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STABILITY OF LINEAR TIME-VARYING SYSTEMS

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Abstract: This paper studies the stabilization of the infinite-dimensional linear time-varying system with state delays

$$\dot{x} = A(t)x + A_1(t)x(t-h) + B(t)u\,.$$

The operator $A(t)$ is assumed to be the generator of a strong evolution operator. In contrast to the previous results, the stabilizability conditions are obtained via solving a Riccati differential equation and do not involve any stability property of the evolution operator. Our conditions are easy to be constructed and verifyed. We provide a step-by-step procedure for finding feedback controllers.

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1. Introduction

Consider a linear control system with state delays

$$\begin{gathered} \dot{x}(t) = A(t)x(t) + A_1(t)x(t-h) + B(t)u(t), \quad t \geq t_0, \\ x(t) = \phi(t), \quad t \in [-h, t_0], \end{gathered} \tag{1.1}$$

where $x \in X$ is the state, $u \in U$ is the control, $h \geq 0$. The stabilizability question consists on finding a feedback control $u(t) = K(t)x(t)$ for keeping the closed-loop system

$$\dot{x}(t) = [A(t) + B(t)K(t)]x(t) + A_1(t)x(t-h)$$

asymptotically stable in the Lyapunov sense. In the qualitative theory of dy-

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namical systems, the stabilizability is one of the most important properties of the systems and has attracted the attention of many researchers; see for example [1, 7, 10, 16, 17, 21] and references therein. It is well known that the main technique for solving stabilizability for control systems is the Lyapunov function method, but finding Lyapunov functions is still a difficult task (see, e.g. [3, 13, 15, 19, 20, 22]). However, for linear control system (1.1), the system can be made exponentially stabilizable if the underlying system $\dot{x}(t) = A(t)x(t)$ is asymptotically stable. In other words, if the evolution operator $E(t,s)$ generated by $A(t)$ is stable, then the delay control system (1.1) is asymptotically stabilizable under appropriate conditions on $A_1(t)$ (see [1, 17, 22]). For infinite-dimensional control systems, the investigation of stabilizability is more complicated and requires sophisticated techniques from semigroup theory. The difficulties increase to the same extent as passing from time-invariant to time-varying systems. Some results have been given in [2, 4, 9, 17] for time-invariant systems in Hilbert spaces.

The paper is organized as follows. In Section 2 we give the notation, and definitions to be used in this paper. Auxiliary propositions are given in Section 3. Sufficient conditions for the stabilizability are presented in Section 4.

2. Notation and Definitions

We will use the following notation: $\mathbb{R}^+$ denotes the set of all non-negative real numbers. X denotes a Hilbert space with the norm $\|.\|_X$ and the inner product $\langle .,. \rangle_X$, etc. $L(X)$ (respectively, $L(X,Y)$) denotes the Banach space of all linear bounded operators S mapping X into X (respectively, X into Y) endowed with the norm

$$\|S\| = \sup\{\|Sx\| : x \in X, \|x\| \leq 1\}.$$

$L_2([t,s],X)$ denotes the set of all strongly measurable square integrable X-valued functions on $[t,s]$. $D(A)$, $\mathrm{Im}(A), A^*$ and A^{-1} denote the domain, the image, the adjoint and the inverse of the operator A, respectively. If A is a matrix, then A^T denotes the conjugate transpose of A. $B_1 = \{x \in X : \|x\| = 1\}$. $\mathrm{cl}\, M$ denotes the closure of a set M; I denotes the identity operator. $C_{[t,s],X}$ denotes the set of all X-valued continuous functions on $[t,s]$. Let X, U be Hilbert spaces. Consider a linear time-varying control undelayed system $[A(t), B(t)]$ given by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad t \geq t_0, \\ x(t_0) &= x_0, \end{aligned} \tag{2.1}$$

where $x(t) \in X$, $u(t) \in U$; $A(t) : X \to X$; $B(t) \in L(U, X)$.

In the sequel, we say that control $u(t)$ is admissible if $u(t) \in L_2([t_0, \infty), U)$.

We make the following assumptions on the system (2.1):

(i) $B(t) \in L(U, X)$ and $B(.)u \in C_{[t_0,\infty),X}$ for all $u \in U$.

(ii) The operator $A(t) : D(A(t)) \subset X \to X$, $\operatorname{cl} D(A(t)) = X$ is a bounded function in $t \in [t_0, \infty)$ and generates a strong evolution operator $E(t, \tau) : \{(t, \tau) : t \geq \tau \geq t_0\} \to L(X)$ (see, e.g. [5, 6]):

$$E(t, t) = I, \quad t \geq t_0, \quad E(t, \tau)E(\tau, r) = E(t, r), \quad \forall t \geq \tau \geq r \geq t_0\,,$$

$E(t, \tau)$ is continuous in t and τ, $E(t, t_0)x = x + \int_{t_0}^{t} E(t, \tau)A(\tau)x d\tau$, for all $x \in D(A(t))$, so that the system (2.1), for every admissible control $u(t)$ has a unique solution given by

$$x(t) = E(t, t_0)x_0 + \int_{t_0}^{t} E(t, \tau)B(\tau)u(\tau)d\tau.$$

Definition. The system $[A(t), B(t)]$ is called globally null-controllable in time $T > 0$, if every state can be transferred to 0 in time T by some admissible control $u(t)$, i.e.,

$$\operatorname{Im} U(T, t_0) \subset L_T(L_2([t_0, T), U),$$

where $L_T = \int_{t_0}^{T} E(T, s)B(s)ds$.

Definition. The system $[A(t), B(t)]$ is called stabilizable if there exists an operator function $K(t) \in L(X, U)$ such that the zero solution of the closed loop system $\dot{x} = [A(t) + B(t)K(t)]x$ is asymptotically stable in the Lyapunov sense.

Following the setting in [2], we give a concept of the Riccati differential equation in a Hilbert space. Consider a differential operator equation

$$\dot{P}(t) + A^*(t)P(t) + P(t)A(t) - P(t)B(t)R^{-1}B^*(t)P(t) + Q(t) = 0, \qquad (2.2)$$

where $P(t), Q(t) \in L(X)$ and $R > 0$ is a constant operator.

Definition. An operator $P(t) \in L(X)$ is said to be a solution of the Riccati differential equation (2.2) if for all $t \geq t_0$ and all $x \in D(A(t))$,

$$\langle \dot{P}x, x\rangle + \langle PAx, x\rangle + \langle Px, Ax\rangle - \langle PBR^{-1}B^*Px, x\rangle + \langle Qx, x\rangle = 0\,.$$

An operator $Q \in L(X)$ is said to be non-negative definite, denote by $Q \geq 0$, if $\langle Qx, x\rangle \geq 0$, for all $x \in X$. If for some $c > 0, \langle Qx, x\rangle > c\|x\|^2$ for all $x \in X$, then Q is called positive definite and is denote by $Q > 0$. Operator $Q \in L(X)$ is called self-adjoint if $Q = Q^*$. The self-adjoint operator is characterized by the fact that its inner product $\langle Qx, x\rangle$ takes only real values and its spectrum is a bounded closed set on the real axis. The least segment that contains the spectrum is $[\lambda_{\min}(Q), \lambda_{\max}(Q)]$, where

$$\lambda_{\min}(Q) = \inf\{\langle Qx, x\rangle : x \in B_1\},$$
$$\lambda_{\max}(Q) = \sup\{\langle Qx, x\rangle : x \in B_1\} = \|Q\|.$$

We denote by $BC([t,\infty], X^+)$ the set of all linear bounded self-adjoint non-negative definite operators in $L(X)$ that are continuous and bounded on $[t,\infty)$.

3. Auxiliary Propositions

To prove the main results we need the following propositions.

Proposition 3.1. (see [5]) *If $Q \in L(X)$ is a self-adjoint positive definite operator, then $\lambda_{\min}(Q) > 0$ and*

$$\lambda_{\min}(Q)\|x\|^2 \le \langle Qx, x\rangle \le \lambda_{\max}(Q)\|x\|^2, \quad \forall x \in X.$$

Proposition 3.2. (see [11]) *Assume that there exist a function $V(t, x_t) : R^+ \times C([t_0, -h]) \to R^+$ and numbers $c_1 > 0, c_2 > 0, c_3 > 0$ such that:*

(i) $c_1\|x(t)\|^2 \le V(t, x_t) \le c_2\|x_t\|^2$, *for all* $t \ge t_0$.

(ii) $\frac{d}{dt}V(t, x_t) \le -c_3\|x(t)\|^2$, *for all* $t \ge t_0$.

Then the system (3.3) is asymptotically stable.

4. Stabilizability Conditions

Consider the linear control delay system (1.1), where $x(t) \in X$, $u(t) \in U$; X, U are infinite-dimensional Hilbert spaces; $A_1(t) : X \to X$ and $A(t), B(t)$ satisfy the assumptions stated in Section 2 so that the control system (1.1) has a unique solution for every initial condition $\phi(t) \in C_{[0,\infty),X}$ and admissible control $u(t)$. Let

$$p = \sup_{t\in[t_0,\infty)} \|P(t)\|.$$

Theorem 4.1. *Assume that for some self-adjoint constant positive definite operator $Q \in L(X)$, the Riccati differential equation (2.2), where $R = I$ has a solution $P(t) \in BC([t_0,\infty), X^+)$ such that*

$$a_1 := \sup_{t\in[t_0,\infty)} \|A_1(t)\| < \frac{\sqrt{\lambda_{\min}(Q)}}{2p}. \tag{4.1}$$

Then the control delay system (1.1) is asymptotically stable.

Proof. For simplicity of expression, let $t_0 = 0$. Let $0 < Q \in L(X)$, $P(t) \in BC([0,\infty), X^+)$ satisfy the Riccati equation (2.2), where $R = I$. Let

$$u(t) = K(t)x(t), \tag{4.2}$$

where $K(t) = -\frac{1}{2}B^*(t)P(t)$, $t \geq 0$.

For some number $\alpha \in (0,1)$ to be chosen later, we consider a Lyapunov function, for the delay system (1.1),

$$V(t, x_t) = \langle P(t)x(t), x(t)\rangle + \alpha \int_{t-h}^{t} \langle Qx(s), x(s)\rangle ds.$$

Since $Q > 0$ and $P(t) \in BC([0,\infty), X^+)$, it is easy to verify that

$$c_1\|x(t)\|^2 \leq V(t, x_t) \leq c_2\|x_t\|^2,$$

for some positive constants c_1, c_2. On the other hand, taking the derivative of $V(t, x_t)$ along the solution $x(t)$ of the system, we have

$$\begin{aligned} \dot{V}(t, x_t) =& \langle \dot{P}(t)x(t), x(t)\rangle + 2\langle P(t)\dot{x}(t), x(t)\rangle \\ &+ \alpha[\langle Qx(t), x(t)\rangle - \langle Qx(t-h), x(t-h)\rangle]. \end{aligned} \tag{4.3}$$

Substituting the control (4.2) into (4.3) gives

$$\begin{aligned} \dot{V}(t, x_t) &= -(1-\alpha)\langle Qx(t), x(t)\rangle + 2\langle P(t)A_1(t)x(t-h), x(t)\rangle \\ &\quad -\alpha\langle Qx(t-h), x(t-h)\rangle. \end{aligned}$$

From Proposition 3.1 it follows that

$$\lambda_{\min}(Q)\|x\|^2 \leq \langle Qx, x\rangle \leq \lambda_{\max}(Q)\|x\|^2, \quad x \in X,$$

where $\lambda_{\min}(Q) > 0$. Therefore,

$$\dot{V}(t, x_t) \leq -\lambda_{\min}(Q)(1-\alpha)\|x\|^2 + 2pa_1\|x(t-h)\|\|x(t)\| - \lambda_{\min}(Q)\alpha\|x(t-h)\|^2.$$

By completing the square, we obtain

$$\begin{aligned} &2pa_1\|x(t-h)\|\|x(t)\| - \lambda_{\min}(Q)\alpha\|x(t-h)\|^2 \\ &\quad = -\Big[\sqrt{\alpha\lambda_{\min}(Q)}\|x(t-h)\| - \frac{pa_1}{\sqrt{\alpha\lambda_{\min}(Q)}}\|x(t)\|\Big]^2 + \frac{p^2a_1^2}{\alpha\lambda_{\min}(Q)}\|x(t)\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \dot{V}(t, x_t) &\leq -\lambda_{\min}(Q)(1-\alpha)\|x(t)\|^2 + \frac{p^2a_1^2}{\alpha\lambda_{\min}(Q)}\|x(t)\|^2 \\ &= -\Big[\lambda_{\min}(Q)(1-\alpha) - \frac{1}{\alpha\lambda_{\min}(Q)}p^2a_1^2\Big]\|x(t)\|^2. \end{aligned}$$

Since the maximum value of $\alpha(1-\alpha)$ in $(0,1)$ is attained at $\alpha = 1/2$, from (4.1) it follows that for some $c_3 > 0$,

$$\dot{V}(t, x_t) \leq -c_3\|x(t)\|^2, \quad \forall t \geq t_0.$$

The the present proof is complete by using Proposition 3.2. □

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