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STABILITY OF THE NULL SOLUTION OF THE EQUATION $\dot{x}(t) = -a(t)x(t) + b(t)x([t])$

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Abstract: The asymptotic stability of the null solution of the equation $\dot{x}(t) = -a(t)x(t)+b(t)x([t])$ with argument $[t]$, where $[t]$ designates the greatest integer function, is studied by means of dichotomic maps.

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1. Introduction

The study of differential equations with piecewise continuous argument has been the subject of many investigations, Cooke et al [1] and in the stability study of this type of equations using dichotomic maps, some literature can be cited, Bena et al [2], Carvalho et al [3] and Marconato [4]. A potential application of this equation is in the stabilization of hybrid control system, by which we mean one with a continuous plant and with a discrete controller, see Seifert [5]. We proved that the null solution of the equation $\dot{x}(t) = cx(t) + bx([t]), t \geq 0$, is asymptotically stable, since $c \leq -\delta < 0$, $|b| < k\delta$, $\delta > 0$ and $k \in (0, 1)$, by using

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dichotomic maps, Marconato [4]. The aim of this work is to extend the result of [4] to the equation $\dot{x}(t) = -a(t)x(t) + b(t)x([t])$, with imposed conditions on the functions $a(t)$ and $b(t)$. The idea of the method of dichotomic maps is due to Razumikhin, which is a simplification of the idea that is behind Liapunov's method. We remark that one of the main advantages of this method is the use of extremely simple functions such as $V(x) = x^2/2$. The differential equation with constant argument in intervals is given by

$$\dot{x}(t) = f(t, x(t), x([t])) \tag{1.1}$$

with $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ continuous and $f(t, 0, 0) = 0$ for all $t \in \mathbb{R}$. We further impose that f takes bounded sets into bounded sets and satisfies enough additional smoothness conditions to ensure the existence, uniqueness and continuous dependence with respect the initial conditions of the solutions of (1.1). The delay of equation (1.1) is sectionally continuous because $r(t) = t - [t]$ is continuous in $[n, n+1)$, for all $n \in \mathbb{Z}$, and discontinuous at integers values.

2. Notations and Preliminaries

2.1. The Solution of (1.1)

We denote by $x(., t_o, \psi)$ the solution of (1.1) with $x_{t_o}(., t_o, \psi) = \psi$ and

$$x_t(., t_o, \psi)(\theta) = x(t + \theta, t_o, \psi), \qquad \theta \in [-1, 0] \tag{2.1}$$

$\psi \in C$, where C denotes the Banach space of the continuous maps from [-1,0] into $\mathbb{R}^n$, endowed with the supremum norm.

Definition 2.1. A solution of (1.1)-(2.1) on $[t_o - 1, b)$ is a function x(t) which satisfies:

(i) $x(t)$ is continuous on $[t_o - 1, b)$,

(ii) the derivative $\dot{x}(t)$ exists at each point $t \in [t_o, b)$, with the possible exception of the points $[t] \in [t_o, b)$ where one-side derivatives exists, and

(iii) wherever $\dot{x}(t)$ exists, it satisfies (1.1)-(2.1).

By defining the function, $g(t, \psi) = f(t, \psi(0), \psi([t] - t)), t \geq t_o$, the equation (1.1) can be represented in the form

$$\dot{x}(t) = g(t, x_t). \tag{2.2}$$

We observe that this function g is not continuous with respect to time t, but considering the continuity of the function f, we can easily see that g satisfies the Carathéodory condition on $\mathbb{R} \times C$. With this hypothesis, we can extend the results of existence, uniqueness and continuous dependence to equation (2.2), and in this case, the solution of (1.1) is an absolutely continuous function on $[t_o, b)$ and satisfies (1.1) almost everywhere on $[t_o, b)$.

2.2. Dichotomic Maps

It is well known that Liapunov functionals are hard to find. This difficulty has inspired some simplifications of this idea, which is contained in Razumikhin's principle, that is a special case of Liapunov's. In this sense, we present an improvement of this idea with the help of a dichotomic maps theory. It follows the definition of dichotomic maps.

Definition 2.2. If $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous and $x(., t, \psi)$ is the solution of (1.1) through (t, ψ), we define the variation of V along (1.1)-(2.1) by

$$V'(t,\psi) = \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, x(t+h,t,\psi)) - V(t, x(t,t,\psi))].$$

Definition 2.3. Let U be a neighborhood of the origin in $\mathbb{R}^n$. A continuous function $V : \mathbb{R} \times U \to \mathbb{R}$ is said to be positive definite in U, when $V(t, 0) = 0$ for $t \in \mathbb{R}$ and there exists a continuous map $\alpha \colon \mathbb{R} \to \mathbb{R}$, strictly increasing on $[0, \infty)$ with $\alpha(0) = 0$, such that $V(t, x) \geq \alpha(|x|)$, $(t, x) \in \mathbb{R} \times U$.

In the next definition, $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a differentiable map, Ω is a neighborhood of the origin in C, $p : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and nondecreasing map and δ is a positive number such that $p(y) > y$ for all $y \in (0, \delta)$.

Definition 2.4. We say that V is dichotomic with respect to (1.1) (in Ω) if, for $t_o \in \mathbb{R}$ we can find $\mu = \mu(t_o) > 0$ such that, for all points of Ω where the variation of V along (1.1)-(2.1) is nonnegative at time t for $t \geq t_o + \mu$, then there exist $T = T(t_o, t)$ and a previous instant $\bar{t}$, $\bar{t} < t$, $t - \bar{t} \leq T$ such that $V(t, x(t)) \leq V(\bar{t}, x(\bar{t}))$.

Definition 2.5. V is strictly dichotomic with respect to (1.1) when:

(i) If V is as above, then we must have p satisfying the condition (iii) such that $p(V(t, x(t)) < V(\bar{t}, x(\bar{t}))$ with $t - \bar{t} \leq M < \infty$, for some real number M, $0 < M < \infty$.

(ii) If the derivative of V with respect to t along the solution tends to zero as $t \to \infty$ and if V tends to the constant function as $t \to \infty$, it must imply that this solution tends to the null solution as $t \to \infty$.

At the study of stability, we know that the direct method of Liapunov requires a Liapunov functional V, that is, the variation of V along the solution of a certain functional differential equation it must be negative semi-definite, everywhere. To use the theory of dichotomic maps, we note that is not required negative variation everywhere; if there exists an instant t which the variation is positive, it is sufficient to have a anterior instant $\overline{t}$ such that the function V presents decreasing in $[\overline{t}, t]$. In the next subsection, we present the theory of stability by using dichotomic maps.

2.3. Stability

The following results are concerned about the stability and asymptotic stability of the null solution of (1.1); see Carvalho et al [3].

Theorem 2.6. *Let $u, v : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous, nondecreasing functions, which are positive for $s > 0$ and $u(0) = v(0) = 0$. If there exists a positive definite dichotomic map with respect to (1.1), $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $u(|x|) \leq V(t, x) \leq v(|x|)$, for all $t, x) \in\in \mathbb{R} \times \mathbb{R}^n$, then the null equilibrium of (1.1) is stable.*

Theorem 2.7. *Let V be a continuously differentiable, strictly dichotomic map with respect to (1.1) in Theorem (2.6). Then, the null equilibrium of (1.1) is asymptotically stable.*

3. Main Result

In Marconato [4], we proved that the null solution of the equation

$$\dot{x}(t) = ax(t) + bx([t]) \tag{3.1}$$

is asymptotically stable, if $a \leq -\delta < 0$, $|b| < k\delta$, $\delta > 0$ and $k \in (0, 1)$.

Now, we proved, under some conditions, the same result to the equation

$$\dot{x}(t) = -a(t)x(t) + b(t)x([t]). \tag{3.2}$$

Theorem 3.1. *Let equations (3.2) where $a : \mathbb{R}_+ \to \mathbb{R}_+^*$ and $b : \mathbb{R}_+ \to \mathbb{R}$ are continuous maps such that $0 < c \leq a(t) < +\infty$ and $\mid b(t) \mid\leq b < \mu.c$ for all t and for some $0 < \mu < 1$. Then the null solution of (3.2) is asymptotically stable.*

Proof. We will prove that $V(x) = x^2/2$ is strictly dichotomic map with respect to (3.2), that is, we will prove that, whenever $V'(x_t) \geq 0$ for some t, there exist, unless one anterior instant $\bar{t}$ and a map $p(y)$ such that $p(V(x(t))) < V(x(\bar{t})$. We suppose that, for some $t \geq 0$, $V'(x_t) = x(t)\dot{x}(t) = x(t)[-a(t)x(t) + b(t)x([t])] \geq 0$. We observe that, in this case, $b(t) \neq 0$. We analyze two possibilities: (i) $x(t) \geq 0$ and (ii) $x(t) < 0$. In the case (i) we have $x(t)[-a(t)x(t) + b(t)x([t])] \geq 0$, that is, $-a(t)x(t) + b(t)x([t]) \geq 0$. Then, $b(t)x([t]) \geq a(t)x(t)$ and since $a(t) > 0$ and $x(t) \geq 0$, we have $b(t)x([t]) \geq 0$. We consider two cases: (i-1) $b(t) > 0$ and $x([t]) \geq 0$ and (i-2) $b(t) < 0$ and $x([t]) < 0$. In the case (i-1) it follows that $x([t]) \geq \frac{a(t)}{b(t)}x(t) \geq \frac{c}{b}x(t) > \frac{1}{\mu}x(t)$ and therefore, $\frac{x^2([t])}{2} > \frac{x^2(t)}{2\mu^2}$. By considering the map $p(y) = \frac{y}{\mu^2}$ for $y > 0$, we have that $p(V(x(t)) < V(x([t]))$. In the case (i-2), that is, $b(t) < 0$ and $x([t]) < 0$ with $V'(x_t) \geq 0$, we have $-x([t]) \geq \frac{a(t)}{-b(t)}x(t) \geq \frac{c}{b}x(t) > \frac{1}{\mu}x(t)$ and therefore, $\frac{x^2([t])}{2} > \frac{x^2(t)}{2\mu^2}$, that is, $p(V(x(t)) < V(x([t]))$. In the case (ii), that is, $x(t) < 0$, we have $-a(t)x(t) + b(t)x([t]) \leq 0$ that is, $b(t)x([t]) \leq a(t)x(t)$. Since $a(t) > 0$ and $x(t) < 0$, it follows that $b(t)x([t]) \leq 0$. We have two cases to consider: (ii-1) $b(t) < 0$ and $x([t]) \geq 0$ and (ii-2) $b(t) > 0$ and $x([t]) < 0$. In the case (ii-1) we have that $x([t]) \geq \frac{a(t)}{b(t)}x(t) = \frac{a(t)}{-b(t)}[-x(t)] \geq \frac{c}{b}[-x(t)] > \frac{1}{\mu}[-x(t)]$. Then, $\frac{x^2([t])}{2} > \frac{[-x(t)]^2}{2\mu^2}$ and $p(V(x(t)) < V(x([t]))$. In the case (ii-2), the proof is analogous. Then, in the two cases, whenever $V'(x_t) \geq 0$ for some t, we have an anterior instant $[t]$, such that $p(V(x(t)) < V(x([t]))$ and therefore, (i) and (ii) of the strict dichotomy definition are satisfied with $M = 1$. We will prove (iii). Let $V^{'}(x_t) \to 0$ and $x(t) \to w$ when $t \to \infty$. Then $0 = \lim_{t\to\infty} x(t)[-a(t)x(t) + b(t)x([t])] \leq \lim_{t\to\infty} x(t)[(-c)x(t) + bx([t])] = w^2[-c+b] \leq 0$. Since $(b - c) < 0$ by hypothesis, we have, necessarily, $w = 0$ and it follows the result. So Definition (2.5) is satisfied and V is a strictly dichotomic map with respect to (3.2). By Theorems 2.6 and 2.7 and considering the functions $u(x) = \frac{x^2}{2}$ and $v(x) = x^2$ for $x > 0$, we have that the null solution of (3.2) is asymptotically stable. □

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