, fs\rangle$ factors through $\leq_M \stackrel{e_M}{\rightarrowtail} M \times M$.

$\Leftarrow$ It suffices to show there exists $\leq_L \stackrel{m}{\longrightarrow} \leq_M$ with $f \times f e_L = e_M m$, as in the Figure 13.

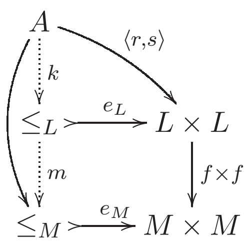

Figure 13: The existence of m

By Lemma 2, it is obvious that m exists. □

It is well known that the image of an arrow f is the smallest subobject (of the codomain f) through which f can factor. And the factorization of f is unique "up to isomorphism" as the following two lemmas show.

Lemma 8. (see [10]) *In a topos, every morphism f has an image m and factors as $f = me$, with e epi.*

Lemma 9. *If $f = me$ and $f' = m'e'$ with m, m' monic and e, e' epi, then each map of the arrow f to the arrow f' extends to a unique map of m, e to m', e'.*

Proof. A map of the arrow f to the arrow f' is a pair of arrows r, t which make the following square commute.

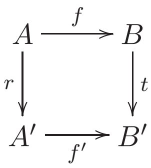

Figure 14:

Given such a pair of arrows and the two $e-m$ factorizations, it suffices to construct a unique arrow s from m to m' which makes both squares in the following diagram commute.

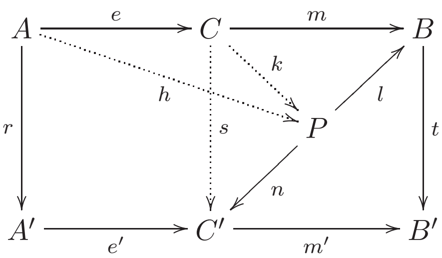

Figure 15:

Take the pullback P of t along m', as is shown in the above diagram, then l is monic. By the definition of the pullback and Figure 14, then, there exists the unique h such that f factors through l, i.e., $f = lh$. By the minimal property of the image, then there exists one unique arrow k, such that $m = lk$. Because l is monic, then $h = ke$. Let $s = nk$, then tm factors through C' via s, as $tm = m's$, and the arrow s is unique because m' is monic. Moreover, we have $se = e'r$ for the same reason, which means the left hand square of the above diagram also commutes. □

Theorem 10. *If a monotone morphism $L \xrightarrow{f} M$ between two partially ordered objects factors as $f = me$ with image m. Then m and e are monotone morphisms.*

Proof. Given $L \xrightarrow{f} M$, which factors as $L \xrightarrow{e} I \rightarrowtail^{m} M$. The proof is just a matter of observing the corresponding partial order on I. Construct the following commutative Figure 16

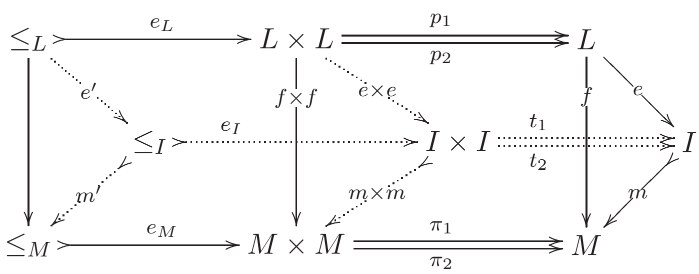

Figure 16: The partial order on I

By the definition of product $L\times L$, $M\times M$, $I\times I$ with projections p_i, π_i, t_i $(i = 1, 2)$ respectively, we have $fp_i = \pi_i f \times f$, $ep_i = t_i e \times e$, $mt_i = \pi_i m \times m$, i.e., the front, back, bottom faces of the right side of the diagram are all commutative. Then, $\pi_i f \times f = fp_i = mep_i = \pi_i m \times m \cdot e \times e$, so $f \times f = m \times m \cdot e \times e$, which means the middle triangle is commutative. Since the smallestness of $m \times m$ is obvious, $f \times f = m \times m \cdot e \times e$ is again an epi-momo factorization, i.e., $m \times m$ is the image of $f \times f$.

We take $\leq_I$ as the pullback of $I \times I \to M \times M$ along e_M, that is, $\leq_I = (I \times I) \cap \leq_M$. It is easy to prove that $\leq_I$ is just both the induced partial order on I and the image of $\leq_L$. This shows the back and the bottom faces of the left side of the diagram are commutative, in other words, $\leq_I \overset{e_L}{\rightarrowtail} I \times I \xrightarrow{m\times m} M \times M$ and $\leq_L \overset{e_L}{\rightarrowtail} L \times L \xrightarrow{e\times e} I \times I$ factor through $\leq_M \overset{e_M}{\rightarrowtail} M \times M$ and $\leq_I \overset{e_I}{\rightarrowtail} I \times I$ respectively. So m, e are all monotone morphisms. □

Acknowledgments

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