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FINITE ELEMENTS APPROXIMATION OF TWO-STEP REACTION COMBUSTION MODEL

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Abstract: Our aim in this article is to study the equations of combustion at the case of two-step combustion model. The model considered is a system of equations coupling Navier-Stokes equations with three non-linear reaction-diffusion equations. We present a finite elements approximations and the existence and uniqueness are established. Optimal error estimates are given.

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1. Introduction and Model Presentation

We study in this paper a problem of combustion at the case of two consecutive reactions, the species A burn and give a product B which undergoes a second

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reaction and gives the species C. For this two-step chemistry $A \longrightarrow B \longrightarrow C$, we will assume that all the fluids acting in those reactions are incompressible; therefore, we introduce a coupling between the hydrodynamic and the combustion variables, thanks to the classical Boussinesq approximation. We mention that this approximation was justified and usually used to study some physical phenomena, see [7, 8].

The model for such process, is given by:

$$(P) \qquad \begin{cases} \partial_t T - \lambda \Delta T + u.\nabla T - C_1 g_1(T) - C_2 g_2(T) = 0, \\ \partial_t C_1 - d_1 \Delta C_1 + u.\nabla C_1 + C_1 g_1(T) = 0, \\ \partial_t C_2 - d_2 \Delta C_2 + u.\nabla C_2 + C_2 g_2(T) - C_1 g_1(T) = 0, \\ \partial_t u - \nu \Delta u + (u\nabla)\, u + \nabla p = \alpha(t) f(T), \\ div\ u = 0, \end{cases}$$

where the unknown factors are speed u, the pressure p, the temperature T, and the concentrations C_1 of A species, C_2 of B species (the concentration C_3 of the species C is given by the equation $C_3 = 1 - C_1 - C_2$); the coefficients ν, λ, d_1 and d_2 are assumed to be a positive constants (physically, d_i indicates the diffusion of the concentration C_i (i=1,2), λ the thermal diffusion and ν the viscosity of the fluid). The datas are a regular function f of $I\!R$ to $I\!R^d$ (typically, the function f is a gravity force proportional to the variations of density, therefore it depends on the temperature) and an other regular function g_i (i=1,2) of $I\!R$ to $I\!R^d$, where $d = 2$, or 3 (typically, the function g_i (i=1,2) is the source term of the reaction depending on the temperature and also on the energy). Those functions are given by the *Arrhenius* law, see [2].

The boundary conditions are of *Dirichlet* type for speed and *Dirichlet-Neumann* type for the temperature same as for the concentration. They are written:

$$\begin{cases} T|_{\Gamma_1} = C_i|_{\Gamma_1} = 0, \ \dfrac{\partial T}{\partial n}|_{\Gamma_2} = \dfrac{\partial C_i}{\partial n}|_{\Gamma_2} = 0, \ \text{ and } \ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0, \ \ T|_{t=0} = T_0, \ \text{ and } \ C_i|_{t=0} = C_{i_0} \ \ i = 1, 2, \end{cases}$$

where Γ_1 and Γ_2 are dis-joined open parts of $\partial\Omega$ such that $\overline{\Gamma_1} \cup \overline{\Gamma_2} = \partial\Omega$.

It will be useful to note that for the following assumption:

$$f(T) = \beta(T - T_0), \ \kappa = 0, \ \alpha(t) = g\gamma \ \text{ and } \ g_i(T) = k_0 \exp(-\frac{E_i}{RT}). \qquad (1.1)$$

In our model we announce that the function g_i is multiplied by the concentration C_i, therefore the reactions are considered of the first order; this functions are obtained by the *Arrhenius* law, see [2]. Here E_i indicates the activation energy of the reactor i, R the universal gas constant, g indicates the constant of gravity, γ is the ascending unit vector, T_0 the initial temperature of the product and

$k_0 \geq 0$.

Under the assumptions (1.1), the system of equations (P) gives the chemical model which is known under the name of *thermo-diffusif* model.

We notice that the numerical analysis for one-step chemical reaction is studied in (see [1]). The authors proved the existence and uniqueness of the solutions, and give some a priori error estames on the unkowns.

In this paper, we establish the existence and the uniqueness of discrete solution and we obtain some error estimates at the same time on speed, the pressure, the temperature and the concentration.

2. Variational Form of the Problem

We assume that:

$$\begin{cases} \text{The reals } \nu, \lambda \ d_1, \ d_2 \text{ and } \lambda \text{ are strictly positive,} \\ \alpha \in W^{1,\infty}(I\!R), \\ g_i \in W^{1,\infty}(I\!R), \ C_{g_i} = \| g_i' \|_{L^\infty(\Omega)}, \text{ and } g_i \geq 0, \ \| g_i \|_{L^\infty(\Omega)} = 1, \ i = 1,2, \\ f \in W^{1,\infty}(I\!R), \ f(0) = 0, \\ \forall (T_1, T_2) \in (H^1_{0,\Gamma_1}(\Omega))^2, \ \| f(T_1) - f(T_2) \|_{L^2(\Omega)} \leq \| \nabla (T_1 - T_2) \|_{L^2(\Omega)} . \end{cases}$$

We specify now the functional framework in which our analysis of the problem is carried out. The speeds space V defined by:

$$V = \{u \in (H^1_0(\Omega))^d; \quad divu = 0 \ \ in \ \Omega\}.$$

The temperature and the concentration space is $H^1_{0,\Gamma_1}(\Omega) = \{v \in H^1(\Omega)/v|_{\Gamma_1} = 0\}$.

We introduce now the constant of *Friedrichs-Poincaré* related to the domain geometry:

$$\rho = \sup_{u \in H^1_0(\Omega)} \frac{\| u \|_{L^2(\Omega)}}{\| \nabla u \|_{L^2(\Omega)}}.$$

We set

$$X = L^2(0,t,(H^1_0(\Omega))^d), \quad M = L^2(0,t,(L^2_0(\Omega))), \quad W = L^2(0,t,H^1_{0,\Gamma_1}(\Omega)),$$

and

$$\overline{X} = \left\{ u \in X; \frac{\partial u}{\partial t} \in L^2(0,t,(H^{-1}(\Omega))^d) \right\},$$

$$\overline{W} = \left\{ T \in W; \frac{\partial T}{\partial t} \in L^2(0,t,(H^1_{0,\Gamma_1}(\Omega))^*) \right\},$$

where $(H^1_{0,\Gamma_1}(\Omega))^*$ is the dual of the space $H^1_{0,\Gamma_1}(\Omega)$.

The variational form of the continued problem can be written as the following:

Find a $(u, p, T, C_1, C_2) \in \overline{X} \times M \times \overline{W}^3$, such as, for all $(v, q, \phi, \xi_1, \xi_2) \in X \times M \times W^3$, we have:

$$(P_v) \quad \left\{ \begin{array}{l} (\partial_t u, v) + \nu(\nabla u, \nabla v) + \dfrac{1}{2}\displaystyle\int_\Omega ((u\nabla)u)\, v - \dfrac{1}{2}\displaystyle\int_\Omega ((u\nabla)v)\, u - \displaystyle\int_\Omega p div(v) \\ \qquad\qquad = \alpha(t)\displaystyle\int_\Omega f(T)v, \\ (\partial_t T, \phi) + \lambda(\nabla T, \nabla \phi) + \dfrac{1}{2}\displaystyle\int_\Omega (u\nabla T)\phi - \dfrac{1}{2}\displaystyle\int_\Omega (u\nabla\phi)T \\ \qquad\qquad = \displaystyle\int_\Omega C_1 g_1(T)\phi + \displaystyle\int_\Omega C_2 g_2(T)\phi, \\ (\partial_t C_1, \xi_1) + \kappa_1(\nabla C_1, \nabla \xi_1) + \dfrac{1}{2}\displaystyle\int_\Omega (u\nabla C_1)\xi - \dfrac{1}{2}\displaystyle\int_\Omega (u\nabla\xi_1)C_1 \\ \qquad\qquad = -\displaystyle\int_\Omega C_1 g_1(T)\xi_1, \\ (\partial_t C_2, \xi_2) + \kappa_2(\nabla C_2, \nabla \xi_2) + \dfrac{1}{2}\displaystyle\int_\Omega (u\nabla C_2)\xi_2 - \dfrac{1}{2}\displaystyle\int_\Omega (u\nabla\xi_2)C_2 \\ \qquad\qquad = -\displaystyle\int_\Omega C_2 g_2(T)\xi_2 + \displaystyle\int_\Omega C_1 g_1(T)\xi_2, \\ \displaystyle\int_\Omega q div u = 0. \end{array} \right.$$

The existence and uniqueness of the continuous weak solution of the problem (P_v) has been established in [6].

3. Presentation of the Semi-Discretized Problem

In this section, we give our functional framework by introducing some spaces. Next, we give our discretized problem. First, we introduce the spaces that we need for our studies:

For any value of the real parameter $h > 0$, we consider three spaces X_h, M_h and W_h such that

$$X_h \subset (H^1_0(\Omega))^d \ , \ M_h \subset L^2_0(\Omega) \ \text{ and } W_h \subset H^1_{0,\Gamma_1}(\Omega).$$

We set

$$V_h := \{v_h, \ \ \forall q_h \in M_h, \int_\Omega q_h div\ v_h dx = 0\},$$

and we assume that they satisfy the following conditions:

1. For any $0 < \sigma \leq 1$, there exists a linear continuous operator P_h from $H^\sigma(\Omega) \cap L_0^2(\Omega)$ onto M_h such that

$$\forall q \in H^\sigma(\Omega) \cap L_0^2(\Omega), \|q - P_h q\|_{0,\Omega} \lesssim h^\sigma |q|_{\sigma,\Omega},$$

2. For all $0 < \sigma \leq 1$, there exists a linear continuous operator $\mathcal{I}_h$ from $(H^{1+\sigma}(\Omega))^d \cap (H_0^1(\Omega))^d$ onto X_h such that

$$\forall u \in (H^{1+\sigma}(\Omega))^d \cap (H_0^1(\Omega))^d, \quad \|u - \mathcal{I}_h u\|_{1,\Omega} \lesssim h^\sigma |u|_{1+\sigma,\Omega}.$$

3. There exists a constant β independent of h, such that

$$\forall q_h \in M_h, \exists v_h \in X_h, \quad \text{such as } (div v_h, q_h)_{0,\Omega} \geq \beta \|q_h\|_{0,\Omega} \|v_h\|_{1,\Omega}.$$

4. For all $0 < \sigma \leq 1$, there exists a linear continuous operator i_h from $H^{1+\sigma}(\Omega) \cap H_0^1(\Omega)$ onto W_h such that

$$\forall T \in H^{1+\sigma}(\Omega) \cap H_0^1(\Omega), \quad \|T - i_h T\|_{1,\Omega} \lesssim h^\sigma |T|_{1+\sigma,\Omega}.$$

Examples of such couple of spaces verifying the last conditions are given in [3, 4].

Throughout the paper, we often use the following notation:

For each ζ, $\eta > 0$: $\zeta \lesssim \eta \Leftrightarrow \exists C^\star > 0 \quad \zeta \leq C^\star \eta$; without further specification, we intend the constant $C^\star$ independent of the mesh-size and the solutions.

We can then state the discretized problem in space, in variational form:

Find $(u_h, p_h, T_h, C_{1_h}, C_2) \in C^1(0, t, X_h) \times C^0(0, t, M_h) \times (C^1(0, t, W_h))^3$ such that:

$$
(P_h)\quad \begin{cases}
\text{For any } (v_h, q_h, \phi_h, \xi_{1_h}, \xi_{2_h}) \in X_h \times M_h \times W_h^3, \\
(\partial_t u_h, v_h) + \nu(\nabla u_h, \nabla v_h) + \frac{1}{2}\int_\Omega ((u_h \nabla) u_h)\, v_h - \frac{1}{2}\int_\Omega ((u_h\nabla) v_h)\, u_h \\
\qquad\qquad - \int_\Omega p_h div v_h = \alpha(t) \int_\Omega f(T_h) v_h, \\
(\partial_t T_h, \phi_h) + \lambda(\nabla T_h, \nabla \phi_h) + \frac{1}{2}\int_\Omega (u_h \nabla T_h)\phi_h \\
\qquad\qquad - \frac{1}{2}\int_\Omega (u_h \nabla \phi_h) T_h = \sum_{i=1}^{2} \int_\Omega C_{i_h} g_i(T_h)\phi_h, \\
(\partial_t C_{1_h}, \xi_{1_h}) + \kappa_1(\nabla C_{1_h}, \nabla \xi_{1_h}) + \frac{1}{2}\int_\Omega (u_h \nabla C_{1_h})\xi_{1_h} - \frac{1}{2}\int_\Omega (u_h \nabla \xi_{1_h}) C_{1_h} \\
\qquad\qquad = -\int_\Omega C_{1_h} g_1(T_h)\xi_{1_h}, \\
(\partial_t C_{2_h}, \xi_{2_h}) + \kappa_2(\nabla C_{2_h}, \nabla \xi_{2_h}) + \frac{1}{2}\int_\Omega (u_h \nabla C_{2_h})\xi_{2_h} - \frac{1}{2}\int_\Omega (u_h \nabla \xi_{2_h}) C_{2_h} \\
\qquad\qquad = -\int_\Omega C_{2_h} g_2(T_h)\xi_{2_h} \\
+ \int_\Omega C_{1_h} g_1(T_h)\xi_{2_h}, \\
\int_\Omega q_h div u_h = 0.
\end{cases}
$$

Further, we denoted by:

$$X = \left(H_0^1(\Omega)\right)^d, \;\; W = H^1_{0,\Gamma_1}(\Omega), \;\; M = L_0^2(\Omega), \;\; V = \{v \in X, div\ v = 0\}$$

and we introduce the forms defined by:

$$X = \left(H_{0,\Gamma_1}^1(\Omega)\right)^d, \;\; W = H^1_{0,\Gamma_1}(\Omega), \;\; M = L_0^2(\Omega), \;\; V = \{v \in X, divv = 0\},$$

for all $(u, v, w) \in X^3$, $(T, \phi, C, \psi) \in W^4$, $p \in M$,

$$a_1(u, v, w) = \frac{1}{2}\left(\int_\Omega (u\nabla) vw\ dx - \int_\Omega (u\nabla) wv\ dx\right),$$

$$a_2(u, T, \phi) = \frac{1}{2}\left(\int_\Omega (u\nabla) T\phi\ dx - \int_\Omega (u\nabla)\phi T\ dx\right),$$

$$d(T, \psi) = \int_\Omega \nabla T \nabla \psi\ dx, \;\; b(p, v) = \int_\Omega p div\ v\ dx,$$

and

$$k_i(C, T, \psi) = \int_\Omega C g_i(T)\psi dx.$$

We can rewrite the semi-discretized problem (P_h) in the following form:
(P_h)

$$\begin{cases} \text{Find } u_h \in C^1(0,t,X_h),\ p_h \in C^0(0,t,M_h),\ (C_{1_h},C_{2_h},T_h) \in (C^1(0,t,W_h))^3, \\ \text{such that:} \\ (\partial_t u_h, v_h) + \nu d(u_h,v_h) + a_1(u_h,u_h,v_h) - b(p_h,v_h) = \alpha(t)(f(T_h),v_h),\ \forall v_h \in V_h, \\ (\partial_t T_h,\phi_h) + \lambda d(T_h,\phi_h) + a_2(u_h,T_h,\phi_h) - \sum_{i=1}^{2} k_i(C_{i_h},T_h,\phi_h) = 0,\ \forall \phi_h \in W_h, \\ (\partial_t C_{1_h},\xi_{1_h}) + d_1 d(C_{1_h},\xi_{1_h}) + a_2(u_h,C_{1_h},\xi_{1_h}) + k_1(C_{1_h},T_h,\xi_{1_h}) = 0,\ \forall \xi_{1_h} \in W_h, \\ (\partial_t C_{2_h},\xi_{2_h}) + d_2 d(C_{2_h},\xi_{2_h}) + a_2(u_h,C_{2_h},\xi_{2_h}) - k_1(C_{1_h},T_h,\xi_{1_h}) \\ \qquad\qquad + k_2(C_{2_h},T_h,\xi_{2_h}) = 0,\ \forall \xi_h \in W_h, \\ b(q_h,u_h) = 0,\ \forall q_h \in M_h, \end{cases}$$

with the initial conditions:

$$u_h(t=0) = u_h^0 \in V_h,\ T_h(t=0) = T_h^0 \in W_h,\ C_{ih}(t=0) = C_{ih}^0 \in W_h,\ i=1,2.$$

In the sequel, we assume, for simplicity, that:

$$u_0 = u_h^0,\quad T_0 = T_h^0 \ \text{ and } \ C_{i0} = C_{ih}^0,\quad i=1,2$$

The main theorem of the two following sections, is the following:

Theorem 3.1. *The problem (P_h) admits a unique solution. Moreover, if we assume that there exist positive constants δ_2, δ_5 and δ_6, such that:*

$$\frac{NM_1}{\nu} < 1 - \delta_5,\quad \delta_5 \in [0,1[, \tag{H_1}$$

$$\delta_6 = 1 - \frac{\sqrt{6} M_C C_g \rho^3}{\lambda} > 0, \tag{H_2}$$

$$\delta_2 = 1 - \frac{\sqrt{6}\rho^2}{\delta_6 \lambda}\left(\frac{2M_C C_g \rho^2}{d_1} + \frac{2M_C C_g \rho^2}{d_2}\sqrt{3 + \frac{12\rho^4}{d_1^2}}\right), \tag{H_3}$$

and

$$\frac{4\rho}{\nu\delta_5}\left(\frac{\sqrt{6}N}{\delta_2\delta_6\lambda}\frac{\rho^2 2\sqrt{2}M_C}{d_1} + \frac{2\rho^2 M_C}{d_2}\sqrt{3 + \frac{24\rho^4}{d_1^2}} + M_1\right) \le \frac{1}{4}, \tag{H_4}$$

and the solution of the problem (P_v) admits the following regularity:

$$u \in L^2(0,t,(H^{1+\sigma}(\Omega))^d) \cap H^1(0,t,(H^\sigma(\Omega))^d),\quad p \in H^1(0,t,H^\sigma(\Omega)),$$

and

$$T, C_i \in L^2(0,t,H^{1+\sigma}(\Omega)) \cap H^1(0,t,H^\sigma(\Omega)),\quad i=1,2,$$

then, we have the following error estimate:

$$\begin{aligned}&\| u-u_h \|_{L^2(0,t,(H^1(\Omega))^d)} + \| T-T_h \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \\ &\qquad + \sum_{i=1}^{2} \| C_i - C_{i_h} \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} + \| p-p_h \|_{L^2(0,t,L^2_0(\Omega))} \\ &\lesssim h^\sigma \Bigg(\| T \|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \| T \|_{H^1(0,t,H^{\sigma}(\Omega))} + \sum_{i=1}^{2} \| C_i \|_{L^2(0,t,H^{1+\sigma}(\Omega))} \\ &\quad + \sum_{i=1}^{2} \| C_i \|_{H^1(0,t,H^{\sigma}(\Omega))} + \| u \|_{L^2(0,t,(H^{1+\sigma}(\Omega))^d)} + \| u \|_{H^1(0,t,(H^{\sigma}(\Omega))^d)} \\ &\qquad\qquad + \| p \|_{H^1(0,t,H^{\sigma}(\Omega))} \Big) ,\end{aligned}$$

for $0 < \sigma \leq 1$. *Here*

$$M_T = \| T_h \|_{L^\infty(0,t,H^1_{0,\Gamma_1}(\Omega))}, \quad M_C = \| C_h \|_{L^\infty(0,t,H^1_{0,\Gamma_1}(\Omega))},$$

and

$$M_1 = \sup \Big(\| u_h \|_{L^\infty(0,t,(H^1_0(\Omega))^d)}, \| u \|_{L^\infty(0,t,(H^1_0(\Omega))^d)} \Big) ,$$

with

$$N = \sup_{u,v,w \in (H^1_0)^d(\Omega)} \frac{a_1(u,v,w)}{\| v \|_{H^1_0(\Omega)} \| u \|_{H^1_0(\Omega)} \| w \|_{H^1_0(\Omega)}}.$$

4. Existence and Uniqueness of Discret Solutions

In this section we are interested of the existence and uniqueness of discretized problem. For this, we begin by some a priori error estimates.

Lemma 4.1. *For any local solution* C_{1_h} *of the problem* (P_h), *we have the a priori estimate:*

$$\| C_{1_h} \|^2_{L^\infty(0,t,L^2(\Omega))} + 2d_1 \| C_{1_h} \|^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \leq 2 \| C^0_{1_h} \|^2_{L^2(\Omega)} . \tag{4.1}$$

Proof. By choosing $\xi_{1_h} = C_{1_h}$, as test function in the third equation of the problem (P_h), we have:

$$\frac{1}{2} \frac{d}{dt} \| C_{1_h} \|^2_{L^2(\Omega)} + d_1 \| \nabla C_{1_h} \|^2_{L^2(\Omega)} + k_1(C_{1_h}, T_h, C_{1_h}) = 0.$$

By integrating the last equality, and by noticing that $k_1(C_{1_h}, T_h, C_{1_h})$ is positive, we obtain:

$$2d_1 \parallel C_{1_h} \parallel^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \leq \parallel C^0_{1_h} \parallel^2_{L^2(\Omega)}$$

and

$$\parallel C_{1_h}(t=s) \parallel^2_{L^2(\Omega)} \leq \parallel C^0_{1_h} \parallel^2_{L^2(\Omega)} .$$

By using the two last inequalities, we obtain the a priori estimate (4.1). □

Lemma 4.2. *For any local solution C_{2_h} of the problem (P_h), we have the a priori estimate:*

$$\parallel C_{2_h} \parallel^2_{L^\infty(0,t,L^2(\Omega))} + d_2 \parallel C_{2_h} \parallel^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \leq \frac{\rho^3}{d_1 d_2} \parallel C^0_{1_h} \parallel^2_{L^2(\Omega)} + 2 \parallel C^0_{2_h} \parallel^2_{L^2(\Omega)} . \quad (4.2)$$

Proof. By choosing $\xi_{2_h} = C_{2_h}$, as test function in the fourth equation of the problem (P_h), we have:

$$\frac{1}{2}\frac{d}{dt} \parallel C_{2_h} \parallel^2_{L^2(\Omega)} + d_2 \parallel \nabla C_{2_h} \parallel^2_{L^2(\Omega)} + k_2(C_{1_h}, T_h, C_{1_h}) - k_1(C_{1_h}, T_h, C_{2_h}) = 0.$$

By noticing that $k_2(C_{2_h}, T_h, C_{2_h})$ is positive and by Young inequality, we have:

$$\frac{1}{2}\frac{d}{dt} \parallel C_{2_h} \parallel^2_{L^2(\Omega)} + d_2 \parallel \nabla C_{2_h} \parallel^2_{L^2(\Omega)} \leq \rho \parallel C_{1_h} \parallel_{L^2(\Omega)} \parallel \nabla C_{2_h} \parallel_{L^2(\Omega)}, \leq \frac{\rho^2}{2d_2} \parallel C_{1_h} \parallel^2_{L^2(\Omega)} + \frac{d_2}{2} \parallel \nabla C_{2_h} \parallel^2_{L^2(\Omega)} .$$

By integrating the last equality, we obtain:

$$\parallel C_{2_h}(t=s) \parallel^2_{L^2(\Omega)} + d_2 \parallel C_{2_h} \parallel^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \leq \frac{\rho^2}{d_2} \parallel C_{1_h} \parallel^2_{L^2(0,t,L^2(\Omega))} + \parallel C^0_{2_h} \parallel^2_{L^2(\Omega)} .$$

By Lemma 4.1, we have

$$\parallel C_{2_h}(t=s) \parallel^2_{L^2(\Omega)} + d_2 \parallel C_{2_h} \parallel^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \leq \frac{\rho^3}{2d_1 d_2} \parallel C^0_{1_h} \parallel^2_{L^2(0,t,L^2(\Omega))} + \parallel C^0_{2_h} \parallel^2_{L^2(\Omega)} .$$

We have also

$$\| C_{2_h} \|^2_{L^\infty(0,t,(L^2(\Omega))} \leq \frac{\rho^3}{2d_1d_2} \| C^0_{1_h} \|^2_{L^2(0,t,L^2(\Omega))} + \| C^0_{2_h} \|^2_{L^2(\Omega)} .$$

By using the two last inequalities, we obtain the a priori estimate (4.2). ☐

Lemma 4.3. *For any local solution T_h of problem (P_h), we have the a priori estimate:*

$$\| T_h \|^2_{L^\infty(0,t,L^2(\Omega))} + \lambda \| T_h \|^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \leq \left(\frac{2\rho^3}{\lambda d_1} + \frac{4\rho^6}{\lambda d_1 d_2} \right) \| C^0_{1_h} \|^2_{L^2(\Omega)} + \frac{8\rho^3}{\lambda d_2} \| C^0_{2_h} \|^2_{L^2(\Omega)} + 2 \| T^0_h \|^2_{L^2(\Omega)} . \quad (4.3)$$

Proof. By choosing $\phi_h = T_h$, as test function in the second equation of the problem (P_h), we have:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \| T_h \|^2_{L^2(\Omega)} + \lambda \| \nabla T_h \|^2_{L^2(\Omega)} &= k_1(C_{1_h}, T_h, T_h) + k_2(C_{2_h}, T_h, T_h) \\ &\leq \rho \left(\| C_{1_h} \|_{L^2(\Omega)} + \| C_{2_h} \|_{L^2(\Omega)} \right) \| \nabla T_h \|_{L^2(\Omega)} \\ &\leq \frac{\rho^2}{\lambda} \| C_{1_h} \|^2_{L^2(\Omega)} + \frac{\rho^2}{\lambda} \| C_{2_h} \|^2_{L^2(\Omega)} + \frac{\lambda}{2} \| \nabla T_h \|^2_{L^2(\Omega)} .\end{aligned}$$

By integrating the last inequality, we obtain:

$$\| T_h(t=s) \|^2_{L^2(\Omega)} + \lambda \| T_h \|^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \leq \frac{2\rho^2}{\lambda} \| C_{1_h} \|^2_{L^2(0,t,L^2(\Omega))} + \frac{2\rho^2}{\lambda} \| C_{2_h} \|^2_{L^2(0,t,L^2(\Omega))} + \| T^0_h \|^2_{L^2(\Omega)} .$$

However, using (4.1) and (4.2), we have:

$$\| T_h(t=s) \|^2_{L^2(\Omega)} + \lambda \| T_h \|^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \leq \frac{2\rho^4}{\lambda d_1} \| C^0_{1_h} \|^2_{L^2(\Omega)} + \frac{2\rho^7}{\lambda d_1 d_2} \| C^0_{1_h} \|^2_{L^2(\Omega)} + \| T^0_h \|^2_{L^2(\Omega)} + \frac{4\rho^4}{\lambda d_2} \| C^0_{2h} \|^2_{L^2(\Omega)},$$

and

$$\| T_h \|^2_{L^\infty(0,t,L^2(\Omega))} \leq \frac{2\rho^3}{\lambda d_1} \| C^0_{1_h} \|^2_{L^2(\Omega)} + \frac{2\rho^7}{\lambda d_1 d_2} \| C^0_{1_h} \|^2_{L^2(\Omega)} + \| T^0_h \|^2_{L^2(\Omega)} + \frac{4\rho^4}{\lambda d_2} \| C^0_{2h} \|^2_{L^2(\Omega)} .$$

By the last inequalities, it is obvious that:

$$\| T_h \|^2_{L^\infty(0,t,L^2(\Omega))} + \lambda \| T_h \|^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \leq \left(\frac{2\rho^3}{\lambda d_1} + \frac{4\rho^6}{\lambda d_1 d_2} \right) \| C^0_{1_h} \|^2_{L^2(\Omega)} + \frac{8\rho^3}{\lambda d_2} \| C^0_{2_h} \|^2_{L^2(\Omega)} + 2 \| T^0_h \|^2_{L^2(\Omega)} . \quad \square$$

Lemma 4.4. *For any local solution u_h of the problem (P_h), we have the estimate:*

$$\| u_h \|^2_{L^\infty(0,t,(L^2(\Omega))^d)} + \nu \| u_h \|^2_{L^2(0,t,(H^1_0(\Omega))^d)} \leq \left(\frac{4\rho^5}{\lambda^2 d_1 \nu} + \frac{8\rho^8}{\lambda^2 d_1 d_2 \nu} \right) \| C^0_{1_h} \|^2_{L^2(\Omega)} + \frac{8\rho^5}{\lambda^2 d_2 \nu} \| C^0_{2_h} \|^2_{L^2(\Omega)} + \frac{2\rho^2}{\lambda\nu} \| T^0_h \|^2_{L^2(\Omega)} + 2 \| u^0_h \|^2_{L^2(\Omega)} . \tag{4.4}$$

Proof. By choosing $v_h = u_h$, as test function in the first equation of the problem (P_h), we have:

$$\frac{1}{2}\frac{d}{dt} \| u_h \|^2_{(L^2(\Omega))^d} + \nu \| \nabla u_h \|^2_{(L^2(\Omega))^d} \leq \rho \| \nabla T_h \|_{L^2(\Omega)} \| \nabla u_h \|_{(L^2(\Omega))^d} \leq \frac{\rho^2}{2\nu} \| \nabla T_h \|^2_{L^2(\Omega)} + \frac{\nu}{2} \| \nabla u_h \|^2_{(L^2(\Omega))^d} .$$

While integrating the last inequality, we obtain:

$$\| u_h(t=s) \|^2_{(L^2(\Omega))^d} + \nu \| u_h \|^2_{L^2(0,t,(H^1_0(\Omega))^d)} \leq \frac{\rho^2}{\nu} \| T_h \|^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} + \| u^0_h \|^2_{L^2(\Omega)} .$$

However, using (4.3), we have:

$$\| u_h(t=s) \|^2_{(L^2(\Omega))^d} + \nu \| u_h \|^2_{L^2(0,t,(H^1_0(\Omega))^d)} \leq \left(\frac{2\rho^5}{\lambda^2 d_1 \nu} + \frac{4\rho^8}{\lambda^2 d_1 d_2 \nu} \right) \| C^0_{1_h} \|^2_{L^2(\Omega)} + \frac{4\rho^5}{\lambda^2 \nu d_2} \| C^0_{1_h} \|^2_{L^2(\Omega)} + \frac{\rho^2}{\lambda\nu} \| T^0_h \|^2_{L^2(\Omega)} + \| u^0_h \|^2_{L^2(\Omega)},$$

then, we have

$$\| u_h \|^2_{L^\infty(0,t,(L^2(\Omega))^d)} \leq \left(\frac{2\rho^5}{\lambda^2 d_1 \nu} + \frac{4\rho^8}{\lambda^2 d_1 d_2 \nu}\right) \| C^0_{1_h} \|^2_{L^2(\Omega)} + \frac{4\rho^5}{\lambda^2 \nu d_2} \| C^0_{1_h} \|^2_{L^2(\Omega)} + \frac{\rho^2}{\lambda\nu} \| T^0_h \|^2_{L^2(\Omega)} + \| u^0_h \|^2_{L^2(\Omega)} .$$

By the two last inequalities, we obtain

$$\| u_h \|^2_{L^\infty(0,t,(L^2(\Omega))^d)} + \nu \| u_h \|^2_{L^2(0,t,(H^1_0(\Omega))^d)} \leq \left(\frac{4\rho^5}{\lambda^2 d_1 \nu} + \frac{8\rho^8}{\lambda^2 d_1 d_2 \nu}\right) \| C^0_{1_h} \|^2_{L^2(\Omega)} + \frac{8\rho^5}{\lambda^2 \nu d_2} \| C^0_{1_h} \|^2_{L^2(\Omega)} + \frac{2\rho^2}{\lambda\nu} \| T^0_h \|^2_{L^2(\Omega)} + 2 \| u^0_h \|^2_{L^2(\Omega)} . \quad \square$$

Now we are able to proof the main theorem of this section more precisely, we have

Theorem 4.1. *The problem (P_h) admits at least a solution*

$$(u_h, p_h, C_{1_h}, C_{2_h}, T_h) \in H^1(0,t,X_h) \times L^2(0,t,M_h) \times \left(\mathcal{C}^1(0,t,W_h)\right)^3 .$$

Proof. Indeed, it is obvious that the problem (P_h) admits a local solution in the interval $(0, t_h)$. For t_h rather small, the a priori estimates (4.1), (4.2), (4.3) and (4.4) show that this solution can be defined on the interval $(0, t)$ for $t > 0$. $\square$

5. A Priori Error Estimations

In this section, we prove some error estimates on the speed, on the pressure, on the temperature and on the concentration. Finally, we obtain our main theorem of the section.

Subsequently, we assume that there exists $0 < \sigma \leq 1$, such that

$$p \in L^2(0,t,H^\sigma(\Omega)), \quad u \in L^2(0,t,(H^{1+\sigma}(\Omega))^d) \cap H^1(0,t,(H^\sigma(\Omega))^d)$$

and

$$T, \ \ C \in L^2(0,t,H^{1+\sigma}(\Omega)) \cap H^1(0,t,H^\sigma(\Omega)).$$

We introduce the operator $R_h : X \to X_h$ such that

$$\int_\Omega \nabla(R_h u - u).\nabla v_h = 0, \quad \forall v_h \in X_h.$$

Let us recall that (see e.g. [4]):

$$\forall\, u \in H_0^1(\Omega), \quad \| \nabla R_h u \|_{L^2(\Omega)} \leq \| \nabla u \|_{L^2(\Omega)},$$

and that if $u \in H_0^1(\Omega) \cap H^{1+\sigma}(\Omega)$ with $0 < \sigma \leq 1$, we have:

$$\|u - R_h u\|_{1,\Omega} \lesssim h^\sigma \|u\|_{1+\sigma,\Omega}.$$

In this section, we assume that the hypothesis (H_1), (H_2), (H_3) and (H_4) are valid.

We have the following technical lemmas:

Lemma 5.1. *We set*

$$M_1 = sup\left(\| u_h \|_{L^\infty(0,t,H_0^1(\Omega))}, \| u \|_{L^\infty(0,t,H_0^1(\Omega))} \right),$$

we assume that the hypothesis (H_1) *is verified. Then, for any* $\theta_0 > 0$, *we have:*

$$\begin{aligned} &\| u - u_h \|_{L^2(0,t,(H_0^1(\Omega))^d)} \\ &\quad \lesssim h^\sigma \left(\| u \|_{L^2(0,t,(H^{1+\sigma}(\Omega))^d)} + \| u \|_{H^1(0,t,(H^\sigma(\Omega))^d)} + \| p \|_{L^2(0,t,H^\sigma(\Omega))} \right) \\ &\qquad + 4\frac{\alpha\rho}{\nu\delta_5} \| T - T_h \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} + \theta_0 \| p - p_h \|_{L^2(0,t,L^2(\Omega))} . \end{aligned} \tag{5.1}$$

Proof. For u and u_h, solutions of the continuous and the discretized problem, we have, respectively:

$$\left\{ \begin{array}{l} (\partial_t u - \partial_t u_h, v_h) + \nu d(u - u_h, v_h) + a_1(u_h, u - u_h, v_h) - b(p - p_h, v_h) \\ +a_1(u - u_h, u, v_h) \ = \ \alpha(f(T) - f(T_h), v_h), \quad \forall v_h \in X_h. \end{array} \right. \tag{5.2}$$

We set $v_h = R_h u - u_h$. By noticing that $d(R_h u - u, R_h u - u_h) = 0$, we obtain:

$$\begin{aligned} &(R_h\partial_t u - \partial_t u_h, v_h) + \nu d(R_h u - u_h, R_h u - u_h) = -(\partial_t u - R_h \partial_t u, v_h) \\ &\quad + \alpha(f(T) - f(T_h), v_h) + b(p - p_h, u - u_h) - a_1(u - R_h u, u, R_h u - u_h) \\ &\quad - a_1(R_h u - u_h, u, R_h u - u_h) - a_1(u_h, u - R_h u, R_h u - u_h) \\ &\quad - a_1(u_h, R_h u - u_h, R_h u - u_h) + b(p - p_h, R_h u - u). \end{aligned}$$

However:

$$b(p - p_h, u - u_h) = b(p - P_h p, u - u_h).$$

Therefore, we have:

$$\frac{1}{2}\frac{d}{dt}\parallel R_h u-u_h\parallel^2_{(L^2(\Omega))^d}+\nu\parallel\nabla(R_h u-u_h)\parallel^2_{(L^2(\Omega))^d}\leq$$
$$\Big(\rho\parallel\partial_t u-R_h\partial_t u\parallel_{(L^2(\Omega))^d}+\alpha\rho^2\parallel\nabla(T-T_h)\parallel_{L^2(\Omega)}\Big)\parallel\nabla(R_h u-u_h)\parallel_{(L^2(\Omega))^d}$$
$$+\parallel p-P_h p\parallel_{L^2(\Omega)}\parallel\nabla(u-u_h)\parallel_{(L^2(\Omega))^d}+\frac{N^2M_1^2}{2\varepsilon}\parallel\nabla(u-R_h u)\parallel^2_{(L^2(\Omega))^d}$$
$$+\Big(NM_1+\frac{\varepsilon}{2}\Big)\parallel\nabla(R_h u-u_h)\parallel^2_{(L^2(\Omega))^d}+\parallel p-p_h\parallel_{L^2(\Omega)}\parallel\nabla(R_h u-u)\parallel_{(L^2(\Omega))^d}.$$

We set $\varepsilon=\frac{\nu-M_1N}{2}$, we obtain then:

$$\frac{1}{2}\frac{d}{dt}\parallel R_h u-u_h\parallel^2_{(L^2(\Omega))^d}+\frac{3\nu(1-\nu^{-1}M_1N)}{4}\parallel\nabla(R_h u-u_h)\parallel^2_{(L^2(\Omega))^d}\leq$$
$$\Big(\rho\parallel\partial_t u-R_h\partial_t u\parallel_{(L^2(\Omega))^d}+\alpha\rho^2\parallel\nabla(T-T_h)\parallel_{L^2(\Omega)}\Big)\parallel\nabla(R_h u-u_h)\parallel_{(L^2(\Omega))^d}$$
$$+\parallel p-P_h p\parallel_{L^2(\Omega)}\parallel\nabla(u-u_h)\parallel_{(L^2(\Omega))^d}+\frac{N^2M_1^2}{\nu-M_1N}\parallel\nabla(u-R_h u)\parallel^2_{(L^2(\Omega))^d}$$
$$+\parallel p-p_h\parallel_{L^2(\Omega)}\parallel\nabla(R_h u-u)\parallel_{(L^2(\Omega))^d}.$$

However, by using the hypothesis (H_1), we can write:

$$\frac{1}{2}\frac{\partial}{\partial t}\parallel R_h u-u_h\parallel^2_{(L^2(\Omega))^d}+\frac{3\nu}{4}\delta_5\parallel\nabla(R_h u-u_h)\parallel^2_{(L^2(\Omega))^d}\leq$$
$$\big(\rho\parallel\partial_t u-R_h\partial_t u\parallel_{(L^2(\Omega))d}+\alpha\rho^2\parallel\nabla(T-T_h)\parallel_{L^2(\Omega)}\big)\parallel\nabla(R_h u-u_h)\parallel_{(L^2(\Omega))^d}$$
$$+\parallel p-P_h p\parallel_{L^2(\Omega)}\parallel\nabla(u-u_h)\parallel_{(L^2(\Omega))^d}+\frac{N^2M_1^2}{\nu-M_1N}\parallel\nabla(u-R_h u)\parallel^2_{(L^2(\Omega))^d}$$
$$+\parallel p-p_h\parallel_{L^2(\Omega)}\parallel\nabla(R_h u-u)\parallel_{(L^2(\Omega))^d}.$$

Therefore:

$$\frac{\partial}{\partial t}\parallel R_h u-u_h\parallel^2_{(L^2(\Omega))^d}+\frac{1}{2}\nu\delta_5\parallel\nabla(R_h u-u_h)\parallel^2_{(L^2(\Omega))^d}$$
$$\leq\frac{2\rho^2}{\nu\delta_5}\parallel\partial_t u-R_h\partial_t u\parallel^2_{(L^2(\Omega))^d}$$
$$+\frac{2\alpha^2\rho^4}{\nu\delta_5}\parallel\nabla(T-T_h)\parallel^2_{L^2(\Omega)}+2\parallel p-P_h p\parallel_{L^2(\Omega)}\parallel\nabla(u-u_h)\parallel_{(L^2(\Omega))^d}$$
$$+\frac{2N^2M_1^2}{\nu-M_1N}\parallel\nabla(u-R_h u)\parallel^2_{(L^2(\Omega))^d}$$
$$+2\parallel p-p_h\parallel_{L^2(\Omega)}\parallel\nabla(R_h u-u)\parallel_{(L^2(\Omega))^d}. \quad (5.3)$$

We multiply (5.3) by $2\delta_5^{-1}\nu^{-1}$. Then we have:

$$
\begin{aligned}
2\delta_5^{-1}\nu^{-1}\frac{\partial}{\partial t} \| R_h u - u_h \|^2_{(L^2(\Omega))^d} + \| \nabla(R_h u - u_h) \|^2_{(L^2(\Omega))^d} \\
\leq \frac{4\rho^2}{\nu^2\delta_5^2} \| \partial_t u - R_h \partial_t u \|^2_{(L^2(\Omega))^d} \\
+ 4\left(\frac{\alpha\rho^2}{\nu\delta_5}\right)^2 \| \nabla(T - T_h) \|^2_{L^2(\Omega)} + \frac{4}{\delta_5\nu} \| p - P_h p \|_{L^2(\Omega)} \| \nabla(u - u_h) \|_{(L^2(\Omega))^d} \\
+ \frac{4N^2M_1^2}{(\nu - M_1 N)\delta_5\nu} \| \nabla(u - R_h u) \|^2_{(L^2(\Omega))^d} \\
+ \frac{4}{\delta_5\nu} \| p - p_h \|_{L^2(\Omega)} \| \nabla(R_h u - u) \|_{(L^2(\Omega))^d} . \quad (5.4)
\end{aligned}
$$

By integrating (5.4), and by using the properties of the operator R_h, we obtain:

$$
\begin{aligned}
&\| \nabla(R_h u - u_h) \|_{L^2(0,t,(L^2(\Omega))^d)} \\
&\lesssim h^\sigma \left(\| u \|_{L^2(0,t,(H^{1+\sigma}(\Omega))^d)} + \| u \|_{H^1(0,t,(H^\sigma(\Omega)^d))} + \| p \|_{L^2(0,t,H^\sigma(\Omega))} \right) \\
&\quad + \frac{\theta_0}{2} \| p - p_h \|_{L^2(0,t,L^2(\Omega))} + \frac{1}{2} \| \nabla(u - u_h) \|_{L^2(0,t,(L^2(\Omega)^d))} \\
&\quad + 2\frac{\alpha\rho^2}{\nu\delta_5} \| T - T_h \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))},
\end{aligned}
$$

where $\theta_0 > 0$ is unspecified, therefore we obtain:

$$
\begin{aligned}
&\| u - u_h \|_{L^2(0,t,(H^1_0(\Omega))^d)} \\
&\lesssim h^\sigma \left(\| u \|_{L^2(0,t,(H^{1+\sigma}(\Omega))^d)} + \| u \|_{H^1(0,t,(H^\sigma(\Omega))^d)} + \| p \|_{L^2(0,t,H^\sigma(\Omega))} \right) \\
&\quad + 4\frac{\alpha\rho}{\nu\delta_5} \| T - T_h \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} + \theta_0 \| p - p_h \|_{L^2(0,t,L^2(\Omega))} . \quad \square
\end{aligned}
$$

Lemma 5.2. *For p, p_h solutions of the problems (P) and (P_h), respectively, we have:*

$$
\begin{aligned}
\| p - p_h \|_{L^2(0,t,L^2(\Omega))} \lesssim h^\sigma \| p \|_{L^2(0,t,H^\sigma(\Omega))} + \beta^{-1}\rho \| \partial_t u - \partial_t u_h \|_{L^2(0,t,(H^{-1}(\Omega))^d)} \\
+ (\nu + 2NM_1)\beta^{-1} \| u - u_h \|_{L^2(0,t,(H^1_0(\Omega))^d)} + \alpha\rho\beta^{-1} \\
\times \| T - T_h \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} . \quad (5.5)
\end{aligned}
$$

Proof. From (5.2), we have:

$$
b(P_h p - p_h, v_h) = b(P_h p - p, v_h) + b(p - p_h, v_h) = b(P_h p - p, v_h)
$$

$$+ (\partial_t u - \partial_t u, v_h) + \nu d(u - u_h, v_h) + a_1(u - u_h, v, v_h)$$
$$+ a_1(u_h, u - u_h, v_h) - \alpha(f(T) - f(T_h), v_h), \quad \forall v_h \in X_h.$$

Therefore:

$$b(P_h p - p_h, v_h) \leq \Big(\| P_h p - p \|_{L^2(\Omega)} + \rho \| \partial_t u - \partial_t u_h \|_{(L^2(\Omega))^d} + (\nu + 2NM_1) \| \nabla(u - u_h) \|_{(L^2(\Omega))^d} \Big) \| v_h \|_{(H_0^1(\Omega))^d} + \alpha \| \nabla(T - T_h) \|_{L^2(\Omega)} \| v_h \|_{(L^2(\Omega))^d} . \quad (5.6)$$

However:

$$\| p - p_h \|_{L^2(\Omega)} \leq \| p - P_h p \|_{L^2(\Omega)} + \frac{1}{\beta} \sup_{v_h \in X_h} \frac{b(P_h p - p_h, v_h)}{\| v_h \|_{(H_0^1(\Omega))^d}}. \quad (5.7)$$

From (5.6) and (5.7) we have:

$$\| p - p_h \|_{L^2(\Omega)} \lesssim \| p - P_h p \|_{L^2(\Omega)} + \beta^{-1} \rho \| \partial_t u - \partial_t u_h \|_{(H^{-1}(\Omega))^d} + (\nu + 2NM_1)\beta^{-1} \| u - u_h \|_{(H_0^1(\Omega))^d} + \alpha\rho\beta^{-1} \| \nabla(T - T_h) \|_{L^2(\Omega)} .$$

Consequently:

$$\| p - p_h \|_{L^2(0,t,L^2(\Omega))} \lesssim h^\sigma \| p \|_{L^2(0,t,H^\sigma(\Omega))} + \beta^{-1} \rho \| \partial_t u - \partial_t u_h \|_{L^2(0,t,(H^{-1}(\Omega))^d)} + (\nu + 2NM_1)\beta^{-1} \| u - u_h \|_{L^2(0,t,(H_0^1(\Omega))^d)} + \alpha\rho\beta^{-1} \| T - T_h \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} . \quad \square$$

To continue our analysis we need the following technical lemma.

Lemma 5.3. *For u, p and T solutions of the problem (P_v), u_h, p_h and T_h solutions of the problem (P_h), we have the following estimate:*

$$\| \partial_t u - \partial_t u_h \|_{L^2(0,t,(H^{-1}(\Omega))^d)} \lesssim h^\sigma \Big(\| u \|_{H^1(0,t,(H^\sigma(\Omega))^d)} + \| p \|_{L^2(0,t,H^\sigma(\Omega))} \Big) + \alpha\rho \| T - T_h \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} + (\nu + 2NM_1) \| u - u_h \|_{L^2(0,t,(H_0^1(\Omega))^d)} . \quad (5.8)$$

Proof. We introduce now the operator:

$$\rho_h : V \to V_h \text{ such that } \frac{\partial}{\partial t}(v - \rho_h v, v_h) = 0, \quad \forall v_h \in V_h, \quad (5.9)$$

and we obtain:

$$\forall v \in H_0^1(0,t,V) \qquad \| \nabla \rho_h v \|_{L^2(\Omega)} \leq \bar{\rho} \| \nabla v \|_{L^2(\Omega)} .$$

In the sequel, to simplify the notations, we will note by ρ the maximum of ρ and $\bar{\rho}$.

We have the fact that:

$$\| \partial_t u - \partial_t u_h \|_{(H^{-1}(\Omega))^d} \leq \| \partial_t u - \rho_h \partial_t u \|_{(H^{-1}(\Omega))^d} + \| \rho_h \partial_t u - \partial_t u_h \|_{(H^{-1}(\Omega))^d} .$$

Then:

$$\| \partial_t u - \partial_t u_h \|_{(H^{-1}(\Omega))^d} \leq \| \partial_t u - \rho_h \partial_t u \|_{(H^{-1}(\Omega))^d} + \sup_{v \in V} \frac{\frac{\partial}{\partial t}(\rho_h u - u_h, v)}{\| v \|_V} . \quad (5.10)$$

From (5.2) and (5.9), we have:

$$\begin{aligned} \frac{\partial}{\partial t}(\rho_h u - u_h, v_h) &= (\partial_t \rho_h u - \partial_t u_h, v_h) = (\partial_t u - \partial_t u_h, v_h) = -\nu d(u - u_h, v_h) \\ &- a_1(u_h, u - u_h, v_h) - b(p - p_h, v_h) - a_1(u - u_h, u, v_h) + \alpha(f(T) - f(T_h), v_h) \\ &\leq \alpha\rho \| \nabla(T - T_h) \|_{L^2(\Omega)} \| v_h \|_{(H_0^1(\Omega))^d} \\ &+ (\nu + 2NM) \| \nabla(u - u_h) \|_{(L^2(\Omega))^d} \| \nabla v_h \|_{(L^2(\Omega))^d} \\ &+ \| p - R_h p \|_{L^2(\Omega)} \| \nabla v_h \|_{(L^2(\Omega))^d} . \quad (5.11) \end{aligned}$$

By integrating (5.11) and we by using (5.10), we obtain:

$$\begin{aligned} \| \partial_t u - \partial_t u_h \|_{L^2(0,t,(H^{-1}(\Omega))^d)} &\lesssim h^\sigma \Big(\| u \|_{H^1(0,t,(H^\sigma(\Omega))^d)} + \| p \|_{L^2(0,t,H^\sigma(\Omega))} \Big) \\ &+ \alpha\rho \| T - T_h \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} + (\nu + 2NM_1) \| u - u_h \|_{L^2(0,t,(H_0^1(\Omega))^d)} . \quad \square \end{aligned}$$

Again, we have:

Lemma 5.4. *For u, u_h solutions of the problems (P_v) and (P_h), respective, we have the following error estimate:*

$$\begin{aligned} &\| u - u_h \|_{L^2(0,t,(H_0^1(\Omega))^d)} \\ &\lesssim h^\sigma \Big(\| u \|_{L^2(0,t,(H^{1+\sigma}(\Omega))^d)} + \| u \|_{H^1(0,t,(H^\sigma(\Omega))^d)} + \| p \|_{L^2(0,t,H^\sigma(\Omega))} \Big) \\ &\quad + \left(\frac{\alpha\rho\theta_0}{\beta} + \frac{\alpha\rho^2\theta_0}{\beta} + 4\frac{\alpha\rho}{\nu\delta_5} \right) \| T - T_h \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \\ &\quad + \frac{\rho\theta_0 + \theta_0}{\beta}(\nu + 2NM_1) \| u - u_h \|_{L^2(0,t,(H_0^1(\Omega))^d)} . \quad (5.12) \end{aligned}$$

Proof. According to the error estimates (5.5) and (5.8), we have:

$$\begin{aligned}\| p-p_h \|_{L^2(0,t,L^2(\Omega))} \lesssim h^{\sigma}(\| u \|_{H^1(0,t,(H^{\sigma}(\Omega))^d)} + \| p \|_{L^2(0,t,(H^{\sigma}(\Omega))^d)}) \\ +(\frac{\alpha\rho}{\beta}+\frac{\alpha\rho^2}{\beta}) \| T-T_h \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \\ +\frac{\rho+1}{\beta}(\nu+2NM_1) \| u-u_h \|_{L^2(0,t,(H^1_0(\Omega))^d)} .\end{aligned}$$

We deduce, by using (5.1), we obtain the following estimate:

$$\begin{aligned}&\| u-u_h \|_{L^2(0,t,(H^1_0(\Omega))^d)} \\ &\lesssim h^{\sigma}\left(\| u \|_{L^2(0,t,(H^{1+\sigma}(\Omega))^d)} + \| u \|_{H^1(0,t,(H^{\sigma}(\Omega))^d)} + \| p \|_{L^2(0,t,H^{\sigma}(\Omega))}\right) \\ &\quad +\left(\frac{\alpha\rho\theta_0}{\beta}+\frac{\alpha\rho^2\theta_0}{\beta}+4\frac{\alpha\rho}{\nu\delta_5}\right) \| T-T_h \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \\ &\quad +\frac{\rho\theta_0+\theta_0}{\beta}(\nu+2NM_1) \| u-u_h \|_{L^2(0,t,(H^1_0(\Omega))^d)} . \quad \square\end{aligned}$$

We have again the following estimate on temperature.

Lemma 5.5. *We set that the hypothesis* (H_2) *is verified. We have:*

$$\begin{aligned}\| T-T_h \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \lesssim h^{\sigma}\left(\| T \|_{H^1(0,t,H^{\sigma}(\Omega))} + \| T \|_{L^2(0,t,H^{1+\sigma}(\Omega))}\right) \\ +\frac{\sqrt{6}\rho^2}{\delta_6\lambda}\left(\| C_1-C_{1_h} \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} + \| C_2-C_{2_h} \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))}\right) \\ +\frac{\sqrt{6}NM_1}{\delta_6\lambda} \| u-u_h \|_{L^2(0,t,(H^1_0(\Omega))^d)} . \quad (5.13)\end{aligned}$$

Proof. For T solution of the problem (P_v) and T_h solution of the problem (P_h), we have:

$$\begin{aligned}(\partial_t T-\partial_t T_h,\phi_h)+\lambda d(T-T_h,\phi_h)+a_2(u-u_h,T,\phi_h)+a_1(u_h,T-T_h,\phi_h) \\ =k_1(C_1,T,\phi_h)-k_1(C_{1_h},T_h,\phi_h)+k_2(C_2,T,\phi_h)-k_2(C_{2_h},T_h,\phi_h),\end{aligned}$$

so:

$$\begin{aligned}&(R_h\partial_t T-\partial_t T_h,\phi_h)+\lambda d(R_hT-T_h,\phi_h)=(R_h\partial_t T-\partial_t T,\phi_h) \\ &-a_2(u-u_h,T,\phi_h) \\ &-a_2(u_h,T-T_h,\phi_h)+k_1(C_1,T,\phi_h)-k_1(C_{1_h},T_h,\phi_h)\end{aligned}$$

$$+ k_2(C_2, T, \phi_h) - k_2(C_{2_h}, T_h, \phi_h).$$

By setting $\phi_h = R_h T - T_h$, we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \| R_h T - T_h \|^2_{L^2(\Omega)} + \lambda \| \nabla(R_h T - T_h) \|^2_{L^2(\Omega)} &= - (\partial_t T - R_h \partial_t T, \phi_h) - a_2(u - u_h, T, \phi_h) \\ &\quad - a_1(u_h, T - R_h T, R_h T - T_h) + k_1(C_1, T, \phi_h) \\ &\quad - k_1(C_{1_h}, T_h, \phi_h) + k_2(C_2, T, \phi_h) - k_2(C_{2_h}, T_h, \phi_h),\end{aligned}$$

then:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \| R_h T - T_h \|^2_{L^2(\Omega)} &+ \lambda \| \nabla(R_h T - T_h) \|^2_{L^2(\Omega)} \leq \rho \| \partial_t T - R_h \partial_t T \|_{L^2(\Omega)} \\ &+ N M_1 \| \nabla(u - u_h) \|_{(L^2(\Omega)^d)} + N M_1 \| \nabla(T - R_h T) \|_{L^2(\Omega)} \\ &+ \| \nabla(R_h T - T_h) \|_{L^2(\Omega)} + |k_1(C_1, T, \phi_h) + k_1(C_{1_h}, T_h, \phi_h)| \\ &+ |k_2(C_2, T, \phi_h) + k_2(C_{2_h}, T_h, \phi_h)|. \qquad (5.14)\end{aligned}$$

We have also:

$$\begin{aligned}|k_1(C_1, T, \phi_h) - k_1(C_{1_h}, T_h, \phi_h)| &\leq \rho^2 \| C_1 - C_{1_h} \|_{H^1_{0,\Gamma_1}(\Omega)} \| \phi_h \|_{H^1_{0,\Gamma_1}(\Omega)} \\ &+ M_C C_g \rho^3 \| T - T_h \|_{H^1_{0,\Gamma_1}(\Omega)} \| \phi_h \|_{H^1_{0,\Gamma_1}(\Omega)}, \qquad (5.15)\end{aligned}$$

$$\begin{aligned}|k_2(C_2, T, \phi_h) - k_2(C_{2_h}, T_h, \phi_h)| &\leq \rho^2 \| C_2 - C_{2_h} \|_{H^1_{0,\Gamma_1}(\Omega)} \| \phi_h \|_{H^1_{0,\Gamma_1}(\Omega)} \\ &+ M_C C_g \rho^3 \| T - T_h \|_{H^1_{0,\Gamma_1}(\Omega)} \| \phi_h \|_{H^1_{0,\Gamma_1}(\Omega)}, \qquad (5.16)\end{aligned}$$

from (5.14), (5.15) and (5.16), we have:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \| R_h T - T_h \|^2_{L^2(\Omega)} &+ \lambda \| \nabla(R_h T - T_h) \|^2_{L^2(\Omega)} \leq \big(\rho \| \partial_t T - R_h \partial_t T \|_{L^2(\Omega)} \\ + N M_1 \| \nabla(u - u_h) \|_{(L^2(\Omega)^d)} &+ N M_1 \| \nabla(T - R_h T) \|_{L^2(\Omega)}\big) \| \nabla(R_h T - T_h) \\ &\times \|_{L^2(\Omega)} \\ + \Big(\rho^2 \| C_1 - C_{1_h} \|_{H^1_{0,\Gamma_1}(\Omega)} &+ \rho^2 \| C_2 - C_{2_h} \|_{H^1_{0,\Gamma_1}(\Omega)} \\ + 2 M_C C_g \rho^3 \| T - T_h \|_{H^1_{0,\Gamma_1}(\Omega)}\Big) &\| \nabla(R_h T - T_h) \|_{L^2(\Omega)} .\end{aligned}$$

We deduced:

$$\frac{1}{2}\frac{d}{dt} \parallel R_hT - T_h \parallel^2_{L^2(\Omega)} + \frac{\lambda}{2} \parallel \nabla(R_hT - T_h) \parallel^2_{L^2(\Omega)} \leq \frac{3\rho^2}{\lambda} \parallel \partial_t T - R_h \partial_t T \parallel^2_{L^2(\Omega)}$$
$$+ \frac{3N^2M_1^2}{\lambda} \parallel \nabla(T - R_hT) \parallel^2_{L^2(\Omega)} + \frac{3\rho^4}{\lambda} \parallel \nabla(C_1 - C_{1_h}) \parallel^2_{L^2(\Omega)}$$
$$+ \frac{3\rho^4}{\lambda} \parallel \nabla(C_2 - C_{2_h}) \parallel^2_{L^2(\Omega)}$$
$$+ \frac{3N^2M_1^2}{\lambda} \parallel \nabla(u - u_h) \parallel^2_{(L^2(\Omega))^d} + \frac{3M_C^2C_g^2\rho^6}{\lambda} \parallel \nabla(T - T_h) \parallel^2_{L^2(\Omega)} .$$

While multiplying by $\frac{2}{\lambda}$, we have:

$$\frac{1}{2}\frac{d}{dt} |||\, R_hT - T_h \parallel^2_{H^1_{0,\Gamma_1}(\Omega)}$$
$$\leq \frac{6\rho^2}{\lambda^2} \parallel \partial_t T - R_h \partial_t T \parallel^2_{L^2(\Omega)} + \frac{6N^2M_1^2}{\lambda^2} \parallel \nabla(T - R_hT) \parallel^2_{L^2(\Omega)}$$
$$+ \frac{6\rho^4}{\lambda^2} \parallel \nabla(C_1 - C_{1_h}) \parallel^2_{L^2(\Omega)} + \frac{6\rho^4}{\lambda^2} \parallel \nabla(C_2 - C_{2_h}) \parallel^2_{L^2(\Omega)}$$
$$+ \frac{6N^2M_1^2}{\lambda^2} \parallel \nabla(u - u_h) \parallel^2_{(L^2(\Omega))^d} + \frac{6M_C^2C_g^2\rho^6}{\lambda^2} \parallel \nabla(T - T_h) \parallel^2_{L^2(\Omega)} .$$

By integrating, we obtain the estimate:

$$\parallel R_hT - T_h \parallel^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \lesssim \left(h^\sigma \parallel T \parallel_{H^1(0,t,H^\sigma(\Omega))} + h^\sigma \parallel T \parallel_{L^2(0,t,H^{1+\sigma}(\Omega))}\right)^2$$
$$+ \frac{6\rho^4}{\lambda^2} \parallel C_1 - C_{1_h} \parallel^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} + \frac{6\rho^4}{\lambda^2} \parallel C_2 - C_{2_h} \parallel^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))}$$
$$+ \frac{6N^2M_1^2}{\lambda^2} \parallel u - u_h \parallel^2_{L^2(0,t,(H^1_0(\Omega))^d)} + 6\frac{M_C^2C_g^2\rho^6}{\lambda^2} \parallel T - T_h \parallel^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} .$$

Therefore:

$$\parallel R_hT - T_h \parallel_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \lesssim \left(h^\sigma \parallel T \parallel_{H^1(0,t,H^\sigma(\Omega))} + h^\sigma \parallel T \parallel_{L^2(0,t,H^{1+\sigma}(\Omega))}\right)$$
$$+ \frac{\sqrt{6}\rho^2}{\lambda} \parallel C_1 - C_{1_h} \parallel_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} + \frac{\sqrt{6}\rho^2}{\lambda} \parallel C_2 - C_{2_h} \parallel_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))}$$
$$+ \frac{\sqrt{6}NM_1}{\lambda} \parallel u - u_h \parallel_{L^2(0,t,(H^1_0(\Omega))^d)} + \frac{\sqrt{6}M_CC_g\rho^3}{\lambda} \parallel T - T_h \parallel_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} . \tag{5.17}$$

However:

$$\parallel T - T_h \parallel_{H^1_{0,\Gamma_1}(\Omega)} \leq \parallel T - R_hT \parallel_{H^1_{0,\Gamma_1}(\Omega)} + \parallel R_hT - T_h \parallel_{H^1_{0,\Gamma_1}(\Omega)},$$

then:

$$\| T - T_h \|_{H^1_{0,\Gamma_1}(\Omega)} \lesssim h^\sigma \| T \|_{H^{1+\sigma}(\Omega)} + \| R_h T - T_h \|_{H^1_{0,\Gamma_1}(\Omega)} . \tag{5.18}$$

Therefore (5.17) and (5.18) allow us to obtain the following estimate:

$$\begin{aligned}\left(1 - \frac{\sqrt{6} M_C C_g \rho^3}{\lambda}\right) \| T - T_h \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \\ \lesssim h^\sigma \left(\| T \|_{H^1(0,t,H^\sigma(\Omega))} + \| T \|_{L^2(0,t,H^{1+\sigma}(\Omega))}\right) \\ + \frac{\sqrt{6}\rho^2}{\lambda} \| C_1 - C_{1_h} \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} + \frac{\sqrt{6}\rho^2}{\lambda} \| C_2 - C_{2_h} \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \\ + \frac{\sqrt{6} N M_1}{\lambda} \| u - u_h \|_{L^2(0,t,(H^1_0(\Omega))^d)} .\end{aligned}$$

Finally, if we set:

$$\delta_6 = 1 - \frac{\sqrt{6} M_C C_g \rho^3}{\lambda} \quad > \quad 0.$$

We obtain the following estimate:

$$\begin{aligned}\| T - T_h \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \lesssim h^\sigma \left(\| T \|_{H^1(0,t,H^\sigma(\Omega))} + \| T \|_{L^2(0,t,H^{1+\sigma}(\Omega))}\right) \\ + \frac{\sqrt{6}\rho^2}{\delta_6 \lambda} \left(\| C_1 - C_{1_h} \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} + \| C_2 - C_{2_h} \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))}\right) \\ + \frac{\sqrt{6} N M_1}{\delta_6 \lambda} \| u - u_h \|_{L^2(0,t,(H^1_0(\Omega))^d)} . \quad \square\end{aligned}$$

On concentrations, we have the following:

Lemma 5.6. *For C, C_h solutions of the problem (P_v) and (P_h), respectively, we have:*

$$\begin{aligned}\| C_1 - C_{1_h} \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \lesssim h^\sigma \left(\| C_1 \|_{H^1(0,t,H^\sigma(\Omega))} + \| C_1 \|_{L^2(0,t,H^{1+\sigma}(\Omega))}\right) \\ + \frac{2\sqrt{2} N M_C}{d_1} \| u - u_h \|_{L^2(0,t,(H^1_0(\Omega))^d)} \\ + 2\frac{M_C C_g \rho^2}{d_1} \| (T - T_h) \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} .\end{aligned} \tag{5.19}$$

Proof. By choosing $\psi_h = R_h C_1 - C_{1_h}$ as a test function, we have:

$$\frac{1}{2}\frac{d}{dt} \parallel R_hC_1 - C_{1_h} \parallel^2_{0,t,L^2(\Omega)} + d_1 \parallel \nabla(R_hC_1 - C_{1_h}) \parallel^2_{0,t,L^2(\Omega)}$$
$$= -(\partial_t C_1 - R_h\partial_t C, \psi_h) - a_2(u - u_h, C_1, \psi_h)$$
$$- a_2(u_h, C_1 - R_hC_1, R_hC_1 - C_{1_h}) - k_1(C_1, T, \psi_h) + k_1(C_{1_h}, T_h, \psi_h). \quad (5.20)$$

However

$$k(C_1, T, \psi_h) - k(C_{1_h}, T_h, \psi_h) = k_1(C_1, T, \psi_h) - k(R_hC_1, T_h, \psi_h) + k_1(\psi_h, T_h, \psi_h)$$

and

$$k_1(C_1, T, \psi_h) - k_1(R_hC_1, T_h, \psi_h) = k(C_1 - R_hC_1, T, \psi_h) - k_1(R_hC_1, T - T_h, \psi_h)$$
$$\leq \rho^2 \parallel \nabla(C_1 - R_hC_1) \parallel_{L^2(\Omega)} \parallel \nabla\psi_h \parallel_{L^2(\Omega)} + \int R_hC_1 \left(g(T) - g(T_h)\right) \psi_h,$$

then:

$$k_1(C_1, T, \psi_h) - k_1(R_hC_1, T_h, \psi_h) \leq \rho^2 \parallel C_1 - R_hC_1 \parallel_{H^1_{0,\Gamma_1}(\Omega)} \parallel \psi_h \parallel_{H^1_{0,\Gamma_1}(\Omega)}$$
$$+ M_C C_g \rho^2 \parallel T - T_h \parallel_{H^1_{0,\Gamma_1}(\Omega)} \parallel \psi_h \parallel_{H^1_{0,\Gamma_1}(\Omega)} . \quad (5.21)$$

Using (5.20) and (5.21), we obtain:

$$\frac{1}{2}\frac{d}{dt} R_hC_1 - C_{1_h} \parallel^2_{L^2(\Omega)} + d_1 \parallel \nabla(R_hC_1 - C_{1_h}) \parallel^2_{L^2(\Omega)}$$
$$\leq \Big(\rho \parallel \partial_t C_1 - R_h\partial_t C_1 \parallel_{L^2(\Omega)} + NM_C \parallel \nabla(u - u_h) \parallel_{(L^2(\Omega))^d}$$
$$+ NM_C \parallel \nabla(C_1 - R_hC_1) \parallel_{L^2(\Omega)}\Big) \parallel \nabla(R_hC_1 - C_{1_h}) \parallel_{L^2(\Omega)}$$
$$+ \Big(\rho^2 \parallel C_1 - R_hC_1 \parallel_{H^1_{0,\Gamma_1}(\Omega)} + M_C C_g \rho^2 \parallel \nabla(T - T_h) \parallel_{L^2(\Omega)}\Big) \parallel \nabla\psi_h \parallel_{L^2(\Omega)} .$$

By using Young inequality, we obtain:

$$\frac{1}{2}\frac{d}{dt} \parallel R_hC_1 - C_{1_h} \parallel^2_{L^2(\Omega)} + \frac{d_1}{2} \parallel \nabla(R_hT - T_h) \parallel^2_{L^2(\Omega)}$$
$$\leq \frac{2\rho^2}{d_1} \parallel \partial_t C_1 - R_h\partial_t C_1 \parallel^2_{L^2(\Omega)}$$
$$+ \frac{2N^2M_C^2}{d_1} \parallel \nabla(C_1 - R_hC_1) \parallel^2_{L^2(\Omega)} + \frac{2\rho^4}{d_1} \parallel \nabla(C_1 - R_hC_1) \parallel^2_{L^2(\Omega)}$$
$$+ \frac{2N^2M_C^2}{d_1} \parallel \nabla(u - u_h) \parallel^2_{(L^2(\Omega))^d} + M_C C_g \rho^2 \parallel \nabla(T - T_h) \parallel_{L^2(\Omega)} \parallel \nabla\psi_h \parallel_{L^2(\Omega)} .$$

Multiplying by $\frac{2}{d_1}$ and integrating, we obtain:

$$\begin{aligned}
&\| R_hC_1 - C_{1_h} \|^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \\
&\qquad \lesssim \left(h^\sigma \| C_1 \|_{H^1(0,t,H^\sigma(\Omega))} + h^\sigma \| C_1 \|_{L^2(0,t,H^{1+\sigma}(\Omega))}\right)^2 \\
&\qquad + \frac{4N^2M_C^2}{d_1^2} \| \nabla(u-u_h) \|^2_{L^2(0,t,(L^2(\Omega))^d)} \\
&\qquad + 2\frac{M_CC_g\rho^2}{d_1} \| \nabla(T-T_h) \|_{L^2(0,t,L^2(\Omega))} \| \nabla\psi_h \|_{L^2(0,t,L^2(\Omega))} .
\end{aligned}$$

So:

$$\begin{aligned}
&\|R_hC_1 - C_{1_h} \|^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \\
&\lesssim \left(h^\sigma \| C_1 \|_{H^1(0,t,H^\sigma(\Omega))} + h^\sigma \| C_1 \|_{L^2(0,t,H^{1+\sigma}(\Omega))}\right)^2 \\
&+ \frac{4N^2M_C^2}{d_1^2} \| \nabla(u-u_h) \|^2_{L^2(0,t,(L^2(\Omega))^d)} + 2\frac{M_C^2C_g^2\rho^4}{d_1^2} \| (T-T_h) \|^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \\
&+ \frac{1}{2} \| R_hC_1 - C_{1_h} \|^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} .
\end{aligned}$$

By the triangular inequality and the error estimates in finite elements, we obtain:

$$\begin{aligned}
&\|C_1 - C_{1_h} \|_{L^2(H^1_{0,\Gamma_1}(\Omega))} \lesssim h^\sigma \left(\| C_1 \|_{H^1(0,t,H^\sigma(\Omega))} + \| C_1 \|_{L^2(0,t,H^{1+\sigma}(\Omega))}\right) \\
&+ \frac{2\sqrt{2}NM_C}{d_1} \| u-u_h \|_{L^2(0,t,(H^1_0(\Omega))^d)} + 2\frac{M_CC_g\rho^2}{d_1} \| (T-T_h) \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} .
\end{aligned}$$

□

Lemma 5.7. *For C_2, C_{2_h} solutions of the problems (P_v) and (P_h), respectively, we have:*

$$\begin{aligned}
&\| C_2 - C_{2_h} \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \\
&\quad \lesssim h^\sigma \left(\| C_1 \|_{H^1(0,t,H^\sigma(\Omega))} + \| C_1 \|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \| C_2 \|_{H^1(0,t,H^\sigma(\Omega))}\right. \\
&+ \left.\| C_2 \|_{L^2(0,t,H^{1+\sigma}(\Omega))}\right) + \sqrt{\frac{12N^2M_C^2}{d_1^2} + \frac{96\rho^4N^2M_C^2}{d_1^2d_2^2}} \| u-u_h \|_{L^2(0,t,(H^1_0(\Omega))^d)} \\
&\qquad + \sqrt{\frac{16M_C^2C_g^2\rho^4}{d_2^2} + \frac{48\rho^8M_C^2C_g^2}{d_1^2d_2^2}} \| (T-T_h) \|_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} . \qquad (5.22)
\end{aligned}$$

Proof. By choosing $\psi_h = R_hC_2 - C_{2_h}$ as a test function, we have:

$$\frac{1}{2}\frac{d}{dt} \parallel R_hC_2 - C_{2_h} \parallel^2_{0,t,L^2(\Omega)} + d_2 \parallel \nabla(R_hC_2 - C_{2_h}) \parallel^2_{0,t,L^2(\Omega)}$$
$$= -(\partial_t C_2 - R_h\partial_t C_2, \psi_h) - a_2(u - u_h, C_2, \psi_h)$$
$$- a_2(u_h, C_2 - R_hC_2, R_hC_2 - C_{2_h}) - k_2(C_2, T, \psi_h) + k_2(C_{2_h}, T_h, \psi_h) + k_1(C_1, T, \psi_h)$$
$$- k_1(C_{1_h}, T_h, \psi_h). \quad (5.23)$$

However

$$k_2(C_2, T, \psi_h) - k_2(C_{2_h}, T_h, \psi_h) = k_2(C_2, T, \psi_h) - k_2(R_hC_2, T_h, \psi_h) + k_2(\psi_h, T_h, \psi_h)$$

and

$$k_2(C_2, T, \psi_h) - k_2(R_hC_2, T_h, \psi_h) = k_2(C_2 - R_hC_2, T, \psi_h) - k_2(R_hC_2, T - T_h, \psi_h)$$
$$\leq \rho^2 \parallel \nabla(C_2 - R_hC_2) \parallel_{L^2(\Omega)} \parallel \nabla\psi_h \parallel_{L^2(\Omega)} + \int R_hC_2 \left(g_2(T) - g_2(T_h)\right) \psi_h,$$

then:

$$k_2(C_2, T, \psi_h) - k_2(R_hC_1, T_h, \psi_h) \leq \rho^2 \parallel C_2 - R_hC_2 \parallel_{H^1_{0,\Gamma_1}(\Omega)} \parallel \psi_h \parallel_{H^1_{0,\Gamma_1}(\Omega)}$$
$$+ M_C C_g \rho^2 \parallel T - T_h \parallel_{H^1_{0,\Gamma_1}(\Omega)} \parallel \psi_h \parallel_{H^1_{0,\Gamma_1}(\Omega)}. \quad (5.24)$$

We have also:

$$k_1(C_1, T, \psi_h) - k_1(C_{1_h}, T_h, \psi_h) = k_1(C_1 - R_hC_1, T, \psi_h)$$
$$+ k_1(R_hC_1 - C_{1_h}, T_h, \psi_h) + k_1(C_{1_h}, T - T_h, \psi_h)$$
$$\leq \rho^2 \parallel C_1 - R_hC_1 \parallel_{H^1_{0,\Gamma_1}(\Omega)} \parallel \psi_h \parallel_{H^1_{0,\Gamma_1}(\Omega)} + \rho^2 \parallel C_{1_h}$$
$$- R_hC_1 \parallel_{H^1_{0,\Gamma_1}(\Omega)} \parallel \psi_h \parallel_{H^1_{0,\Gamma_1}(\Omega)} + M_C C_g \rho^2 \parallel T - T_h \parallel_{H^1_{0,\Gamma_1}(\Omega)} \parallel \psi_h \parallel_{H^1_{0,\Gamma_1}(\Omega)}.$$

Using (5.23) and (5.24), we obtain:

$$\frac{1}{2}\frac{d}{dt} \parallel R_hC_2 - C_{2_h} \parallel^2_{L^2(\Omega)} + d_2 \parallel \nabla(R_hC_2 - C_{2_h}) \parallel^2_{L^2(\Omega)}$$
$$\leq \Big(\rho \parallel \partial_t C_2 - R_h\partial_t C_2 \parallel_{L^2(\Omega)} + NM_C \parallel \nabla(u - u_h) \parallel_{(L^2(\Omega))^d}$$
$$+ NM_C \parallel \nabla(C_2 - R_hC_2) \parallel_{L^2(\Omega)}\Big) \parallel \nabla(R_hC_2 - C_{2_h}) \parallel_{L^2(\Omega)}$$
$$+ \Big(\rho^2 \parallel C_1 - R_hC_1 \parallel_{H^1_{0,\Gamma_1}(\Omega)} + \rho^2 \parallel C_{1_h} - R_hC_1 \parallel_{H^1_{0,\Gamma_1}(\Omega)}$$

$$+ 2M_C C_g \rho^2 \parallel \nabla(T-T_h) \parallel_{L^2(\Omega)} + \rho^2 \parallel C_2 - R_h C_2 \parallel_{H^1_{0,\Gamma_1}(\Omega)} \Big) \parallel \nabla \psi_h \parallel_{L^2(\Omega)} .$$

By using Young inequality, we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \parallel R_h C_2 - C_{2_h} \parallel^2_{L^2(\Omega)} + \frac{d_2}{2} \parallel \nabla(R_h C_2 - C_{2_h}) \parallel^2_{L^2(\Omega)} \\
&\qquad \le \frac{3\rho^2}{d_2} \parallel \partial_t C_2 - R_h \partial_t C_2 \parallel^2_{L^2(\Omega)} + \frac{3N^2 M_C^2}{d_2} \parallel \nabla(C_2 - R_h C_2) \parallel^2_{L^2(\Omega)} \\
&+ \frac{3\rho^4}{d_2}\big(\parallel \nabla(C_1 - R_h C_1) \parallel^2_{L^2(\Omega)} + \parallel \nabla(C_{1_h} - R_h C_1) \parallel^2_{L^2(\Omega)} \\
&\qquad\qquad + \parallel \nabla(C_2 - R_h C_2) \parallel^2_{L^2(\Omega)} \big) \\
&+ \frac{3N^2 M_C^2}{d_2} \parallel \nabla(u-u_h) \parallel^2_{(L^2(\Omega))^d} + 2M_C C_g \rho^2 \parallel \nabla(T-T_h) \parallel_{L^2(\Omega)} \parallel \nabla \psi_h \parallel_{L^2(\Omega)} .
\end{aligned}$$

Multiplying by $\frac{2}{d_2}$ and integrating, we obtain

$$\begin{aligned}
&\parallel R_h C_2 - C_{2_h} \parallel^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \lesssim \big(h^\sigma \parallel C_1 \parallel_{H^1(0,t,H^\sigma(\Omega))} \\
&\qquad + h^\sigma \parallel C_1 \parallel_{L^2(0,t,H^{1+\sigma}(\Omega))} + h^\sigma \parallel C_2 \parallel_{H^1(0,t,H^\sigma(\Omega))} \\
&\quad + h^\sigma \parallel C_2 \parallel_{L^2(0,t,H^{1+\sigma}(\Omega))}\big)^2 + \frac{6N^2 M_C^2}{d_2^2} \parallel \nabla(u-u_h) \parallel^2_{L^2(0,t,(L^2(\Omega))^d)} \\
&\qquad + 4\frac{M_C C_g \rho^2}{d_2} \parallel \nabla(T-T_h) \parallel_{L^2(0,t,L^2(\Omega))} \parallel \nabla \psi_h \parallel_{L^2(0,t,L^2(\Omega))} \\
&+ \frac{6\rho^4}{d_2^2}\left(\frac{8N^2 M_C^2}{d_1^2} \parallel \nabla(u-u_h) \parallel^2_{L^2(0,t,(L^2(\Omega))^d)} \right.\\
&\qquad\qquad \left. + \frac{4M_C^2 C_g^2 \rho^4}{d_1^2} \parallel \nabla T - T_h) \parallel^2_{L^2(0,t,(L^2(\Omega))^d)} \right).
\end{aligned}$$

So

$$\begin{aligned}
&\parallel R_h C_2 - C_{2_h} \parallel^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} \lesssim \big(h^\sigma \parallel C_1 \parallel_{H^1(0,t,H^\sigma(\Omega))} + h^\sigma \parallel C_1 \parallel_{L^2(0,t,H^{1+\sigma}(\Omega))} \\
&\qquad + h^\sigma \parallel C_2 \parallel_{H^1(0,t,H^\sigma(\Omega))} + h^\sigma \parallel C_2 \parallel_{L^2(0,t,H^{1+\sigma}(\Omega))}\big)^2 \\
&\qquad + \left(\frac{6N^2 M_C^2}{d_2^2} + \frac{6\rho^4 8N^2 M_C^2}{d_1^2 d_2^2}\right) \parallel \nabla(u-u_h) \parallel^2_{L^2(0,t,(L^2(\Omega))^d)} \\
&\qquad + \left(\frac{24M_C^2 C_g^2 \rho^8}{d_1^2 d_2^2} + \frac{8M_C^2 C_g^2 \rho^4}{d_2^2}\right) \parallel (T-T_h) \parallel^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))}
\end{aligned}$$

$$+\frac{1}{2}\parallel R_hC_2 - C_{2_h}\parallel^2_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))}.$$

By the triangular inequality and the error estimates in finite elements, we obtain:

$$\begin{aligned}\parallel C_2 - C_{2_h}\parallel_{L^2(H^1_{0,\Gamma_1}(\Omega))} &\lesssim h^\sigma\big(\parallel C_1\parallel_{H^1(0,t,H^\sigma(\Omega))} + \parallel C_1\parallel_{L^2(0,t,H^{1+\sigma}(\Omega))}\\ &\quad + \parallel C_2\parallel_{H^1(0,t,H^\sigma(\Omega))} + \parallel C_2\parallel_{L^2(0,t,H^{1+\sigma}(\Omega))}\big)\\ &\quad \sqrt{\frac{12N^2M_C^2}{d_1^2} + \frac{96\rho^4N^2M_C^2}{d_1^2d_2^2}}\parallel u - u_h\parallel_{L^2(0,t,(H^1_0(\Omega))^d)}\\ &\quad + \sqrt{\frac{16M_C^2C_g^2\rho^4}{d_2^2} + \frac{48\rho^8M_C^2C_g^2}{d_1^2d_2^2}}\parallel (T - T_h)\parallel_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))}. \quad \square\end{aligned}$$

In the next lemma, we need to use the hypothesis (H_2), more precisely:

Lemma 5.8. *We have the following estimate:*

$$\begin{aligned}&\parallel T - T_h\parallel_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))}\\ &\quad\lesssim h^\sigma\big(\parallel T\parallel_{L^2(0,t,H^{1+\sigma}(\Omega))} + \parallel T\parallel_{H^1(0,t,H^\sigma(\Omega))} + \parallel C_1\parallel_{L^2(0,t,H^{1+\sigma}(\Omega))}\\ &\quad + \parallel C_1\parallel_{H^1(0,t,H^\sigma(\Omega))} + \parallel C_2\parallel_{L^2(0,t,H^{1+\sigma}(\Omega))} + \parallel C_2\parallel_{H^1(0,t,H^\sigma(\Omega))}\big)\\ &\quad + \left(\frac{\sqrt{6}N}{\delta_2\delta_6\lambda}\frac{\rho^2 2\sqrt{2}M_C}{d_1} + \frac{2\rho^2M_C}{d_2}\sqrt{3 + \frac{24\rho^4}{d_1^2}} + M_1\right)\parallel u - u_h\\ &\quad \times\parallel_{L^2(0,t,(H^1_0(\Omega))^d)}. \qquad (5.21)\end{aligned}$$

Proof. By Lemmas 5.5, 5.6 and 5.7, we have:

$$\begin{aligned}&\parallel T - T_h\parallel_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))}\\ &\quad\lesssim h^\sigma\big(\parallel C_1\parallel_{H^1(0,t,H^\sigma(\Omega))} + \parallel C_1\parallel_{L^2(0,t,H^{1+\sigma}(\Omega))} + \parallel C_2\parallel_{H^1(0,t,H^\sigma(\Omega))}\\ &\quad + \parallel C_2\parallel_{L^2(0,t,H^{1+\sigma}(\Omega))} + \parallel T\parallel_{H^1(0,t,H^\sigma(\Omega))} + \parallel T\parallel_{L^2(0,t,H^{1+\sigma}(\Omega))}\big)\\ &\quad + \left(\sqrt{6\rho^2}\delta_6\lambda\left(\frac{2\sqrt{2}NM_C}{d_1} + \frac{2NM_C}{d_2}\sqrt{3 + \frac{24\rho^4}{d_1^2}}\right) + \frac{\sqrt{6}NM_1}{\delta_6\lambda}\right)\\ &\quad \times\parallel u - u_h\parallel_{L^2(0,t,(H^1_0(\Omega))^d)}\\ &+ \frac{\sqrt{6}\rho^2}{\delta_6\lambda}\left(\frac{2M_CC_g\rho^2}{d_1} + \frac{2M_CC_g\rho^2}{d_2}\sqrt{3 + \frac{12\rho^4}{d_1^2}}\right)\parallel T - T_h\parallel_{L^2(0,t,(H^1_{0,\Gamma_1}(\Omega))^d)}.\end{aligned}$$

We set

$$\delta_2 = 1 - \frac{\sqrt{6}\rho^2}{\delta_6\lambda}\left(\frac{2M_C C_g \rho^2}{d_1} + \frac{2M_C C_g \rho^2}{d_2}\sqrt{3 + \frac{12\rho^4}{d_1^2}}\right) \quad > \quad 0.$$

Therefore:

$$\begin{aligned}
\delta_2 \parallel T - T_h \parallel_{L^2(0,t,H^1_{0,\Gamma_1}(\Omega))} &\lesssim h^\sigma \big(\parallel T \parallel_{L^2(0,t,H^{1+\sigma}(\Omega))} + \parallel T \parallel_{H^1(0,t,H^\sigma(\Omega))} \\
&\quad + \parallel C_1 \parallel_{L^2(0,t,(H^{1+\sigma}(\Omega))^d)} \\
&+ \parallel C_1 \parallel_{H^1(0,t,H^\sigma(\Omega))} + \parallel C_2 \parallel_{L^2(0,t,(H^{1+\sigma}(\Omega))^d)} + \parallel C_2 \parallel_{H^1(0,t,H^\sigma(\Omega))}\Big) \\
&+ \left(\sqrt{6\rho^2}\delta_6\lambda\left(\frac{2\sqrt{2}NM_C}{d_1} + \frac{2NM_C}{d_2}\sqrt{3 + \frac{24\rho^4}{d_1^2}}\right) + \frac{\sqrt{6}NM_1}{\delta_6\lambda}\right) \\
&\qquad \times \parallel u - u_h \parallel_{L^2(0,t,(H^1_0(\Omega))^d)} . \quad \square
\end{aligned}$$

Lemma 5.9. *We assume that the hypothesis* (H_1), (H_2), (H_3) *and* (H_4) *are verified so*

$$\begin{aligned}
&\parallel u - u_h \parallel_{L^2(0,t,(H^1_0(\Omega))^d)} \\
&\quad \lesssim h^\sigma \big(\parallel T \parallel_{L^2(0,t,H^{1+\sigma}(\Omega))} + \parallel T \parallel_{H^1(0,t,H^\sigma(\Omega))} + \parallel C_1 \parallel_{L^2(0,t,H^{1+\sigma}(\Omega))} \\
&\quad\quad + \parallel C_1 \parallel_{H^1(0,t,H^\sigma(\Omega))} + \parallel C_2 \parallel_{L^2(0,t,H^{1+\sigma}(\Omega))} + \parallel C_2 \parallel_{H^1(0,t,H^\sigma(\Omega))} \\
&\quad + \parallel u \parallel_{L^2(0,t,(H^{1+\sigma}(\Omega))^d)} + \parallel u \parallel_{H^1(0,t,(H^\sigma(\Omega))^d)} + \parallel p \parallel_{L^2(0,t,H^\sigma(\Omega))}\Big). \qquad (5.26)
\end{aligned}$$

Proof. First of all, by using inequalities (5.12) and (5.25), we obtain:

$$\begin{aligned}
&\|u - u_h \parallel_{L^2(0,t,(H^1_0(\Omega))^d)} \\
&\lesssim h^\sigma \big(\parallel T \parallel_{L^2(0,t,H^{1+\sigma}(\Omega))} + \parallel T \parallel_{H^1(0,t,H^\sigma(\Omega))} + \parallel C_1 \parallel_{L^2(0,t,H^{1+\sigma}(\Omega))} \\
&+ \parallel C_1 \parallel_{H^1(0,t,H^\sigma(\Omega))} + \parallel C_2 \parallel_{L^2(0,t,H^{1+\sigma}(\Omega))} + \parallel C_2 \parallel_{H^1(0,t,H^\sigma(\Omega))} \\
&+ \parallel u \parallel_{L^2(0,t,(H^{1+\sigma}(\Omega))^d)} + \parallel u \parallel_{H^1(0,t,(H^\sigma(\Omega))^d)} + \parallel p \parallel_{L^2(0,t,H^\sigma(\Omega))}\Big) \\
&+ \left(\frac{\rho\theta_0}{\beta} + \frac{\rho^2\theta_0}{\beta} + \frac{4rho}{\nu\delta_5}\right)\left(\frac{\sqrt{6}N}{\delta_2\delta_6\lambda}\frac{\rho^2 2\sqrt{2}M_C}{d_1} + \frac{2\rho^2 M_C}{d_2}\sqrt{3 + \frac{24\rho^4}{d_1^2}} + M_1\right) \\
&\quad \times \parallel u - u_h \parallel_{L^2(0,t,(H^1_0(\Omega))^d)} \\
&+ \frac{\rho\theta_0 + \theta_0}{\beta}(\nu + 2NM_1) \parallel u - u_h \parallel_{L^2(0,t,(H^1_0(\Omega))^d)} .
\end{aligned}$$

We set $A = \left(\frac{\sqrt{6}N}{\delta_2\delta_6\lambda} \frac{\rho^2 2\sqrt{2}M_C}{d_1} + \frac{2\rho^2 M_C}{d_2}\sqrt{3 + \frac{24\rho^4}{d_1^2}} + M_1 \right)$.
We assume that:

$$\frac{4\rho}{\nu\delta_5} A \leq 1 - \delta_7 \text{ with } 0 < \delta_7 < 1.$$

For example, if $\delta_7 = \frac{3}{4}$, we obtain:

$$\begin{aligned}
&\|u - u_h\|_{L^2(0,t,(H_0^1(\Omega))^d)} \\
&\lesssim h^\sigma \big(\| T \|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \| T \|_{H^1(0,t,H^\sigma(\Omega))} + \| C_1 \|_{L^2(0,t,H^{1+\sigma}(\Omega))} \\
&+ \| C_1 \|_{H^1(0,t,H^\sigma(\Omega))} + \| C_2 \|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \| C_2 \|_{H^1(0,t,H^\sigma(\Omega))} \\
&+ \| u \|_{L^2(0,t,(H^{1+\sigma}(\Omega))^d)} + \| u \|_{H^1(0,t,(H^\sigma(\Omega))^d)} + \| p \|_{L^2(0,t,H^\sigma(\Omega))} \big) \\
&+ 2\theta_0 \left(\frac{\rho + \rho^2}{\beta} A + \frac{\rho + 1}{\beta} (\nu + 2NM_1) \right) \| u - u_h \|_{L^2(0,t,(H_0^1(\Omega))^d)} .
\end{aligned}$$

For

$$\theta_0 = \frac{1}{4} \left(\frac{\rho + \rho^2}{\beta} A + \frac{\rho + 1}{\beta} (\nu + 2NM_1) \right)^{-1},$$

We obtain

$$\begin{aligned}
&\|u - u_h\|_{L^2(0,t,(H_0^1(\Omega))^d)} \\
&\lesssim h^\sigma \big(\| T \|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \| T \|_{H^1(0,t,H^\sigma(\Omega))} + \| C_1 \|_{L^2(0,t,H^{1+\sigma}(\Omega))} \\
&+ \| C_1 \|_{H^1(0,t,H^\sigma(\Omega))} + \| C_2 \|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \| C_2 \|_{H^1(0,t,H^\sigma(\Omega))} \\
&+ \| u \|_{L^2(0,t,(H^{1+\sigma}(\Omega))^d)} + \| u \|_{H^1(0,t,(H^\sigma(\Omega))^d)} + \| p \|_{L^2(0,t,H^\sigma(\Omega))} \big) . \qquad \square
\end{aligned}$$

Finally, using all this lemmas, we obtain the a priori error estimate at the same time on the speed, the pressure, the temperature and the concentration.

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