International Journal of Pure and Applied Mathematics

Volume 63 No. 2 2010, 213-222

ON SPLITTING OF EXACT DIFFERENTIAL FORMS

Vladislav N. Dumachev

Department of Mathematics
Voronezh Institute of the MVD of Russia
53, Pr. Patriotov, Voronezh, 394065, RUSSIA
e-mail: dumv@comch.ru

Abstract: In this work the internal structure of de Rham cohomology is considered. As examples the phase flows in $\mathbb{R}^3$ admitting the Nambu Poisson structure are studied.

AMS Subject Classification: 14F40, 37K05, 53D17, 58A15
Key Words: Poisson structure, Hamiltonian vector fields, de Rham cohomology

1. Introduction

Let $\Lambda^k(\mathbb{M})$ – be the exterior graded algebra of differential forms with de Ram complex

$$0 \to \Lambda^0(\mathbb{M}) \to \Lambda^1(\mathbb{M}) \to ... \to \Lambda^{n-1}(\mathbb{M}) \to \Lambda^n(\mathbb{M}) \to 0.$$

Remember (see [1]), that differentual form $\omega \in \Lambda^k(\mathbb{M})$ is called closed if $d\omega = 0$, and exact if $\omega = d\nu$ for some $\nu \in \Lambda^{k-1}(\mathbb{M})$. The quotient of closed k-forms on manifolds $\mathbb{M}$ by exact k-forms is the k-th de Rham cohomology group

$$H^k(\mathbb{M}) = \frac{\ker\left(d : \Lambda^k(\mathbb{M}) \to \Lambda^{k+1}(\mathbb{M})\right)}{\operatorname{im}\left(d : \Lambda^{k-1}(\mathbb{M}) \to \Lambda^k(\mathbb{M})\right)}.$$

It is known, that cohomology $\mathbb{M} = \mathbb{R}^n$ is vanish. It means, that for anyone $\omega \in \Lambda^k(\mathbb{R}^n)$ such that $d\omega = 0$ there exists a $\nu \in \Lambda^{k-1}(\mathbb{R}^n)$ such that $\omega = d\nu$. In any case the dimension of cohomology group is determined by Betti numbers:

$$b^k = \frac{\dim\left(\omega : d\omega = 0\right)}{\dim\left(d\nu\right)}.$$

Received: July 6, 2010

2. Cohomology in $\Lambda^2(\mathbb{R}^n)$

Let us connect with de Ram complex the differential module $\{C, d\}$, then

$$Z(C) = \ker d = \{x \in C |\ dx = 0\}$$

are called cocycles of module $\{C, d\}$ (space of closed forms),

$$B(C) = \mathrm{im} d = dC = \{x = dy |\ y \in C\}$$

are called coboundary of module $\{C, d\}$ (space of exact forms). In given designations the group i-cohomology H^i be the quotient i-cocycles by i-coboundary

$$H^i = Z^i / B^i.$$

Assume that $\omega \in B^2 \subset \Lambda^2(\mathbb{R}^n)$. This means that

$$d\omega = 0, \quad \omega = d\nu, \quad \text{where} \quad \nu \in \Lambda^1(\mathbb{R}^n).$$

The received quotient we can write as

$$H_2^2(\mathbb{R}^n) = \frac{\ker\left(d : \Lambda^2(\mathbb{R}^n) \to \Lambda^3(\mathbb{R}^n)\right)}{\mathrm{im}\left(d : \Lambda^1(\mathbb{R}^n) \to \Lambda^2(\mathbb{R}^n)\right)} = Z^2/B^2.$$

Note however that in $\Lambda^2(\mathbb{R}^n)$ there exists a form

$$\omega = \lambda_1 \wedge \lambda_2, \quad \text{where} \quad \lambda_1, \lambda_2 \in \Lambda^1(\mathbb{R}^n) \quad \text{such that} \quad d\lambda_i = 0.$$

Let $\lambda_i \notin H^1(\mathbb{R}^n)$ (i.e. $\lambda_i = d\mu_i$), then

$$d\omega = 0, \qquad \omega = d\mu_1 \wedge d\mu_2.$$

In other words, exact $\omega \in \Lambda^2(\mathbb{R}^n)$ be wedge product of the other exact forms. We can write this space as

$$B^{1,1} = \{x = dy_1 \wedge dy_2 | y_i \in C\},$$

then

$$H_{1,1}^2(\mathbb{R}^n) = \frac{\ker\left(d : \Lambda^2(\mathbb{R}^n) \to \Lambda^3(\mathbb{R}^n)\right)}{\mathrm{im}\left(d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)\right) \oplus \mathrm{im}\left(d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)\right)} = Z^2/B^{1,1},$$

or

$$b_{1,1}^2(\mathbb{R}^n) = \frac{\dim\left(\omega : d\omega = 0\right)}{\dim\left(d\mu \wedge d\mu\right)}.$$

It is obvious that from $B^2 = \{d\mu_1 \wedge d\mu_2, d\nu\}$, and $B^{1,1} = \{d\mu_1 \wedge d\mu_2\}$ it follows that $B^{1,1} \subset B^2$. This means that quotient

$$B^2/B^{1,1} \simeq H_{1,1}^2/H_2^2$$

should characterize presence of obstacles (topological defects) for existence of the exact forms in $\Lambda^2(\mathbb{R}^n)$, which are wedge product of exact form from $\Lambda^1(\mathbb{R}^n)$.

Example 1. Consider the dynamical systems in $\mathbb{R}^3$

$$\dot{x} = -xz; \qquad \dot{y} = yz; \qquad \dot{z} = x^2 - y^2.$$

According to [2], this phase flow has one vectorial

$$\mathbf{h} = \frac{1}{4}\left((-x^2y + y^3 + yz^2)dx + (x^3 - y^2x + xz^2)dy - 2xyzdz\right)$$

and two scalar Hamiltonians

$$H = \frac{1}{2}(x^2 + y^2 + z^2), \qquad F = xy.$$

These Hamiltonians are connected by expressions

$$d\mathbf{h} = dH \wedge dF.$$

This means that our system admit Poisson structure with vectorial Hamiltonian

$$\dot{x}_i = \{\mathbf{h}, x_i\} = X_h \rfloor dx_i,$$

where

$$X_h = -xz\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} + (x^2 - y^2)\frac{\partial}{\partial z},$$

and Poisson structure (Nambu [3])

$$\dot{x}_i = \{H, F, x_i\} = X_H \rfloor dF \wedge dx_i = -X_F \rfloor dH \wedge dx_i,$$

where

$$\begin{aligned} X_H &= z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}, \\ X_F &= y\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + x\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}. \end{aligned}$$

Example 2. Divergence-free Lorenz set

$$\dot{x} = y - z; \qquad \dot{y} = -x + xz; \qquad \dot{z} = x - xy$$

has one vectorial

$$\begin{aligned} \mathbf{h} &= \left(\frac{x}{4}(z^2 + y^2) - \frac{x}{3}(z + y)\right) dx \\ &+ \left(\frac{1}{3}(x^2 - yz + z^2) - \frac{1}{4}x^2y\right) dy \\ &+ \left(\frac{1}{3}(y^2 - zy + x^2) - \frac{1}{4}x^2z\right) dz \end{aligned}$$

and two scalar Hamiltonians

$$H = \frac{1}{2}(x^2 + y^2 + z^2), \quad F = \left(y - \frac{y^2}{2}\right) + \left(z - \frac{z^2}{2}\right)$$

connected by expressions

$$d\mathbf{h} = dH \wedge dF.$$

This means that our system admit Poisson structure with vectorial Hamiltonian

$$\dot{x}_i = \{\mathbf{h}, x_i\} = X_h \rfloor dx_i,$$

where

$$X_h = (y - z)\frac{\partial}{\partial x} + (-x + xz)\frac{\partial}{\partial y} + (x - xy)\frac{\partial}{\partial z},$$

and Poisson structure in two forms

$$\dot{x}_i = \{H, F, x_i\} = X_H \rfloor dF \wedge dx_i = -X_F \rfloor dH \wedge dx_i,$$

where

$$\begin{aligned} X_H &= z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}, \\ X_F &= (1 - z)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + (1 - y)\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}. \end{aligned}$$

Example 3. Phase flow

$$\dot{x} = xy; \qquad \dot{y} = x - z; \qquad \dot{z} = -zy$$

has one vectorial

$$\mathbf{h} = \frac{1}{12}\begin{pmatrix} z(3y^2 + 4x - 4z \\ -6xyz \\ x(3y + 4z - 4x) \end{pmatrix},$$

one scalar Hamiltonian

$$H = x - \frac{y^2}{2} + z$$

and pre-Hamiltonian form

$$\Theta = -zdx + xdz$$

connected by expressions

$$d\mathbf{h} = dH \wedge \Theta.$$

This means that our system admit Poisson structure with vectorial Hamiltonian

$$\dot{x}_i = \{\mathbf{h}, x_i\} = X_h \rfloor dx_i,$$

where

$$X_h = xy\frac{\partial}{\partial x} + (x - z)\frac{\partial}{\partial y} - yz\frac{\partial}{\partial z},$$

and scalar Poisson structure in the form

$$\dot{x}_i = X_H \rfloor \Theta \wedge dx_i,$$

where

$$X_H = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} - y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}.$$

Therefore,

$$dh \in B^2 \setminus B^{1,1}.$$

For completeness of a statement we shall notice that as $d\Theta \neq 0$, but $d\Theta \wedge \Theta = 0$, then Pfaff equation on prehamiltonians form Θ has solved with integrating factor

$$dF = \frac{\Theta}{x^2 + z^2}, \qquad \Rightarrow \qquad F = \arctan \frac{z}{x}.$$

The caused of global non-integrability of the given system is the presence holes $(x = 0,\ z = 0)$ in $x0z$ planes. It is obvious that vanishing of second Hamiltonians has not admitted to enter of Nambu structure with a bracket $\{H, F, G\}$.

3. Cohomology in $\Lambda^3(\mathbb{R}^n)$

Further, the top index of the any form will denote its degree, i.e. $\omega^k \in \Lambda^k(\mathbb{R}^n)$. For standard de Rham complexes in $\mathbb{R}^n$ we get $\omega^3 \in B^3 \subset \Lambda^3(\mathbb{R}^n)$. This means that

$$d\omega^3 = 0, \quad \omega^3 = d\nu^2, \quad \text{where} \quad \nu^2 \in \Lambda^2(\mathbb{R}^n).$$

Thus

$$H_3^3(\mathbb{R}^n) = \frac{\ker\left(d : \Lambda^3(\mathbb{R}^n) \to \Lambda^4(\mathbb{R}^n)\right)}{\operatorname{im}\left(d : \Lambda^2(\mathbb{R}^n) \to \Lambda^3(\mathbb{R}^n)\right)} = Z^3/B^3,$$

$$b_3^3 = \frac{\dim\left(\omega^3 : d\omega^3 = 0\right)}{\dim\left(d\nu^2\right)}.$$

But in $\Lambda^3(\mathbb{R}^n)$ there exists are forms

$$\omega_1^3 = \lambda_1^1 \wedge \lambda_2^1 \wedge \lambda_3^1, \quad \text{where} \quad \lambda_i^1 \in \Lambda^1(\mathbb{R}^n), \quad \text{such that} \quad d\lambda_i^1 = 0.$$

Let $\lambda_i^1 \notin H^1(\mathbb{R}^n)$ (i.e. $\lambda_i^1 = d\mu_i^0$), then

$$d\omega_1^3 = 0, \qquad \omega_1^3 = d\mu_1^0 \wedge d\mu_2^0 \wedge d\mu_3^0.$$

In other words, exact $\omega_1^3 \in \Lambda^3(\mathbb{R}^n)$ be wedge product of the other exact forms. We can write this quotient as

$$H_{1,1,1}^3(\mathbb{R}^n) = \frac{\ker\left(d : \Lambda^3(\mathbb{R}^n) \to \Lambda^4(\mathbb{R}^n)\right)}{\bigoplus_{k=1}^{3} \operatorname{im}_k\left(d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)\right)} = Z^3/B^{1,1,1},$$

$$b^3_{1,1,1} = \frac{\dim\big(\omega^3 : d\omega^3 = 0\big)}{\dim\,(d\mu^0 \wedge d\mu^0 \wedge d\mu^0)}.$$

At the same time in $\Lambda^3(\mathbb{R}^n)$ it is possible also to construct the forms

$$\omega^3_2 = \lambda^1 \wedge \lambda^2, \quad \text{where} \quad \lambda^k \in \Lambda^k(\mathbb{R}^n)$$

such that $d\lambda^k = 0$. Let $\lambda^k \notin H^k(\mathbb{R}^n)$ (i.e. $\lambda^k_i = d\mu^k_i$), then

$$d\omega^3_2 = 0, \qquad \omega^3_2 = d\mu^0 \wedge d\mu^1.$$

In other words, exact ω^3_2 be wedge product of the other exact forms. We can write this quotient as

$$H^3_{2,1}(\mathbb{R}^n) = \frac{\ker\big(d : \Lambda^3(\mathbb{R}^n) \to \Lambda^4(\mathbb{R}^n)\big)}{\mathrm{im}\,(d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)) \oplus \mathrm{im}\,(d : \Lambda^1(\mathbb{R}^n) \to \Lambda^2(\mathbb{R}^n))} = Z^3/B^{2,1},$$

$$b^3_{1,2} = \frac{\dim\big(\omega^3 : d\omega^3 = 0\big)}{\dim\,(d\mu^0 \wedge d\mu^1)}.$$

Evidently that

$$H^3_{1,1,1}(\mathbb{R}^n) \supset H^3_{1,2}(\mathbb{R}^n) \supset H^3_3(\mathbb{R}^n),$$

such that quotients $H^3_{1,1,1}(\mathbb{R}^n)/H^3_3(\mathbb{R}^n)$ and $H^3_{1,1,1}(\mathbb{R}^n)/H^3_{2,1}(\mathbb{R}^n)$ should characterize presence of obstacles (topological defects) for existence of the exact forms in $\Lambda^3(\mathbb{R}^n)$, which are wedge product of exact form from $\Lambda^1(\mathbb{R}^n)$ or from $\Lambda^2(\mathbb{R}^n)$.

4. Cohomology in $\Lambda^k(\mathbb{R}^n)$

Generalizing the previous calculations we consider de Rham complex in $\mathbb{R}^n$ and get $\omega^k \in \Lambda^k(\mathbb{R}^n) \notin H^k(\mathbb{R}^n)$. Then

$$d\omega^k = 0, \quad \omega^k = d\nu^{k-1}, \quad \text{where} \quad \nu^{k-1} \in \Lambda^{k-1}(\mathbb{R}^n),$$

and

$$H^k_k(\mathbb{R}^n) = \frac{\ker\big(d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n)\big)}{\mathrm{im}\,(d : \Lambda^{k-1}(\mathbb{R}^n) \to \Lambda^k(\mathbb{R}^n))} = Z^k/B^k.$$

However, into $\Lambda^k(\mathbb{R}^n)$ there exists are forms

$$\omega^k = \bigwedge_{i=1}^{k} \lambda^1_i, \quad \text{where} \quad \lambda^1 \in \Lambda^1(\mathbb{R}^n)$$

such that $d\lambda^1 = 0$. Suppose that $\lambda^1 \notin H^1(\mathbb{R}^n)$ (i.e. $\lambda^1 = d\mu^0$), then

$$d\omega^k = 0, \qquad \omega^k = \bigwedge_{i=1}^{k} d\mu^0_i.$$

In other words, exact $\omega^k \in \Lambda^k(\mathbb{R}^n)$ be wedge product of the other exact forms. We can write this quotient as

$$H^k_{1,1,\dots,1}(\mathbb{R}^n) = \frac{\ker\left(d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{n+1}(\mathbb{R}^n)\right)}{\bigoplus_{i=1}^{k} \mathrm{im}_i \left(d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)\right)} = Z^k/B^{1,1,\dots,1},$$

$$b^k_{1,1,\dots,1} = \frac{\dim\left(\omega^k : d\omega^k = 0\right)}{\dim\left(\bigwedge_{i=1}^{k} d\mu_i\right)}.$$

Continuing we receive, similarly:

For $\omega^k = d\mu^1 \wedge \bigwedge_{i=1}^{k-2} d\mu_i^0$

$$H^k_{2,1,1,\dots,1}(\mathbb{R}^n) = \frac{\ker\left(d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n)\right)}{\mathrm{im}\left(d : \Lambda^1(\mathbb{R}^n) \to \Lambda^2(\mathbb{R}^n)\right) \bigoplus_{i=1}^{k-2} \mathrm{im}_i \left(d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)\right)},$$

$$b^k_{2,1,1,\dots,1} = \frac{\dim\left(\omega^k : d\omega^k = 0\right)}{\dim\left(d\mu^1 \wedge \bigwedge_{i=1}^{k-2} d\mu_i^0\right)}.$$

For $\omega^k = d\mu^2 \wedge \bigwedge_{i=1}^{k-3} d\mu_i^0$

$$H^k_{3,1,1,\dots,1}(\mathbb{R}^n) = \frac{\ker\left(d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n)\right)}{\mathrm{im}\left(d : \Lambda^2(\mathbb{R}^n) \to \Lambda^3(\mathbb{R}^n)\right) \bigoplus_{i=1}^{k-3} \mathrm{im}_i \left(d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)\right)},$$

where

$$b^k_{3,1,1,\dots,1} = \frac{\dim\left(\omega^k : d\omega^k = 0\right)}{\dim\left(d\mu^2 \wedge \bigwedge_{i=1}^{k-3} d\mu_i^0\right)}.$$

For $\omega^k = d\mu^3 \wedge \bigwedge_{i=1}^{k-4} d\mu_i^0$

$$H^k_{4,1,1,\dots,1}(\mathbb{R}^n) = \frac{\ker\left(d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n)\right)}{\mathrm{im}\left(d : \Lambda^3(\mathbb{R}^n) \to \Lambda^4(\mathbb{R}^n)\right) \bigoplus_{i=1}^{k-4} \mathrm{im}_i \left(d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)\right)},$$

where

$$b^k_{4,1,1,\dots,1} = \frac{\dim\left(\omega^k : d\omega^k = 0\right)}{\dim\left(d\mu^3 \wedge \bigwedge_{i=1}^{k-4} d\mu_i^0\right)};$$

$$\vdots$$

for $\omega^k = d\mu^1 \wedge d\mu^1 \wedge \bigwedge_{i=1}^{k-4} d\mu_i^0$

$$H^k_{2,2,1,\dots,1}(\mathbb{R}^n) = \frac{\ker\left(d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n)\right)}{\bigoplus\limits_{i=1}^{2} \mathrm{im}_i\left(d : \Lambda^1(\mathbb{R}^n) \to \Lambda^2(\mathbb{R}^n)\right) \bigoplus\limits_{i=1}^{k-4} \mathrm{im}_i\left(d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)\right)},$$

where

$$b^k_{2,2,1,1,\dots,1} = \frac{\dim\left(\omega^k : d\omega^k = 0\right)}{\dim\left(d\mu^1 \wedge d\mu^1 \wedge \bigwedge_{i=1}^{k-4} d\mu_i^0\right)}.$$

$$\vdots$$

etc.

For simplification of record we shall enter a multiindex

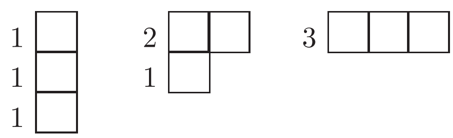
$$\#m = \{m_1, m_2, \dots, m_i\}, \quad m_1 + m_2 + \dots + m_i = k, \quad m_1 \geq m_2 \geq \dots \geq m_i \geq 0,$$

which is formed by a rule of construction of the Young diagrams. Then

$$H^k_{\#m}(\mathbb{R}^n) = \frac{\ker\left(d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n)\right)}{\bigoplus\limits_{\#m} \mathrm{im}_{\#m}\left(d : \Lambda^{\#m}(\mathbb{R}^n) \to \Lambda^{\#m+1}(\mathbb{R}^n)\right)} = Z^k / B^{\#m},$$

$$b^k_{\#m} = \frac{\dim\left(\omega^k : d\omega^k = 0\right)}{\dim\left(\bigwedge^{\#m} d\mu_i^0\right)}.$$

So, for $k = 3$ we shall receive



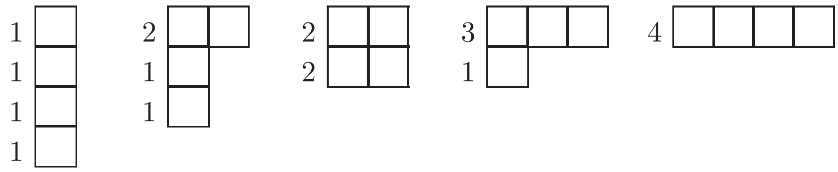
i.e.

$$\begin{aligned} \#m &= \{1,1,1\} \quad \text{or} \quad H^k_{\#m}(\mathbb{R}^n) = H^3_{1,1,1}(\mathbb{R}^n), \\ \#m &= \{2,1\} \quad \text{or} \quad H^k_{\#m}(\mathbb{R}^n) = H^3_{2,1}(\mathbb{R}^n), \\ \#m &= \{3\} \quad \text{or} \quad H^k_{\#m}(\mathbb{R}^n) = H^3_{3}(\mathbb{R}^n). \end{aligned}$$

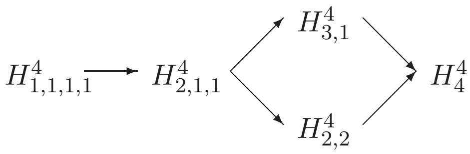
Then the filtered complex of cohomology can be represented as follows

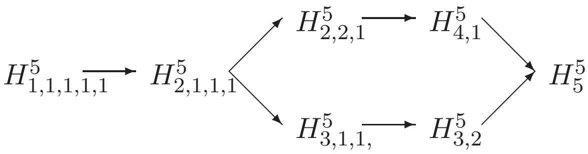
$$H^3_{1,1,1} \to H^3_{2,1} \to H^3_3.$$

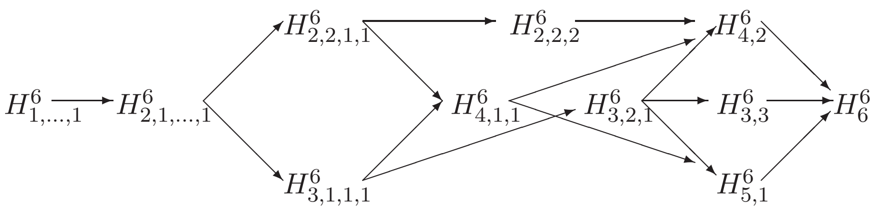
For $k = 4$ we get



i.e. $\#m = \{1,1,1,1\}$, $\#m = \{2,1,1\}$, $\#m = \{2,2\}$, $\#m = \{3,1\}$ or $\#m = \{4\}$. Let us notice, that at $k \geq 4$ the structure of cohomology $H^k_{\#m}(\mathbb{R}^n)$ is not linear and for some small $k = 4,5,6$ is shown in figures:







We can see that in the general case the cohomology sequences are not filtered. This means to define cohomology of cogomology is obviously impossible.

References

[1] R. Bott, L.W. Tu, *Differential Forms in Algebraic Topology. Graduate Texts in Mathematics 82*, Springer-Verlag, New York (1982).

[2] V.N. Dumachev, Phase flows and vector Lagrangians in $J^3(\pi)$, *IJPAM*, **55** (2009), 147-152.

[3] Y. Nambu, Generalized Hamiltonian dynamics, *Physical Review D.*, **7** (1973), 2405-2412.