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ON THE GEOMETRIC LOCUS OF CURVATURE CENTRALS OF THE BERTRAND CURVE OFFSETS

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Abstract: In this paper, some results relevant geometric locus of the curvature centrals of the Bertrand curve offsets are given.

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1. Introduction

In the study of Bertrand curve offsets (in fact nearly all curves) in elementary classical differential geometry, it is always assumed that the curvature is nowhere zero. Also, every student of elementary classical differential geometry meets early in his course the subject of Bertrand curve offsets, discovered in 1850 by J. Bertrand. Bertrand curve offsets, in many papers have been studied, Bioche [1], Whittemore [5], Burke [2], Lai [4], Izmuya and Takeuchi [3].

The purpose of this short paper is to give some results relevant geometric locus of the curvature centrals of the Bertrand curve offsets.

2. Some Preliminary Remarks

We now review some basic concepts on classical differential geometry of space curves in Euclidean space. We denote by E^3 a 3-dimensional Euclidean space

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and by $\overrightarrow{\alpha}$ the position vector in E^3. A parametrized curve $\overrightarrow{\alpha}=\overrightarrow{\alpha}$ (t), $t \in$ I in E^3 is said to be regular if its tangent vector d $\overrightarrow{\alpha}$ /dt is continuous and is nowhere zero. A consequence of this definition is that a regular curve can be parametrized by its arc length, i.e., let $\overrightarrow{\alpha}: I \to E^3$ can be a regular curve. The arc-length of the curve $\overrightarrow{\alpha}$, measured from $\overrightarrow{\alpha}$ $(t_0), t_0 \in I$ is

$$s(t) = \int_{t_0}^{t} \|\dot{\overrightarrow{\alpha}}(u)\| du.$$

Then a parameter s is determined such that $\|\dot{\overrightarrow{\alpha}}(s)\| = 1$, where $\dot{\overrightarrow{\alpha}}(s) = d\overrightarrow{\alpha}$ (s)/ds. So we say that a curve $\overrightarrow{\alpha}$ is parametrised by the arc-length if it satisfies $\|\dot{\overrightarrow{\alpha}}(s)\| = 1$. Throughout in this paper, we denote s the arc-length of space curves. Let us denote $\overrightarrow{T}(s) = \dot{\overrightarrow{\alpha}}(s)$ and we call $\overrightarrow{T}(s)$ a unit tangent vector of $\overrightarrow{\alpha}$ at s. A curvature of a curve $\overrightarrow{\alpha}$ is defined by $\kappa(s) = \|\ddot{\alpha}(s)\|$ and the curvature radius of $\overrightarrow{\alpha}$ is defined by $R = \frac{1}{\kappa}$. If $\kappa(s)$ is nowhere zero ($\kappa(s)$ is everywhere positive), then the unit principal normal vector $\overrightarrow{N}(s)$ of the curve $\overrightarrow{\alpha}$ at s is given by $\ddot{\overrightarrow{\alpha}}(s) = \kappa(s)\overrightarrow{N}(s)$. The unit vector $\overrightarrow{B}(s) = \overrightarrow{T}(s) \times \overrightarrow{N}(s)$ called a unit binormal vector of the curve $\overrightarrow{\alpha}$ at s. The plane constitution of $(\overrightarrow{T}, \overrightarrow{N})$ is called osculating plane of the curve $\overrightarrow{\alpha}$. The plane constitution of $(\overrightarrow{B}, \overrightarrow{N})$ is called normal plane of the curve $\overrightarrow{\alpha}$. The plane constitution of $(\overrightarrow{T}, \overrightarrow{B})$ is called rectifian plane of the curve $\overrightarrow{\alpha}$.

Also, the following Frenet-Serret formula holds:

$$\begin{aligned} &\dot{\overrightarrow{T}}(s) = \kappa(s)\overrightarrow{N}(s), \\ &\dot{\overrightarrow{N}}(s) = -\kappa(s)\overrightarrow{T}(s) + \tau(s)\overrightarrow{B}(s), \\ &\dot{\overrightarrow{B}}(s) = -\tau(s)\overrightarrow{N}(s). \end{aligned} \tag{1}$$

Here $\tau(s)$ is torsion of the curve $\overrightarrow{\alpha}$ at s, Izmuya et al [3].

On the other hand , a curve $\overrightarrow{\alpha}: I \to E^3$ is called a Bertrand curve if there exists a curve $\overrightarrow{\beta}: I \to E^3$ such that the principal normals of $\overrightarrow{\alpha}$ are the principal normals of $\overrightarrow{\beta}$ at $s \in I$. In this case, we say that dual $(\overrightarrow{\alpha}, \overrightarrow{\beta})$ is called Bertrand curve offsets. For a space curve $\overrightarrow{\alpha}$ (s) with $\tau(s) \neq 0$, the curve $\overrightarrow{\alpha}$ is a Bertrand curve if and only if there exist nonzero real numbers n, m such that $n\kappa(s) + m\tau(s) = 1$ for any $s \in I$. So a circular helix is a Bertrand curve, Izmuya and Takeuchi [3].

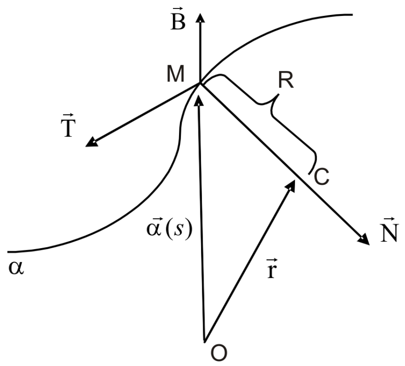

Figure 1

3. The Geometric Locus of Curvature Central of a Space Curve

Definition 3.1. The distance $R = \frac{1}{\kappa}$ taken on the principal normal at a point of a space curve is called curvature central of the space curve.

In Figure 1 $\overrightarrow{MC} = R\,\overrightarrow{N}$, $\quad (\| \overrightarrow{N} \| = 1)$.

From here, we may write as

$$\overrightarrow{r}\,(s) = \overrightarrow{\alpha}\,(s) + R(s)\,\overrightarrow{N}\,(s) \tag{2}$$

the locus vector of the point C. While drawing the curve $\overrightarrow{\alpha}$ as dependent on s the point M, with aid of (2) relationship, the point C draws other a curve $\overrightarrow{\alpha}_1$. The curve $\overrightarrow{\alpha}_1$ is called geometric locus of curvature central of curve $\overrightarrow{\alpha}$.

Differentiating (2) and using (1), we get

$$\frac{d\,\overrightarrow{r}}{ds} = \frac{dR}{ds}\,\overrightarrow{N} + R\tau\,\overrightarrow{B}\,. \tag{3}$$

From (3), we may give the following theorem and results.

Theorem 1. *The tangents of the geometric locus of curvature central of a space curve lie in a normal plane of the space curve.*

Result 1. If $\tau = 0$ then the tangents of $\overrightarrow{\alpha}_1$ are parallel to $\overrightarrow{N}$.

Result 2. If R is a constant then the tangents of $\overrightarrow{\alpha}_1$ are parallel to $\overrightarrow{B}$.

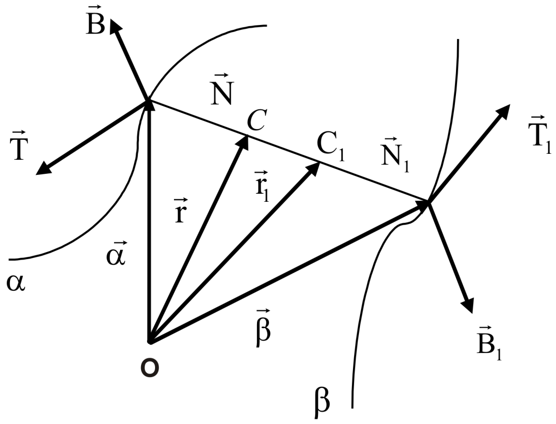

Figure 2

4. The Geometric Locus of Curvature Centrals of Bertrand Curve Offsets

Let dual $(\vec{\alpha},\vec{\beta})$ be Bertrand curve offsets and Frenet vectors of curves $\vec{\alpha}$ and $\vec{\beta}$, respectively, $\vec{T},\vec{N},\vec{B}$ and $\vec{T}_1,\vec{N}_1,\vec{B}_1$ (see Figure 2).

From Figure 2, the geometric locus of curvature centrals of $(\vec{\alpha},\vec{\beta})$ spatial Bertrand curve offsets can be written as

$$\begin{aligned} &\vec{r}_1=\vec{r}+C\,\vec{C}_1, \quad C\,\vec{C}_1=\lambda\,\vec{N}, \quad \lambda\in \mathrm{IR},\\ &\vec{r}_1=\vec{r}+\lambda\,\vec{N}\,. \end{aligned} \tag{4}$$

Differentiating (4), using (1) and (3), we get

$$\begin{aligned} &\frac{d\,\vec{r}_1}{ds}=\frac{dr}{ds}+\frac{d\lambda}{ds}\,\vec{N}+\lambda\frac{d\,\vec{N}}{ds},\\ &\frac{d\,\vec{r}_1}{ds}=(-\lambda\kappa)\,\vec{T}+\left(\frac{dR}{ds}+\frac{d\lambda}{ds}\right)\vec{N}+(R\tau+\lambda\tau)\,\vec{B}\,. \end{aligned} \tag{5}$$

From (5), we can give the following theorem and results.

Theorem 2. *Let dual $(\vec{\alpha},\vec{\beta})$ be spatial Bertrand curve offsets. Then the tangents of the geometric locus of curvature central of curve β cannot be tangents, principal normals and binormals of curve α.*

Result 3. If R and λ constants then the tangents of the geometric locus of curvature central of curve β lie rectifian plane of the curve $\overrightarrow{\alpha}$.

Result 4. If $\tau = 0$ then the tangents of the geometric locus of curvature central of curve β lie osculating plane of the curve $\overrightarrow{\alpha}$.

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