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LAX PAIR FOR 2-D SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract: The algorithm of construction of Lax pair for 2-d system of linear differential equations is represent.

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1. Introduction

Consider a system of 2 linear differential equations

$$\dot{\mathbf{x}} = \mathbf{a}\mathbf{x}. \tag{1}$$

Lax pair give us the representation of these system in form

$$\dot{\mathbf{L}} = [\mathbf{M}\mathbf{L}]$$

provided that invariant I of system (1) coincides with a trace of a matrix $\mathbf{L}^2$:

$$\frac{1}{2}\,\mathrm{Tr}\mathbf{L}^2 = I.$$

Example 1. Consider the harmonic oscillator with frequency ω:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Lax pair (see [1]) for this system has the form

$$\mathbf{L} = \begin{pmatrix} x\omega & y \\ y & -x\omega \end{pmatrix}, \qquad \mathbf{M} = \frac{1}{2}\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix},$$

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and

$$I = \frac{1}{2}\,\mathrm{Tr}\mathbf{L}^2 = x^2\omega^2 + y^2.$$

2. Undamped System

Notes, then system (1) in jet bundle $J^1(1,2)$ can be always rewrited as one equation in $J^2(1,1)$:

$$\ddot{x} - \dot{x}\,\mathrm{Tr}\mathbf{a} + \Delta x = 0,$$

where $\mathrm{Tr}\,\mathbf{a}$ and Δ are traces and determinant of matrix $\mathbf{a}$. If $\mathrm{Tr}\,\mathbf{a} = 0$, then $a_{22} = -a_{11}$ and we have undamped dynamical system. Further, for convenience of calculations we consider system (1) in form

$$\begin{cases} \dot{x} = a_{11}x + a_{12}y, \\ \dot{y} = -a_{21}x - a_{11}y, \end{cases}$$

which has invariant

$$I = a_{21}x^2 + 2a_{11}xy + a_{12}y^2. \tag{2}$$

Assume that the Lax pair for system (1) has the form

$$\mathbf{L} = \begin{pmatrix} A_{11}x + B_{11}y & A_{21}x + B_{21}y \\ A_{21}x + B_{21}y & -A_{11}x - B_{11}y \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 0 & M_1 \\ -M_2 & 0 \end{pmatrix};$$

then its invariants can be written as

$$I_1 = \mathrm{Tr}\,\mathbf{L} = 0,$$

$$I_2 = \frac{1}{2}\mathrm{Tr}\,\mathbf{L}^2 = (A_{11}^2 + A_{21}^2)x^2 + 2\,(A_{11}B_{11} + A_{21}B_{21})\,xy + (B_{11}^2 + B_{21}^2)y^2. \tag{3}$$

Comparing (2) and (3) we shall receive system for definition of coefficients A_{ik} and B_{ik}:

$$\begin{cases} a_{21} = A_{11}^2 + A_{21}^2, \\ a_{11} = A_{11}B_{11} + A_{21}B_{21}, \\ a_{12} = B_{11}^2 + B_{21}^2. \end{cases}$$

As the given system is overfull, we have 4 elementary cases (if $A_{ik} = 0$ or $B_{ik} = 0$):

$$\begin{array}{llll} A_{11} = 0, & A_{21} = \sqrt{a_{21}}, & B_{11} = \sqrt{\frac{\Delta}{a_{21}}}, & B_{21} = \frac{a_{11}}{\sqrt{a_{21}}}, \\ A_{11} = \sqrt{a_{21}}, & A_{21} = 0, & B_{11} = \frac{a_{11}}{\sqrt{a_{21}}}, & B_{21} = \sqrt{\frac{\Delta}{a_{21}}}, \end{array} \tag{4}$$

$$A_{11}=\sqrt{\tfrac{\Delta}{a_{12}}},\quad A_{21}=\tfrac{a_{11}}{\sqrt{a_{12}}},\quad B_{11}=0,\quad B_{21}=\sqrt{a_{12}},$$
$$A_{11}=\tfrac{a_{11}}{\sqrt{a_{12}}},\quad A_{21}=\sqrt{\tfrac{\Delta}{a_{12}}},\quad B_{11}=\sqrt{a_{12}},\quad B_{21}=0.$$

Comparing $\dot{\mathbf{L}}$ and $[\mathbf{ML}]$ we find that:

$$\begin{aligned}
\dot{x} &= -\frac{M_1+M_2}{\sqrt{\Delta}}\left(a_{11}x+a_{12}y\right), && \dot{y}=\frac{M_1+M_2}{\sqrt{\Delta}}\left(a_{21}x+a_{11}y\right),\\
\dot{x} &= \frac{2M_1}{\sqrt{\Delta}}a_{11}x+\frac{M_1+M_2}{\sqrt{\Delta}}a_{12}y, && \dot{y}=-\frac{2M_1}{\sqrt{\Delta}}\left(a_{21}x+a_{11}y\right),\\
\dot{x} &= \frac{M_1+M_2}{\sqrt{\Delta}}\left(a_{11}x+a_{12}y\right), && \dot{y}=-\frac{M_1+M_2}{\sqrt{\Delta}}\left(a_{21}x+a_{11}y\right),\\
\dot{x} &= -\frac{2M_1}{\sqrt{\Delta}}\left(a_{11}x+a_{12}y\right), && \dot{y}=\frac{M_1+M_2}{\sqrt{\Delta}}a_{21}x+\frac{2M_1}{\sqrt{\Delta}}a_{11}y.
\end{aligned}$$

If we assume that $M_2=M_1$ then

$$M_1=-\frac{\sqrt{\Delta}}{2},\qquad M_1=\frac{\sqrt{\Delta}}{2},\qquad M_1=\frac{\sqrt{\Delta}}{2},\qquad M_1=-\frac{\sqrt{\Delta}}{2}. \tag{5}$$

Example 2. Find the Lax pair to the phase flows

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}0 & a_{12}\\ -a_{21} & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

From (4) and (5) we have

$$A_{11}=\frac{a_{11}}{\sqrt{a_{12}}}=0,\qquad A_{21}=\sqrt{\frac{\Delta}{a_{12}}},\qquad B_{11}=\sqrt{a_{12}},\qquad B_{21}=0,$$

and $M_1=-\frac{\sqrt{\Delta}}{2}$, then

$$\mathbf{L}=\begin{pmatrix}y\sqrt{a_{12}} & x\sqrt{a_{21}}\\ x\sqrt{a_{21}} & -y\sqrt{a_{12}}\end{pmatrix},\qquad \mathbf{M}=\frac{\sqrt{\Delta}}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Example 3. Find the Lax pair to the phase flows

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}a_{11} & a_{12}\\ -a_{21} & -a_{11}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

From (4) and (5) we have

$$A_{11}=0,\quad A_{21}=\sqrt{a_{21}},\quad B_{11}=\sqrt{\frac{\Delta}{a_{21}}},\quad B_{21}=\frac{a_{11}}{\sqrt{a_{21}}},$$

and $M_1 = -\frac{\sqrt{\Delta}}{2}$, then

$$\mathbf{L} = \begin{pmatrix} \sqrt{\frac{\Delta}{a_{21}}}y & \sqrt{a_{21}}x + \frac{a_{11}}{\sqrt{a_{21}}}y \\ \sqrt{a_{21}}x + \frac{a_{11}}{\sqrt{a_{21}}}y & -\sqrt{\frac{\Delta}{a_{21}}}y \end{pmatrix},$$

$$\mathbf{M} = \frac{\sqrt{\Delta}}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

3. Damped System

In the general case the dynamic system (1)

$$\begin{cases} \dot{x} = b_{11}x + b_{12}y, \\ \dot{y} = b_{21}x + b_{22}y \end{cases}$$

contain damped component (if $\operatorname{Tr}\mathbf{b} \neq 0$ and $b_{22} \neq b_{11}$).

Here, it is convenient to decompose the operator $\mathbf{b}$ into two parts called spherical and deviatoric tensors:

$$b_{ik} = \left(b_{ik} - \frac{1}{2}\delta_{ik}b_{jj}\right) + \frac{1}{2}\delta_{ik}b_{jj}.$$

Then

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{b_{11}-b_{22}}{2} & b_{12} \\ b_{21} & -\frac{b_{11}-b_{22}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{b_{11}+b_{22}}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and Lax pair may be written into Kac-Moody form [2]

$$\dot{\mathbf{L}} = [\mathbf{ML}] + \lambda\mathbf{L},$$

where

$$\lambda = \frac{b_{11}+b_{22}}{2}.$$

Example 4. Find the Lax pair to the phase flows

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

After decompose the operator $\mathbf{b}$ into spherical and deviatoric parts:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{b_{11}-b_{22}}{2} & b_{12} \\ b_{21} & -\frac{b_{11}-b_{22}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \frac{b_{11}+b_{22}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

the change of entries

$$\begin{pmatrix} \frac{b_{11}-b_{22}}{2} & b_{12} \\ b_{21} & -\frac{b_{11}-b_{22}}{2} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ -a_{21} & -a_{11} \end{pmatrix}$$

is necessary.

From (4) and (5) we have

$$A_{11} = \sqrt{\frac{\Delta}{a_{12}}}, \quad A_{21} = \frac{a_{11}}{\sqrt{a_{12}}}, \quad B_{11} = 0, \quad B_{21} = \sqrt{a_{12}},$$

and $M_1 = \frac{\sqrt{\Delta}}{2}$, then

$$\mathbf{L} = \begin{pmatrix} \sqrt{\frac{\Delta}{a_{12}}}x & \frac{a_{11}}{\sqrt{a_{12}}}x + \sqrt{a_{12}}y \\ \frac{a_{11}}{\sqrt{a_{12}}}x + \sqrt{a_{12}}y & -\sqrt{\frac{\Delta}{a_{12}}}x \end{pmatrix}, \qquad \mathbf{M} = \frac{\sqrt{\Delta}}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Returning to the old entries we obtain

$$\mathbf{L} = \begin{pmatrix} \sqrt{\frac{\Delta}{b_{12}}}x & \frac{b_{11}-b_{22}}{2}\frac{1}{\sqrt{b_{12}}}x + \sqrt{b_{12}}y \\ \frac{b_{11}-b_{22}}{2}\frac{1}{\sqrt{b_{12}}}x + \sqrt{b_{12}}y & -\sqrt{\frac{\Delta}{b_{12}}}x \end{pmatrix},$$

where

$$\Delta = -b_{12}b_{21} - \left(\frac{b_{11}-b_{22}}{2}\right)^2.$$

4. Unentangled System

If our system is unentangled (has not cross entries), then

$$\begin{cases} \dot{x} = ax, \\ \dot{y} = by, \end{cases}$$

and we have two time-depended invariants

$$I_1 = xe^{-at}, \qquad I_2 = ye^{-bt}$$

for which Lax pair does not exist.

References

[1] O. Babelon, D. Bernard, M. Talon, *Introduction to Classical Integrable System*, Cambridge University Press (2003).

[2] A. Goriely, *Integrability and Nonintegrability of Dynamical Systems*, World Scientific Publishing, Singapore (2001).