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TIME-DEPENDENT ANALYSIS FOR THE $M^{[X]}/G/1$ RETRIAL QUEUEING MODEL WITH SERVER BREAKDOWNS AND CONSTANT RATE OF REPEATED ATTEMPTS

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Abstract: By using the Hille-Yosida Theorem, Phillips Theorem and Fattorini Theorem we prove that the $M^{[X]}/G/1$ retrial queueing model with server breakdowns and constant rate of repeated attempts has a unique nonnegative time-dependent solution which satisfies probability condition.

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Key Words: $M^{[X]}/G/1$ retrial queueing model, server breakdowns and constant rate of repeated attempts, C_0-semigroup, dispersive operator

1. Introduction

According to Atencia et al [2], the $M^{[X]}/G/1$ retrial queue with server breakdowns and constant rate of repeated attempts can be described by the following system of equations:

$$\frac{\mathrm{d}p_{0,0}(t)}{\mathrm{d}t} = -\lambda p_{0,0}(t) + \int_0^\infty p_{1,0}(x,t)b_1(x)\mathrm{d}x, \tag{1.1}$$

$$\frac{\mathrm{d}p_{0,n}(t)}{\mathrm{d}t} = -[\lambda+\alpha]p_{0,n}(t) + \int_0^\infty p_{1,n}(x,t)b_1(x)\mathrm{d}x, \quad n \geq 1, \tag{1.2}$$

$$\frac{\partial p_{1,0}(x,t)}{\partial t} + \frac{\partial p_{1,0}(x,t)}{\partial x} = -[\lambda + v + b_1(x)]p_{1,0}(x,t)$$

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$$+\int_0^\infty p_{2,0}(x,y,t)b_2(y)\mathrm{d}y, \tag{1.3}$$

$$\frac{\partial p_{1,n}(x,t)}{\partial t}+\frac{\partial p_{1,n}(x,t)}{\partial x}=-[\lambda+v+b_1(x)]p_{1,n}(x,t)+\lambda\sum_{k=1}^{n}c_kp_{1,n-k}(x,t)$$

$$+\int_0^\infty p_{2,n}(x,y,t)b_2(y)\mathrm{d}y,\quad n\geq 1, \tag{1.4}$$

$$\frac{\partial p_{2,0}(x,y,t)}{\partial t}+\frac{\partial p_{2,0}(x,y,t)}{\partial y}=-[\lambda+b_2(y)]p_{2,0}(x,y,t), \tag{1.5}$$

$$\frac{\partial p_{2,n}(x,y,t)}{\partial t}+\frac{\partial p_{2,n}(x,y,t)}{\partial y}=-[\lambda+b_2(y)]p_{2,n}(x,y,t)$$

$$+\lambda\sum_{k=1}^{n}c_kp_{2,n-k}(x,y,t),\quad n\geq 1, \tag{1.6}$$

$$p_{1,n}(0,t)=\lambda\sum_{k=1}^{n+1}c_kp_{0,n+1-k}(t)+\alpha p_{0,n+1}(t),\quad n\geq 0, \tag{1.7}$$

$$p_{2,n}(x,0,t)=vp_{1,n}(x,t),\quad n\geq 0, \tag{1.8}$$

$$p_{0,0}(0)=1,\ p_{0,n}(0)=0,\quad n\geq 1, \tag{1.9}$$

$$p_{0,k}(x,0)=0,\ p_{2,k}(x,y,0)=0,\quad k\geq 0. \tag{1.10}$$

Here $(x,y,t)\in[0,\infty)\times[0,\infty)\times[0,\infty)$; $p_{0,0}(t)$ represents the probability that there are no repeated customers in the system and the server is idle at time t; $p_{0,n}(t)$ $(n\geq 1)$ represents the probability that at time t the server is free and there are n repeated customers in the system; $p_{1,n}(x,t)dx$ represents the probability that at time t the server is busy and there are n repeated customers in the system with elapsed service time of the customer currently being served lying in $[x,x+\mathrm{d}x)$; $p_{2,n}(x,y,t)dy$ represents the probability that at time t there are n repeated customers with elapsed service time for the customer under service x and the server is breakdown with elapsed repair time lying in $[y,y+\mathrm{d}y)$; λ is the mean arrival rate of customers; α is the repeated rate of customers; v is the service rate of the server; c_k $(k\geq 1)$ represents the probability that at every arrival epoch a batch of k external customers arrives and satisfies $\sum_{k=1}^{\infty}c_k=1$. $b_1(x)$ is the service completion rate at time x satisfying

$$b_1(x)\geq 0,\quad \int_0^\infty b_1(x)dx=\infty.$$

$b_2(y)$ is the repair rate at time y satisfying

$$b_2(y) \geq 0, \quad \int_0^\infty b_2(y)dy = \infty.$$

In 2008, Atencia et al [2] established the above mathematical model of the $M^{[X]}/G/1$ retrial queue with server breakdowns and constant rate of repeated attempts by using supplementary variable technique and studied the joint distribution of the server state and the orbit size in steady-state case under the following hypothesis:

$$\begin{aligned}
&\lim_{t\to\infty} p_{0,n}(t) = p_{0,n}, \quad n \geq 0,\\
&\lim_{t\to\infty} p_{1,n}(x,t) = p_{1,n}(x), \quad n \geq 0,\\
&\lim_{t\to\infty} p_{2,n}(x,y,t) = p_{2,n}(x,y), \quad n \geq 0.
\end{aligned}$$

By reading the paper we find that first of all it needs to study the well-posedness of the above model in view of partial differential equations and then study the above hypothesis. But, any other results have not been found about time-dependent solution in the literature until now. In this paper, by using the idea of Gupur et al [5] and Gupur [4] we study well-posedness of the above queueing model. First of all, by selecting a suitable Banach space as state space and introducing operators corresponding to the model we convert the queueing model into a Cauchy problem in a Banach space, next we prove that the operator corresponding to the model generates a positive contraction C_0-semigroup. Thirdly, we verify that the C_0-semigroup is isometric for the initial value of the model. Finally, we obtain that the model has a unique positive time-dependent solution which satisfies probability condition, which is essential to study the above hypothesis (see Gupur et al [5]).

For simplicity, we introduce two notations as follows:

$$\Gamma_1 = \begin{pmatrix} \lambda c_1 & \alpha & 0 & 0 & \cdots \\ \lambda c_2 & \lambda c_1 & \alpha & 0 & \cdots \\ \lambda c_3 & \lambda c_2 & \lambda c_1 & \alpha & \cdots \\ \lambda c_4 & \lambda c_3 & \lambda c_2 & \lambda c_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} v & 0 & 0 & 0 & \cdots \\ 0 & v & 0 & 0 & \cdots \\ 0 & 0 & v & 0 & \cdots \\ 0 & 0 & 0 & v & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Take a state space as follows:

$$X = \left\{ (p_0, p_1, p_2) \;\middle|\; \begin{array}{l} p_0 \in l^1,\ p_1 \in Y_1,\ p_2 \in Y_2, \\ \|(p_0,p_1,p_2)\| = \|p_0\|_{l^1} + \|p_1\|_{Y_1} + \|p_2\|_{Y_2} < \infty \end{array} \right\},$$

where

$$Y_1 = \left\{ p_1 \;\middle|\; \begin{array}{l} p_1 = (p_{1,0}, p_{1,1}, \cdots) \in L^1[0,\infty) \times L^1[0,\infty) \times \cdots, \\ \|p_1\| = \sum\limits_{n=0}^{\infty} \|p_{1,n}\|_{L^1[0,\infty)} < \infty \end{array} \right\},$$

$$Y_2 = \left\{ p_2 \;\middle|\; \begin{array}{l} p_2 = (p_{2,0}, p_{2,1}, p_{2,2}, \cdots) \\ \quad \in L^1([0,\infty)\times[0,\infty)) \times L^1([0,\infty)\times[0,\infty)) \times \cdots, \\ \|p_2\| = \sum\limits_{n=0}^{\infty} \|p_{2,n}\|_{L^1([0,\infty)\times[0,\infty))} < \infty \end{array} \right\}.$$

It is obvious that X is a Banach space. In the following we define operators and their domains.

$$A(p_0, p_1, p_3) = \left(\begin{pmatrix} -\lambda & 0 & 0 & \cdots \\ 0 & -(\lambda+\alpha) & 0 & \cdots \\ 0 & 0 & -(\lambda+\alpha) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{0,0} \\ p_{0,1} \\ p_{0,2} \\ \vdots \end{pmatrix}, \right.$$

$$\begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}x} & 0 & 0 & \cdots \\ 0 & -\frac{\mathrm{d}}{\mathrm{d}x} & 0 & \cdots \\ 0 & 0 & -\frac{\mathrm{d}}{\mathrm{d}x} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{1,0}(x) \\ p_{1,1}(x) \\ p_{1,2}(x) \\ \vdots \end{pmatrix},$$

$$\left. \begin{pmatrix} -\frac{\partial}{\partial y} & 0 & 0 & \cdots \\ 0 & -\frac{\partial}{\partial y} & 0 & \cdots \\ 0 & 0 & -\frac{\partial}{\partial y} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{2,0}(x,y) \\ p_{2,1}(x,y) \\ p_{2,2}(x,y) \\ \vdots \end{pmatrix} \right),$$

$$D(A) = \left\{ (p_0, p_1, p_2) \in X \;\middle|\; \begin{array}{l} \frac{\mathrm{d}p_{1,n}}{\mathrm{d}x} \in L^1[0,\infty), \quad n \ge 0, \\ \frac{\partial p_{2,n}}{\partial y} \in L^1([0,\infty)\times[0,\infty)), \ n \ge 0, \\ p_{1,n}(x) \text{ and } p_{2,n}(x,y) \text{ are absolutely} \\ \text{continuous and } p_1(0) = \Gamma_1 p_0, \\ p_2(x,0) = \Gamma_2 p_1(x), \end{array} \right\},$$

$$U(p_0, p_1, p_2) = \left(\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{0,0} \\ p_{0,1} \\ p_{0,2} \\ \vdots \end{pmatrix}, \begin{pmatrix} -\tilde{v} & 0 & 0 & \cdots \\ \lambda c_1 & -\tilde{v} & 0 & \cdots \\ \lambda c_2 & \lambda c_1 & -\tilde{v} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{1,0}(x) \\ p_{1,1}(x) \\ p_{1,2}(x) \\ \vdots \end{pmatrix}, \right.$$

$$\begin{pmatrix} -\tilde{u} & 0 & 0 & \cdots \\ \lambda c_1 & -\tilde{u} & 0 & \cdots \\ \lambda c_2 & \lambda c_1 & -\tilde{u} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{2,0}(y) \\ p_{2,1}(y) \\ p_{2,2}(y) \\ \vdots \end{pmatrix} \Bigg), \quad D(U) = X,$$

here $\tilde{v} = \lambda + v + b_1(x), \quad \tilde{u} = \lambda + b_2(y)$.

$$E(p_0, p_1, p_2) = \left(\begin{pmatrix} \psi_1 & 0 & 0 & \cdots \\ 0 & \psi_1 & 0 & \cdots \\ 0 & 0 & \psi_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{1,0}(x) \\ p_{1,1}(x) \\ p_{1,2}(x) \\ \vdots \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} \psi_2 & 0 & 0 & \cdots \\ 0 & \psi_2 & 0 & \cdots \\ 0 & 0 & \psi_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{2,0}(x,y) \\ p_{2,1}(x,y) \\ p_{2,2}(x,y) \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \right), \; D(E) = X,$$

where

$$\psi_1 : L^1[0,\infty) \to \mathbb{C}, \quad \psi_1(f) := \int_0^\infty b_1(x) f(x) dx, \quad \forall f \in L^1[0,\infty),$$

$$\psi_2 : L^1([0,\infty) \times [0,\infty)) \to \mathbb{C},$$

$$\psi_2(g) := \int_0^\infty b_2(y) g(x,y) dy, \; \forall g \in L^1([0,\infty) \times [0,\infty))).$$

Then the above system of equations (1.1)–(1.10) can be expressed as a Cauchy problem in the Banach space X.

$$\begin{cases} \frac{\mathrm{d}(p_0,p_1,p_2)(t)}{\mathrm{d}t} = (A + U + E)(p_0, p_1, p_2)(t), \quad \forall t \in (0, \infty), \\ (p_0, p_1, p_2)(0) = \left(\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix} \right). \end{cases} \tag{1.11}$$

2. Main Results

Theorem 2.1. *If* $b_1 = \sup_{x \in [0,\infty)} b_1(x) < \infty$ *and* $b_2 = \sup_{y \in [0,\infty)} b_2(y) < \infty$, *then* $A + U + E$ *generates a positive contraction* C_0*-semigroup* $T(t)$.

Proof. We split the proof of this theorem into four steps. First of all we will prove that $(\gamma I - A)^{-1}$ exists and is bounded for some γ, next we will prove that $D(A)$ is dense in X and therefore we will obtain that A generates a C_0-

semigroup. Thirdly we will show that U and E are bounded linear operators. Thus we will deduce that $A+U+E$ generates a C_0-semigroup $T(t)$. Lastly, we will prove that $A+U+E$ is a dispersive operator. From which together with the above steps we deduce that $T(t)$ is positive contractive.

For any given $(y_0, y_1, y_2) \in X$, we consider $(\gamma I - A)(p_0, p_1, p_2) = (y_0, y_1, y_2)$. It is equivalent to

$$(\gamma+\lambda)p_{0,0} = y_{0,0}, \tag{2.1}$$

$$(\gamma+\lambda+\alpha)p_{0,n} = y_{0,n}, \quad n \geq 1, \tag{2.2}$$

$$\frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x} = -\gamma p_{1,n}(x) + y_{1,n}(x), \quad n \geq 0, \tag{2.3}$$

$$\frac{\partial p_{2,n}(x,y)}{\partial y} = -\gamma p_{2,n}(x,y) + y_{2,n}(x,y), \quad n \geq 0, \tag{2.4}$$

$$p_{1,n}(0) = \lambda \sum_{k=1}^{n+1} c_k p_{0,n+1-k} + \alpha p_{0,n+1}, \quad n \geq 0, \tag{2.5}$$

$$p_{2,n}(x,0) = v p_{1,n}(x), \quad n \geq 0. \tag{2.6}$$

By solving (2.1)–(2.4) we have

$$p_{0,0} = \frac{1}{\gamma+\lambda} y_{0,0}, \tag{2.7}$$

$$p_{0,n} = \frac{1}{\gamma+\lambda+\alpha} y_{0,n}, \quad n \geq 1, \tag{2.8}$$

$$p_{1,n}(x) = a_{1,n} e^{-\gamma x} + e^{-\gamma x} \int_0^x y_{1,n}(\tau) e^{\gamma \tau} \mathrm{d}\tau, \quad n \geq 0, \tag{2.9}$$

$$p_{2,n}(x,y) = a_{2,n} e^{-\gamma y} + e^{-\gamma y} \int_0^y y_{2,n}(x,\tau) e^{\gamma \tau} \mathrm{d}\tau, \quad n \geq 0. \tag{2.10}$$

From (2.9), (2.10), (2.5) and (2.6) we know

$$a_{1,0} = p_{1,0}(0) = \lambda c_1 p_{0,0} + \alpha p_{0,1}, \tag{2.11}$$

$$a_{1,1} = p_{1,1}(0) = \lambda c_1 p_{0,1} + \lambda c_2 p_{0,0} + \alpha p_{0,2}, \tag{2.12}$$

$$a_{1,2} = p_{1,2}(0) = \lambda c_1 p_{0,2} + \lambda c_2 p_{0,1} + \lambda c_3 p_{0,0} + \alpha p_{0,3}, \tag{2.13}$$

$$\cdots$$

$$\begin{aligned} a_{1,n} &= p_{1,n}(0) \\ &= \lambda c_1 p_{0,n} + \lambda c_2 p_{0,n-1} + \cdots + \lambda c_n p_{0,1} + \lambda c_{n+1} p_{0,0} + \alpha p_{0,n+1}, \end{aligned} \tag{2.14}$$

$$a_{2,n} = p_{2,n}(x,0) = v p_{1,n}(x). \tag{2.15}$$

By using (2.7), (2.8), (2.11)-(2.15) and $\sum_{k=1}^{\infty} c_k = 1$ we deduce

$$\begin{aligned}
\sum_{n=0}^{\infty} |a_{1,n}| &\leq (\lambda c_1 |p_{0,0}| + \alpha |p_{0,1}|) + (\lambda c_1 |p_{0,1}| + \lambda c_2 |p_{0,0}| + \alpha |p_{0,2}|) \\
&\quad + (\lambda c_1 |p_{0,2}| + \lambda c_2 |p_{0,1}| + \lambda c_3 |p_{0,0}| + \alpha |p_{0,3}|) + \cdots \\
&\quad + (\lambda c_1 |p_{0,n-1}| + \lambda c_2 |p_{0,n-2}| + \cdots \\
&\quad + \lambda c_{n-1} |p_{0,1}| + \lambda c_n |p_{0,0}| + \alpha |p_{0,n}|) + (\lambda c_1 |p_{0,n}| + \lambda c_2 |p_{0,n-1}| + \cdots \\
&\quad + \lambda c_n |p_{0,1}| + \lambda c_{n+1} |p_{0,0}| + \alpha |p_{0,n+1}|) + \cdots\cdots \\
&= \alpha (|p_{0,1}| + |p_{0,2}| + |p_{0,3}| + \cdots) \\
&\quad + (\lambda c_1 + \lambda c_2 + \lambda c_3 + \cdots) |p_{0,0}| + (\lambda c_1 + \lambda c_2 + \lambda c_3 + \cdots) |p_{0,1}| \\
&\quad + (\lambda c_1 + \lambda c_2 + \lambda c_3 + \cdots) |p_{0,2}| + \cdots\cdots \\
&= \alpha \sum_{n=1}^{\infty} |p_{0,n}| + \lambda \sum_{k=1}^{\infty} c_k \sum_{n=0}^{\infty} |p_{0,n}| = \lambda |p_{0,0}| + (\lambda + \alpha) \sum_{n=1}^{\infty} |p_{0,n}| \\
&= \frac{\lambda}{\gamma + \lambda} |y_{0,0}| + \frac{\lambda + \alpha}{\gamma + \lambda + \alpha} \sum_{n=1}^{\infty} |y_{0,n}|, \qquad (2.16)
\end{aligned}$$

$$\sum_{n=0}^{\infty} |a_{2,n}| \leq v \sum_{n=0}^{\infty} |p_{1,n}(x)|. \qquad (2.17)$$

By combining (2.9) and (2.10) with (2.16), (2.17) and the Fubini Theorem we estimate (assume $\gamma > 0$)

$$\begin{aligned}
&\sum_{n=0}^{\infty} \|p_{1,n}\|_{L^1[0,\infty)} \\
&\leq \sum_{n=0}^{\infty} |a_{1,n}| \int_0^{\infty} e^{-\gamma x} \mathrm{d}x + \sum_{n=0}^{\infty} \int_0^{\infty} e^{-\gamma x} \int_0^x |y_{1,n}(\tau)| e^{\gamma \tau} \mathrm{d}\tau \mathrm{d}x \\
&= \frac{1}{\gamma} \sum_{n=0}^{\infty} |a_{1,n}| + \sum_{n=0}^{\infty} \int_0^{\infty} |y_{1,n}(\tau)| e^{\gamma \tau} \int_0^{\tau} e^{-\gamma x} \mathrm{d}x \mathrm{d}\tau \\
&= \frac{1}{\gamma} \sum_{n=0}^{\infty} |a_{1,n}| + \frac{1}{\gamma} \sum_{n=0}^{\infty} \|y_{1,n}\|_{L^1[0,\infty)} \\
&\leq \frac{\lambda}{\gamma(\gamma + \lambda)} |y_{0,0}| + \frac{\lambda + \alpha}{\gamma(\gamma + \lambda + \alpha)} \sum_{n=1}^{\infty} |y_{0,n}| + \frac{1}{\gamma} \sum_{n=0}^{\infty} \|y_{1,n}\|_{L^1[0,\infty)} \qquad (2.18) \\
&\Longrightarrow
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=0}^{\infty}\|p_{2,n}\|_{L^1([0,\infty)\times[0,\infty))}\\
&\leq \sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}|a_{2,n}|e^{-\gamma y}\mathrm{d}y\mathrm{d}x\\
&\quad+\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}e^{-\gamma y}\int_0^{y}|y_{2,n}(x,\tau)|e^{\gamma\tau}\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\\
&=\frac{v}{\gamma}\sum_{n=0}^{\infty}\int_0^{\infty}|p_{1,n}(x)|\mathrm{d}x\\
&\quad+\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}|y_{2,n}(x,\tau)|e^{\gamma\tau}\int_0^{\tau}e^{-\gamma y}\mathrm{d}y\mathrm{d}\tau\mathrm{d}x\\
&=\frac{v}{\gamma}\sum_{n=0}^{\infty}\int_0^{\infty}|p_{1,n}(x)|\mathrm{d}x+\frac{1}{\gamma}\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}|y_{2,n}(x,\tau)|\mathrm{d}\tau\mathrm{d}x\\
&\leq\frac{v}{\gamma}\sum_{n=0}^{\infty}\|p_{1,n}\|_{L^1[0,\infty)}+\frac{1}{\gamma}\sum_{n=0}^{\infty}\|y_{2,n}\|_{L^1([0,\infty)\times[0,\infty))}\\
&\leq\frac{\lambda v}{\gamma^2(\gamma+\lambda)}|y_{0,0}|+\frac{v(\lambda+\alpha)}{\gamma^2(\gamma+\lambda+\alpha)}\sum_{n=1}^{\infty}|y_{0,n}|\\
&\quad+\frac{v}{\gamma^2}\sum_{n=0}^{\infty}\|y_{1,n}\|_{L^1[0,\infty)}+\frac{1}{\gamma}\sum_{n=0}^{\infty}\|y_{2,n}\|_{L^1([0,\infty)\times[0,\infty))}.
\end{aligned}\tag{2.19}$$

(2.7), (2.8), (2.18) and (2.19) give (assume $\gamma > v$)

$$\begin{aligned}
\|(p_0,p_1,p_2)\| &= \sum_{n=0}^{\infty}|p_{0,n}|+\sum_{n=0}^{\infty}\|p_{1,n}\|_{L^1[0,\infty)}+\sum_{n=0}^{\infty}\|p_{2,n}\|_{L^1([0,\infty)\times[0,\infty))}\\
&\leq\frac{1}{\gamma+\lambda}|y_{0,0}|+\frac{1}{\gamma+\lambda+\alpha}\sum_{n=1}^{\infty}|y_{0,n}|+\frac{\lambda}{\gamma(\gamma+\lambda)}|y_{0,0}|\\
&\quad+\frac{\lambda+\alpha}{\gamma(\gamma+\lambda+\alpha)}\sum_{n=1}^{\infty}|y_{0,n}|+\frac{1}{\gamma}\sum_{n=0}^{\infty}\|y_{1,n}\|_{L^1[0,\infty)}\\
&\quad+\frac{\lambda v}{\gamma^2(\gamma+\lambda)}|y_{0,0}|+\frac{v(\lambda+\alpha)}{\gamma^2(\gamma+\lambda+\alpha)}\sum_{n=1}^{\infty}|y_{0,n}|\\
&\quad+\frac{v}{\gamma^2}\sum_{n=0}^{\infty}\|y_{1,n}\|_{L^1[0,\infty)}+\frac{1}{\gamma}\sum_{n=0}^{\infty}\|y_{2,n}\|_{L^1([0,\infty)\times[0,\infty))}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{\frac{1}{\gamma+\lambda} + \frac{\lambda}{\gamma(\gamma+\lambda)} + \frac{\lambda v}{\gamma^2(\gamma+\lambda)}\right\} |y_{0,0}| \\
&\quad + \left\{\frac{1}{\gamma+\lambda+\alpha} + \frac{\lambda+\alpha}{\gamma(\gamma+\lambda+\alpha)} + \frac{v(\lambda+\alpha)}{\gamma^2(\gamma+\lambda+\alpha)}\right\} \sum_{n=1}^{\infty} |y_{0,n}| \\
&\quad + \left\{\frac{1}{\gamma} + \frac{v}{\gamma^2}\right\} \sum_{n=0}^{\infty} \|y_{1,n}\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\gamma} \sum_{n=0}^{\infty} \|y_{2,n}\|_{L^1([0,\infty)\times[0,\infty))} \\
&= \frac{\gamma^2+\gamma\lambda+\lambda v}{\gamma^2(\gamma+\lambda)} |y_{0,0}| + \frac{\gamma(\gamma+\lambda+\alpha)+v(\lambda+\alpha)}{\gamma^2(\gamma+\lambda+\alpha)} \sum_{n=1}^{\infty} |y_{0,n}| \\
&\quad + \frac{\gamma+v}{\gamma^2} \sum_{n=0}^{\infty} \|y_{1,n}\|_{L^1[0,\infty)} + \frac{1}{\gamma} \sum_{n=0}^{\infty} \|y_{2,n}\|_{L^1([0,\infty)\times[0,\infty))} \\
&\le \frac{1}{\gamma - v} \left\{ \sum_{n=0}^{\infty} |y_{0,n}| + \sum_{n=0}^{\infty} \|y_{1,n}\|_{L^1[0,\infty)} \right. \\
&\quad \left. + \sum_{n=0}^{\infty} \|y_{2,n}(x,y)\|_{L^1([0,\infty)\times[0,\infty))} \right\} \\
&= \frac{1}{\gamma - v} \|(y_0, y_1, y_2)\|, \qquad (2.20)
\end{aligned}
$$

in which we used the following inequalities

$$
\begin{aligned}
&\gamma > v > 0 \Rightarrow 0 < \gamma^2 - v^2 < \gamma^2 \Rightarrow (\gamma - v)(\gamma + v) < \gamma^2 \\
&\Rightarrow \frac{\gamma+v}{\gamma^2} < \frac{1}{\gamma - v}. \\
&0 < \gamma^2 + v(\lambda+\alpha) \Rightarrow \gamma(\lambda+\alpha) < \gamma^2 + \gamma(\lambda+\alpha) + v(\lambda+\alpha) \\
&\Rightarrow \gamma(\lambda+\alpha) < \gamma(\gamma+\lambda+\alpha) + v(\lambda+v) \\
&\Longrightarrow \gamma v(\lambda+\alpha) < v[\gamma(\gamma+\lambda+\alpha) + v(\lambda+\alpha)] \\
&\Longrightarrow \gamma v(\lambda+\alpha) - v[\gamma(\gamma+\lambda+\alpha) + v(\lambda+\alpha)] < 0 \\
&\Longrightarrow \gamma^2(\gamma+\lambda+\alpha) + \gamma v(\lambda+\alpha) - v[\gamma(\gamma+\lambda+\alpha) + v(\lambda+\alpha)] \\
&< \gamma^2(\gamma+\lambda+\alpha) \\
&\Longrightarrow \gamma[\gamma(\gamma+\lambda+\alpha) + v(\lambda+\alpha)] - v[\gamma(\gamma+\lambda+\alpha) + v(\lambda+\alpha)] \\
&< \gamma^2(\gamma+\lambda+\alpha)[\gamma(\gamma+\lambda+\alpha) + v(\lambda+\alpha)](\gamma - v) < \gamma^2(\gamma+\lambda+\alpha) \\
&\Longrightarrow \frac{\gamma(\gamma+\lambda+\alpha) + v(\lambda+\alpha)}{\gamma^2(\gamma+\lambda+\alpha)} < \frac{1}{\gamma - v}.
\end{aligned}
$$

(2.20) shows that $(\gamma I - A)^{-1}$ exists for $\gamma > v$, and

$$(\gamma I - A)^{-1} : X \to D(A), \quad \left\|(\gamma I - A)^{-1}\right\| \leq \frac{1}{\gamma - v}.$$

In the following we will prove that $D(A)$ is dense in X. Since $\forall (p_0, p_1, p_2) \in X$ implies

$$\sum_{n=0}^{\infty} |p_{0,n}| + \sum_{n=0}^{\infty} \|p_{1,n}\|_{L^1[0,\infty)} + \sum_{n=0}^{\infty} \|p_{2,n}\|_{L^1([0,\infty)\times[0,\infty))} < \infty,$$

for any $\epsilon > 0$ there exists a finite positive integer N such that

$$\sum_{n=N}^{\infty} |p_{0,n}| < \epsilon, \quad \sum_{n=N}^{\infty} \|p_{1,n}\|_{L^1[0,\infty)} < \epsilon, \quad \sum_{n=N}^{\infty} \|p_{2,n}\|_{L^1([0,\infty)\times[0,\infty))} < \epsilon.$$

Which shows that the set

$$L = \left\{ (p_0, p_1, p_2) \;\middle|\; \begin{array}{l} p_0 = (p_{0,0}, p_{0,1}, p_{0,2}, \cdots, p_{0,N}, 0, 0, \cdots), \\ p_1(x) = (p_{1,0}(x), p_{1,1}(x), \cdots, p_{1,N}(x), 0, 0, \cdots), \\ p_2(x,y) = (p_{2,0}(x,y), \cdots, p_{2,N}(x,y), 0, 0, \cdots), \\ p_{0,n} \in \mathbb{R}, p_{1,n} \in L^1[0,\infty), \\ p_{2,n} \in L^1([0,\infty)\times[0,\infty)), \quad n = 0, 1, \cdots, N, \\ N \text{ is a finite positive integer} \end{array} \right\},$$

is dense in X. If we set

$$Z = \left\{ (p_0, p_1, p_2) \;\middle|\; \begin{array}{l} p_0 = (p_{0,0}, p_{0,1}, p_{0,2}, \cdots, p_{0,N}, 0, 0, \cdots), \\ p_1(x) = (p_{1,0}(x), p_{1,1}(x), \cdots, p_{1,N}(x), 0, 0, \cdots), \\ p_2(x,y) = (p_{2,0}(x,y), \cdots, p_{2,N}(x,y), 0, 0, \cdots), \\ p_{1,n} \in C_0^{\infty}[0,\infty), p_{2,n} \in C_0^{\infty}([0,\infty)\times[0,\infty)) \\ \text{there exist positive numbers } d_{1,i} > 0, \; d_{2,i} > 0 \\ \text{such that } p_{1,i}(x) = 0, \; x \in [0, d_{1,i}], \\ p_{2,i}(x,y) = 0, \; y \in [0, d_{2,i}]; \; i = 0, 1, \cdots, N, \\ N \text{ is a finite positive integer} \end{array} \right\},$$

then from Adams [1] we know that Z is dense in L. Therefore, in order to obtain denseness of $D(A)$, it is sufficient to prove that $D(A)$ is dense in Z.

Take any fixed $(p_0, p_1, p_2) \in Z$, then there are a finite positive integer N and positive numbers $d_{1,i} > 0, \; d_{2,i} > 0$ $(i = 0, 1, \cdots, N)$ such that

$$\begin{aligned} p_0 &= (p_{0,0}, p_{0,1}, p_{0,2}, \cdots, p_{0,N}, 0, 0, \cdots), \\ p_1(x) &= (p_{1,0}(x), p_{1,1}(x), p_{1,2}(x), \cdots, p_{1,N}(x), 0, 0, \cdots), \\ p_{1,i}(x) &= 0 \quad \text{for} \quad x \in [0, d_{1,i}], \; i = 0, 1, \cdots, N, \\ p_2(x,y) &= (p_{2,0}(x,y), p_{2,1}(x,y), p_{2,2}(x,y), \cdots, p_{2,N}(x,y), 0, 0, \cdots), \\ p_{2,i}(x,y) &= 0 \quad \text{for} \quad y \in [0, d_{2,i}], \; i = 0, 1, \cdots, N. \end{aligned}$$

This implies

$$\begin{aligned} p_{1,i}(x) =& 0, \quad \text{for} \quad x \in [0,s],\ i = 0,1,\cdots,N, \\ p_{2,i}(x,y) =& 0, \quad \text{for} \quad y \in [0,s],\ i = 0,1,\cdots,N, \end{aligned}$$

where $0 < s < \min\{d_{1,0}, d_{1,1}, \cdots, d_{1,N}, d_{2,0}, d_{2,1}, \cdots, d_{2,N}\}$. Define

$$f_1^s(0) = \begin{pmatrix} f_{1,0}^s(0) \\ f_{1,1}^s(0) \\ \vdots \\ f_{1,N-1}^s(0) \\ f_{1,N}^s(0) \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda c_1 p_{0,0} + \alpha p_{0,1} \\ \lambda c_1 p_{0,1} + \lambda c_2 p_{0,0} + \alpha p_{0,2} \\ \vdots \\ \lambda c_1 p_{0,N-1} + \lambda c_2 p_{0,N-2} + \cdots + \lambda c_N p_{0,0} + \alpha p_{0,N} \\ \lambda c_1 p_{0,N} + \lambda c_2 p_{0,N-1} + \cdots + \lambda c_{N+1} p_{0,0} \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

$$f_2^s(x,0) = \begin{pmatrix} f_{2,0}^s(x,0) \\ f_{2,1}^s(x,0) \\ \vdots \\ f_{2,N}^s(x,0) \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} v f_{1,0}^s(x) \\ v f_{1,1}^s(x) \\ \vdots \\ v f_{1,N}^s(x) \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

$$f_1^s(x) = \begin{pmatrix} f_{1,0}^s(x) \\ f_{1,1}^s(x) \\ \vdots \\ f_{1,N}^s(x) \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad f_2^s(x,y) = \begin{pmatrix} f_{2,0}^s(x,y) \\ f_{2,1}^s(x,y) \\ \vdots \\ f_{2,N}^s(x,y) \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

Here

$$f_{1,i}^s(x) = \begin{cases} f_{1,i}^s(0)\left(1 - \frac{x}{s}\right)^2 & \text{if } x \in [0,s) \\ p_{1,i}(x) & \text{if } x \in [s,\infty) \end{cases}, \quad i = 0,1,\cdots,N;$$

$$f_{2,i}^s(x,y) = \begin{cases} f_{2,i}^s(x,0)\left(1 - \frac{y}{s}\right)^2 & \text{if } y \in [0,s) \\ p_{2,i}(x,y) & \text{if } y \in [s,\infty) \end{cases}, \quad i = 0,1,\cdots,N.$$

It is easy to verify that $(p_0, f_1^s, f_2^s) \in D(A)$. Moreover

$$\|(p_0,p_1,p_2)-(p_0,f_1^s,f_2^s)\| = \sum_{i=0}^{N}\int_0^\infty |p_{1,i}(x)-f_{1,i}^s(x)|\mathrm{d}x$$

$$+\sum_{i=0}^{N}\int_0^\infty\int_0^\infty |p_{2,i}(x,y)-f_{2,i}^s(x,y)|\mathrm{d}y\mathrm{d}x$$

$$=\sum_{i=0}^{N}\int_0^s |f_{1,i}^s(0)|\left(1-\frac{x}{s}\right)^2\mathrm{d}x+\sum_{i=0}^{N}\int_0^\infty\int_0^s |f_{2,i}^s(x,0)|\left(1-\frac{y}{s}\right)^2\mathrm{d}y\mathrm{d}x$$

$$=\sum_{i=0}^{N}|f_{1,i}^s(0)|\frac{s}{3}+\frac{s}{3}\sum_{i=0}^{N}\int_0^\infty |vf_{1,i}^s(x)|\mathrm{d}x$$

$$=\sum_{i=0}^{N}|f_{1,i}^s(0)|\frac{s}{3}+\frac{s}{3}v\sum_{i=0}^{N}\left\{\int_0^s |f_{1,i}^s(0)|\left(1-\frac{x}{s}\right)^2dx+\int_s^\infty |p_{1,i}(x)|dx\right\}$$

$$=\sum_{i=0}^{N}|f_{1,i}^s(0)|\frac{s}{3}+\frac{s}{3}v\sum_{i=0}^{N}\left\{|f_{1,i}^s(0)|\frac{s}{3}+\int_s^\infty |p_{1,i}(x)|dx\right\}\to 0,\quad \text{as}\quad s\to 0.$$

This means that $D(A)$ is dense in Z. In other words, $D(A)$ is dense in X. From the above two steps and the Hille-Yosida Theorem we conclude that A generates a C_0-semigroup (see Gupur et al [5]).

As far as the third step is concerned, from the definitions of U and E, and $\sum\limits_{k=1}^{\infty} c_k = 1$ we have, for $\forall (p_0,p_1,p_2)\in X$,

$$\|U(p_0,p_1,p_2)\| \le \sum_{n=0}^{\infty}\int_0^\infty(\lambda+v+b_1(x))|p_{1,n}(x)|\mathrm{d}x+\lambda c_1\int_0^\infty |p_{1,0}(x)|\mathrm{d}x$$

$$+\left(\lambda c_2\int_0^\infty |p_{1,0}(x)|\mathrm{d}x+\lambda c_1\int_0^\infty |p_{1,1}(x)|\mathrm{d}x\right)$$

$$+\left(\lambda c_3\int_0^\infty |p_{1,0}(x)|\mathrm{d}x+\lambda c_2\int_0^\infty |p_{1,1}(x)|\mathrm{d}x+\lambda c_1\int_0^\infty |p_{1,2}(x)|\mathrm{d}x\right)$$

$$+\cdots$$

$$+\sum_{n=0}^{\infty}\int_0^\infty\int_0^\infty(\lambda+b_2(x))|p_{2,n}(x,y)|\mathrm{d}y\mathrm{d}x$$

$$+\lambda c_1\int_0^\infty\int_0^\infty |p_{2,0}(x,y)|\mathrm{d}y\mathrm{d}x+\left(\lambda c_2\int_0^\infty\int_0^\infty |p_{2,0}(x,y)|\mathrm{d}y\mathrm{d}x\right.$$

$$
\begin{aligned}
&+\lambda c_1 \int_0^\infty \int_0^\infty |p_{2,1}(x,y)(x)| \mathrm{d}y \mathrm{d}x \Big) + \Big(\lambda c_3 \int_0^\infty \int_0^\infty |p_{2,0}(x,y)(x)| \mathrm{d}y \mathrm{d}x \\
&+ \lambda c_2 \int_0^\infty \int_0^\infty |p_{2,1}(x,y)| \mathrm{d}y \mathrm{d}x + \lambda c_1 \int_0^\infty \int_0^\infty |p_{2,2}(x,y)| \mathrm{d}y \mathrm{d}x \Big) \\
&+ \cdots \\
\le\ & (\lambda + v + b_1) \sum_{n=0}^\infty \int_0^\infty |p_{1,n}(x)| \mathrm{d}x + (\lambda c_1 + \lambda c_2 + \lambda c_3 + \cdots) \int_0^\infty |p_{1,0}(x)| \mathrm{d}x \\
&+ (\lambda c_1 + \lambda c_2 + \lambda c_3 + \cdots) \int_0^\infty |p_{1,1}(x)| \mathrm{d}x \\
&+ (\lambda c_1 + \lambda c_2 + \lambda c_3 + \cdots) \int_0^\infty |p_{1,2}(x)| \mathrm{d}x \\
&+ \cdots \\
&+ (\lambda + b_2) \sum_{n=0}^\infty \int_0^\infty \int_0^\infty |p_{2,n}(x,y)| \mathrm{d}y \mathrm{d}x \\
&+ (\lambda c_1 + \lambda c_2 + \lambda c_3 + \cdots) \int_0^\infty \int_0^\infty |p_{2,0}(x,y)| \mathrm{d}y \mathrm{d}x \\
&+ (\lambda c_1 + \lambda c_2 + \lambda c_3 + \cdots) \int_0^\infty \int_0^\infty |p_{2,1}(x,y)| \mathrm{d}y \mathrm{d}x \\
&+ (\lambda c_1 + \lambda c_2 + \lambda c_3 + \cdots) \int_0^\infty \int_0^\infty |p_{2,2}(x,y)| \mathrm{d}y \mathrm{d}x \\
&+ \cdots\cdots \\
=\ & (\lambda + v + b_1) \sum_{n=0}^\infty \int_0^\infty |p_{1,n}(x)| \mathrm{d}x + \lambda \sum_{k=1}^\infty c_k \sum_{n=0}^\infty \int_0^\infty |p_{1,n}(x)| \mathrm{d}x \\
&+ (\lambda + b_2) \sum_{n=0}^\infty \int_0^\infty \int_0^\infty |p_{2,n}(x,y)| \mathrm{d}y \mathrm{d}x \\
&+ \lambda \sum_{k=1}^\infty c_k \sum_{n=0}^\infty \int_0^\infty \int_0^\infty |p_{2,n}(x,y)| \mathrm{d}y \mathrm{d}x \\
=\ & (2\lambda + v + b_1) \sum_{n=0}^\infty \|p_{1,n}\|_{L^1[0,\infty)} + (2\lambda + b_2) \sum_{n=0}^\infty \|p_{2,n}\|_{L^1([0,\infty)\times[0,\infty))} \\
\le\ & \max\{2\lambda + v + b_1, 2\lambda + b_2\} \|(p_0, p_1, p_2)\|,
\end{aligned}
$$

$$
\|E(p_0,p_1,p_2)\| \le \sum_{n=0}^\infty \int_0^\infty |p_{1,n}(x)| b_1(x) \mathrm{d}x + \sum_{n=0}^\infty \int_0^\infty \left| \int_0^\infty p_{2,n}(x,y) b_2(y) \mathrm{d}y \right| dx
$$

$$\leq b_1 \sum_{n=0}^{\infty} \|p_{1,n}\|_{L^1[0,\infty)} + b_2 \sum_{n=0}^{\infty} \|p_{2,n}\|_{L^1([0,\infty)\times[0,\infty))}$$
$$\leq \max\{b_1, b_2\}\|(p_0, p_1, p_2)\|.$$

The above two formulas show that U and E are bounded operators. It is easy to see that U and E are linear operators. Hence, from the perturbation theorem of a C_0-semigroup (see Gupur et al [5]) we know that $A + U + E$ generates a C_0- semigroup $T(t)$.

Fourthly, by using the idea of Gupur et al [5] or Gupur [4] we will prove that $A + U + E$ is a dispersive operator. For $(p_0, p_1, p_2) \in D(A)$, we choose

$$\phi_0 = \left(\frac{[p_{0,0}]^+}{p_{0,0}}, \frac{[p_{0,1}]^+}{p_{0,1}}, \frac{[p_{0,2}]^+}{p_{0,2}}, \cdots\right),$$
$$\phi_1(x) = \left(\frac{[p_{1,0}(x)]^+}{p_{1,0}(x)}, \frac{[p_{1,1}(x)]^+}{p_{1,1}(x)}, \frac{[p_{1,2}(x)]^+}{p_{1,2}(x)}, \cdots\right),$$
$$\phi_2(x,y) = \left(\frac{[p_{2,0}(x,y)]^+}{p_{2,0}(x,y)}, \frac{[p_{2,1}(x,y)]^+}{p_{2,1}(x,y)}, \frac{[p_{2,2}(x,y)]^+}{p_{2,2}(x,y)}, \cdots\right),$$

where

$$[p_{0,n}]^+ = \begin{cases} p_{0,n} & \text{if} \quad p_{0,n} > 0 \\ 0 & \text{if} \quad p_{0,n} \leq 0 \end{cases}, \quad n \geq 0,$$
$$[p_{1,n}(x)]^+ = \begin{cases} p_{1,n}(x) & \text{if} \quad p_{1,n}(x) > 0 \\ 0 & \text{if} \quad p_{1,n}(x) \leq 0 \end{cases}, \quad n \geq 0,$$
$$[p_{2,n}(x,y)]^+ = \begin{cases} p_{2,n}(x,y) & \text{if} \quad p_{2,n}(x,y) > 0 \\ 0 & \text{if} \quad p_{2,n}(x,y) \leq 0 \end{cases}, \quad n \geq 0.$$

The boundary conditions on $(p_0, p_1, p_2) \in D(A)$ imply

$$\sum_{n=0}^{\infty}[p_{1,n}(0)]^+ \leq \lambda \sum_{n=0}^{\infty}\sum_{k=1}^{n+1} c_k[p_{0,n+1-k}]^+ + \alpha \sum_{n=0}^{\infty}[p_{0,n+1}]^+, \tag{2.21}$$
$$\sum_{n=0}^{\infty}[p_{2,n}(x,0)]^+ \leq v \sum_{n=0}^{\infty}[p_{1,n}(x)]^+. \tag{2.22}$$

If we define $U_{1,n} = \{x \in [0,\infty) \mid p_{1,n}(x) > 0\}$ and $W_{1,n} = \{x \in [0,\infty) \mid p_{1,n}(x) \leq 0\}$ for $n \geq 0$, then we have

$$\int_0^{\infty} \frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x} \frac{[p_{1,n}(x)]^+}{p_{1,n}(x)} \mathrm{d}x$$
$$= \int_{U_{1,n}} \frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x} \frac{[p_{1,n}(x)]^+}{p_{1,n}(x)} \mathrm{d}x + \int_{W_{1,n}} \frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x} \frac{[p_{1,n}(x)]^+}{p_{1,n}(x)} \mathrm{d}x$$

$$= \int_{U_{1,n}} \frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x} \frac{[p_{1,n}(x)]^+}{p_{1,n}(x)} \mathrm{d}x = \int_{U_{1,n}} \frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x} \mathrm{d}x$$

$$= \int_0^\infty \frac{\mathrm{d}[p_{1,n}(x)]^+}{\mathrm{d}x} \mathrm{d}x = -[p_{1,n}(0)]^+, \quad n \geq 0. \tag{2.23}$$

Similarly, if we define $U_{2,n} = \{(x, y) \in [0, \infty) \times [0, \infty) \mid p_{2,n}(x, y) > 0\}$ and $W_{2,n} = \{(x, y) \in [0, \infty) \times [0, \infty) \mid p_{2,n}(x, y) \leq 0\}$ for $n \geq 0$, then

$$\int_0^\infty \int_0^\infty \frac{\partial p_{2,n}(x,y)}{\partial y} \frac{[p_{2,n}(x,y)]^+}{p_{2,n}(x,y)} \mathrm{d}y\mathrm{d}x$$

$$= \int_0^\infty \int_{U_{2,n}} \frac{\partial p_{2,n}(x,y)}{\partial y} \frac{[p_{2,n}(x,y)]^+}{p_{2,n}(x,y)} \mathrm{d}y\mathrm{d}x$$

$$+ \int_0^\infty \int_{W_{2,n}} \frac{\partial p_{2,n}(x,y)}{\partial y} \frac{[p_{2,n}(x,y)]^+}{p_{2,n}(x,y)} \mathrm{d}y\mathrm{d}x$$

$$= \int_0^\infty \int_{U_{2,n}} \frac{\partial p_{2,n}(x,y)}{\partial y} \frac{[p_{2,n}(x,y)]^+}{p_{2,n}(x,y)} \mathrm{d}y\mathrm{d}x = \int_0^\infty \int_{U_{2,n}} \frac{\partial p_{2,n}(x,y)}{\partial y} \mathrm{d}y\mathrm{d}x$$

$$= \int_0^\infty \int_0^\infty \frac{\partial [p_{2,n}(x,y)]^+}{\partial y} \mathrm{d}y\mathrm{d}x = -\int_0^\infty [p_{2,n}(x,0)]^+ \mathrm{d}x, \ n \geq 0. \tag{2.24}$$

By using (2.21)–(2.24), the Fubini Theorem and $\sum\limits_{k=1}^{\infty} c_k = 1$, for $(p_0, p_1, p_2) \in D(A)$ and (ϕ_0, ϕ_1, ϕ_2) we have

$$\langle (A + U + E)(p_0, p_1, p_2), (\phi_0, \phi_1, \phi_2) \rangle$$

$$= \frac{[p_{0,0}]^+}{p_{0,0}} \left\{ -\lambda p_{0,0} + \int_0^\infty p_{1,0}(x) b_1(x) \mathrm{d}x \right\}$$

$$+ \sum_{n=1}^\infty \frac{[p_{0,n}]^+}{p_{0,n}} \left\{ -(\lambda + \alpha) p_{0,n} + \int_0^\infty p_{1,n}(x) b_1(x) \mathrm{d}x \right\}$$

$$+ \int_0^\infty \left\{ -\frac{\mathrm{d}p_{1,0}(x)}{\mathrm{d}x} - (\lambda + v + b_1(x)) p_{1,0}(x) \right.$$

$$\left. + \int_0^\infty p_{2,0}(x,y) b_2(y) \mathrm{d}y \right\} \frac{[p_{1,0}(x)]^+}{p_{1,0}(x)} \mathrm{d}x$$

$$+ \sum_{n=1}^\infty \int_0^\infty \left\{ -\frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x} - (\lambda + v + b_1(x)) p_{1,n}(x) \right.$$

$$\left. + \lambda \sum_{k=1}^n c_k p_{1,n-k}(x) + \int_0^\infty p_{2,n}(x,y) b_2(y) \mathrm{d}y \right\} \frac{[p_{1,n}(x)]^+}{p_{1,n}(x)} \mathrm{d}x$$

$$+ \int_0^\infty \int_0^\infty \left\{ -\frac{\partial p_{2,0}(x,y)}{\partial y} - (\lambda + b_2(y)) p_{2,0}(x,y) \right\} \frac{[p_{2,0}(x,y)]^+}{p_{2,0}(x,y)} \mathrm{d}y\mathrm{d}x$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}\int_0^{\infty}\int_0^{\infty}\Big\{-\frac{\partial p_{2,n}(x,y)}{\partial y}-(\lambda+b_2(y))p_{2,n}(x,y)\\
&+\lambda\sum_{k=1}^{n}c_k p_{2,n-k}(x,y)\Big\}\frac{[p_{2,n}(x,y)]^+}{p_{2,n}(x,y)}\mathrm{d}y\mathrm{d}x\\
=&-\lambda[p_{0,0}]^+-(\lambda+\alpha)\sum_{n=1}^{\infty}[p_{0,n}]^+\\
&+\sum_{n=0}^{\infty}\frac{[p_{0,n}]^+}{p_{0,n}}\int_0^{\infty}p_{1,n}(x)b_1(x)\mathrm{d}x-\sum_{n=0}^{\infty}\int_0^{\infty}\frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x}\frac{[p_{1,n}(x)]^+}{p_{1,n}(x)}\mathrm{d}x\\
&-\sum_{n=0}^{\infty}\int_0^{\infty}[\lambda+v+b_1(x)][p_{1,n}(x)]^+\mathrm{d}x\\
&+\lambda\sum_{n=1}^{\infty}\sum_{k=1}^{n}c_k\int_0^{\infty}p_{1,n-k}(x)\frac{[p_{1,n}(x)]^+}{p_{1,n}(x)}\mathrm{d}x\\
&+\sum_{n=0}^{\infty}\int_0^{\infty}\left(\int_0^{\infty}p_{2,n}(x,y)b_2(y)\mathrm{d}y\right)\frac{[p_{1,n}(x)]^+}{p_{1,n}(x)}\mathrm{d}x\\
&-\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{\partial p_{2,n}(x,y)}{\partial y}\frac{[p_{2,n}(x,y)]^+}{p_{2,n}(x,y)}\mathrm{d}y\mathrm{d}x\\
&-\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}(\lambda+b_2(y))[p_{2,n}(x,y)]^+\mathrm{d}y\mathrm{d}x\\
&+\lambda\sum_{n=1}^{\infty}\sum_{k=1}^{n}c_k\int_0^{\infty}\int_0^{\infty}p_{2,n-k}(x,y)\frac{[p_{2,n}(x,y)]^+}{p_{2,n}(x,y)}\mathrm{d}y\mathrm{d}x\\
=&-\lambda[p_{0,0}]^+-(\lambda+\alpha)\sum_{n=1}^{\infty}[p_{0,n}]^+\\
&+\sum_{n=0}^{\infty}\frac{[p_{0,n}]^+}{p_{0,n}}\int_0^{\infty}p_{1,n}(x)b_1(x)\mathrm{d}x+\sum_{n=0}^{\infty}[p_{1,n}(0)]^+\\
&-\sum_{n=0}^{\infty}\int_0^{\infty}[\lambda+v+b_1(x)][p_{1,n}(x)]^+\mathrm{d}x\\
&+\lambda\sum_{k=1}^{\infty}c_k\sum_{n=k}^{\infty}\int_0^{\infty}p_{1,n-k}(x)\frac{[p_{1,n}(x)]^+}{p_{1,n}(x)}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=0}^{\infty}\int_0^{\infty}\left(\int_0^{\infty}p_{2,n}(x,y)b_2(y)\mathrm{d}y\right)\frac{[p_{1,n}(x)]^+}{p_{1,n}(x)}\mathrm{d}x\\
&+\sum_{n=0}^{\infty}\int_0^{\infty}[p_{2,n}(x,0)]^+\mathrm{d}x-\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}(\lambda+b_2(y))[p_{2,n}(x,y)]^+\mathrm{d}y\mathrm{d}x\\
&+\lambda\sum_{k=1}^{\infty}c_k\sum_{n=k}^{\infty}\int_0^{\infty}p_{2,n-k}(x,y)\frac{[p_{2,n}(x,y)]^+}{p_{2,n}(x,y)}\mathrm{d}y\mathrm{d}x\\
\le&-\lambda[p_{0,0}]^+-(\lambda+\alpha)\sum_{n=1}^{\infty}[p_{0,n}]^++\sum_{n=0}^{\infty}\frac{[p_{0,n}]^+}{p_{0,n}}\int_0^{\infty}[p_{1,n}(x)]^+b_1(x)\mathrm{d}x\\
&+\lambda\sum_{n=0}^{\infty}\sum_{k=1}^{n+1}c_k[p_{0,n+1-k}]^++\alpha\sum_{n=0}^{\infty}[p_{0,n+1}]^+\\
&-\sum_{n=0}^{\infty}\int_0^{\infty}(\lambda+v+b_1(x))[p_{1,n}(x)]^+\mathrm{d}x\\
&+\lambda\sum_{k=1}^{\infty}c_k\sum_{n=k}^{\infty}\int_0^{\infty}[p_{1,n-k}(x)]^+\mathrm{d}x\\
&+\sum_{n=0}^{\infty}\int_0^{\infty}\left(\int_0^{\infty}[p_{2,n}(x,y)]^+b_2(y)\mathrm{d}y\right)\frac{[p_{1,n}(x)]^+}{p_{1,n}(x)}\mathrm{d}x\\
&+v\sum_{n=0}^{\infty}\int_0^{\infty}[p_{1,n}(x)]^+\mathrm{d}x-\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}(\lambda+b_2(y))[p_{2,n}(x,y)]^+\mathrm{d}y\mathrm{d}x\\
&+\lambda\sum_{k=1}^{\infty}c_k\sum_{n=k}^{\infty}\int_0^{\infty}\int_0^{\infty}[p_{2,n-k}(x,y)]^+\mathrm{d}y\mathrm{d}x\\
\le&-\lambda\sum_{n=0}^{\infty}[p_{0,n}]^+-\alpha\sum_{n=1}^{\infty}[p_{0,n}]^+\\
&+\sum_{n=0}^{\infty}\left(\frac{[p_{0,n}]^+}{p_{0,n}}-1\right)\int_0^{\infty}[p_{1,n}(x)]^+b_1(x)dx\\
&+\lambda\sum_{k=1}^{\infty}c_k\sum_{n=k-1}^{\infty}[p_{0,n+1-k}]^++\alpha\sum_{n=0}^{\infty}[p_{0,n+1}]^+\\
&-\lambda\sum_{n=0}^{\infty}\int_0^{\infty}[p_{1,n}(x)]^+dx+\lambda\sum_{k=1}^{\infty}c_k\sum_{n=0}^{\infty}\int_0^{\infty}[p_{1,n}(x)]^+dx
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}[p_{2,n}(x,y)]^+ b_2(y)dydx \\
&-\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}[p_{2,n}(x,y)]^+ (\lambda+b_2(y))dydx \\
&+\lambda\sum_{k=1}^{\infty}c_k\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}[p_{2,n}(x,y)]^+ dydx \\
=&-\lambda\sum_{n=0}^{\infty}[p_{0,n}]^+ \\
&+\sum_{n=0}^{\infty}\left(\frac{[p_{0,n}]^+}{p_{0,n}}-1\right)\int_0^{\infty}[p_{1,n}(x)]^+ b_1(x)dx \\
&+\lambda\sum_{k=1}^{\infty}c_k\sum_{n=0}^{\infty}[p_{0,n}]^+ \\
&-\lambda\sum_{n=0}^{\infty}\int_0^{\infty}[p_{1,n}(x)]^+dx+\lambda\sum_{n=0}^{\infty}\int_0^{\infty}[p_{1,n}(x)]^+dx \\
&-\lambda\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}[p_{2,n}(x,y)]^+ dydx \\
&+\lambda\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}[p_{2,n}(x,y)]^+ dydx \\
=&\sum_{n=0}^{\infty}\left(\frac{[p_{0,n}]^+}{p_{0,n}}-1\right)\int_0^{\infty}b_1(x)[p_{1,n}(x)]^+\mathrm{d}x \\
\le&\ 0. \qquad (2.25)
\end{aligned}
$$

From this together with the definition of dispersive operator we know that $A+U+E$ is a dispersive operator. Therefore, from the first step, second step and the Phillips Theorem we know that $A+U+E$ generates a positive contraction C_0-semigroup (see Gupur et al [5]). From the uniqueness theorem of a C_0-semigroup (see Gupur et al [5]) it follows that this semigroup is just $T(t)$. □

It is not difficult to verify that X^*, the dual space of X, is as follows

$$
X^*=\left\{(q_0^*,q_1^*,q_2^*)\ \middle|\ \begin{array}{l} q_0^*\in l^{\infty},\ q_1^*\in Y_1^*,\ q_2^*\in Y_2^*, \\ (q_0^*,q_1^*,q_2^*)=\sup\{\|q_0\|_{l^{\infty}},\|q_1\|_{Y_1^*},\|q_2\|_{Y_2^*}\} \end{array}\right\},
$$

where

$$Y_1^* = \left\{ q_1^* \;\middle|\; \begin{array}{l} q_1^* \in L^\infty[0,\infty) \times L^\infty[0,\infty) \times \cdots, \\ \|q_1^*\| = \sup\limits_{n\geq 0} \|q_{1,n}^*\|_{L^1[0,\infty)} < \infty \end{array} \right\},$$

$$Y_2^* = \left\{ q_2^* \;\middle|\; \begin{array}{l} q_2^* \in L^\infty([0,\infty)\times[0,\infty)) \times L^\infty([0,\infty)\times[0,\infty)) \times \cdots, \\ \|q_2^*\| = \sup\limits_{n\geq 0} \|q_{2,n}\|_{L^\infty([0,\infty)\times[0,\infty))} < \infty \end{array} \right\}.$$

It is easy to check that X^* is a Banach space. If we take a set S in X as

$$S = \left\{ (p_0, p_1, p_2) \in X \;\middle|\; \begin{array}{l} p_{0,n} \geq 0,\ n \geq 0 \\ p_{1,n}(x) \geq 0,\ n \geq 0,\ x \in [0,\infty) \\ p_{2,n}(x,y) \geq 0,\ n \geq 0,\ (x,y) \in [0,\infty)\times[0,\infty) \end{array} \right\},$$

then S is a cone in X. For $(p_0, p_1, p_2) \in D(A) \cap S$ we take

$$(q_0^*, q_1^*, q_2^*) = \|(p_0, p_1, p_2)\| \left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} \right) \in X^*,$$

then we have

$$\begin{aligned} &\langle (p_0, p_1, p_2), (q_0^*, q_1^*, q_2^*) \rangle \\ &= \|(p_0, p_1, p_2)\| \sum_{n=0}^{\infty} p_{0,n} + \|(p_0, p_1, p_2)\| \sum_{n=0}^{\infty} \int_0^\infty p_{1,n}(x)\mathrm{d}x \\ &\quad + \|(p_0, p_1, p_2)\| \sum_{n=0}^{\infty} \int_0^\infty \int_0^\infty p_{2,n}(x,y)\mathrm{d}y\mathrm{d}x \\ &= \|(p_0, p_1, p_2)\|^2 = (q_0^*, q_1^*, q_2^*)^2, \end{aligned}$$

that is, $(q_0^*, q_1^*, q_2^*) \in \theta((p_0, p_1, p_2))$, where

$$\theta((p_0, p_1, p_2)) = \left\{ (q_0^*, q_1^*, q_2^*) \in X^* \;\middle|\; \begin{array}{l} \langle (p_0, p_1, p_2), (q_0^*, q_1^*, q_2^*) \rangle \\ = \|(p_0, p_1, p_2)\|^2 \\ = (q_0^*, q_1^*, q_2^*)^2 \end{array} \right\}.$$

For such (q_0^*, q_1^*, q_2^*), by using boundary conditions on $(p_0, p_1, p_2) \in D(A) \cap S$ and $\sum\limits_{k=1}^{\infty} c_k = 1$ we have

$$\begin{aligned} &\langle (A+U+E)(p_0, p_1, p_2), (q_0^*, q_1^*, q_2^*) \rangle \\ &= \|(p_0, p_1, p_2)\| \left\{ -\lambda p_{0,0} + \int_0^\infty p_{1,0}(x) b_1(x)\mathrm{d}x \right\} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}\|(p_0,p_1,p_2)\|\left\{-(\lambda+\alpha)p_{0,n}+\int_0^{\infty}p_{1,n}(x)b_1(x)\mathrm{d}x\right\}\\
&+\int_0^{\infty}\|(p_0,p_1,p_2)\|\left\{-\frac{\mathrm{d}p_{1,0}(x)}{\mathrm{d}x}-(\lambda+v+b_1(x))p_{1,0}(x)\right.\\
&\left.+\int_0^{\infty}p_{2,0}(x,y)b_2(y)\mathrm{d}y\right\}\mathrm{d}x\\
&+\sum_{n=1}^{\infty}\int_0^{\infty}\|(p_0,p_1,p_2)\|\left\{-\frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x}-(\lambda+v+b_1(x))p_{1,n}(x)\right.\\
&\left.+\lambda\sum_{k=1}^{n}c_kp_{1,n-k}(x)+\int_0^{\infty}p_{2,n}(x,y)b_2(y)\mathrm{d}y\right\}\mathrm{d}x\\
&+\int_0^{\infty}\int_0^{\infty}\|(p_0,p_1,p_2)\|\left\{-\frac{\mathrm{d}p_{2,0}(x)}{\mathrm{d}x}-(\lambda+b_1(y))p_{2,0}(x,y)\right\}\mathrm{d}y\mathrm{d}x\\
&+\sum_{n=1}^{\infty}\int_0^{\infty}\int_0^{\infty}\|(p_0,p_1,p_2)\|\left\{-\frac{\partial p_{2,n}(x,y)}{\partial y}-(\lambda+b_2(y))p_{2,n}(x,y)\right.\\
&\left.+\lambda\sum_{k=1}^{n}c_kp_{2,n-k}(x,y)\right\}\mathrm{d}y\mathrm{d}x\\
=&\|(p_0,p_1,p_2)\|\left\{-\lambda p_{0,0}+\int_0^{\infty}p_{1,0}(x)b_1(x)\mathrm{d}x\right\}\\
&+\|(p_0,p_1,p_2)\|\sum_{n=1}^{\infty}\left\{-(\lambda+\alpha)p_{0,n}+\int_0^{\infty}p_{1,n}(x)b_1(x)\mathrm{d}x\right\}\\
&+\|(p_0,p_1,p_2)\|\sum_{n=0}^{\infty}\int_0^{\infty}\left\{-\frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x}-(\lambda+v+b_1(x))p_{1,n}(x)\right.\\
&\left.+\int_0^{\infty}p_{2,n}(x,y)b_2(y)\mathrm{d}y\right\}\mathrm{d}x\\
&+\|(p_0,p_1,p_2)\|\sum_{n=1}^{\infty}\int_0^{\infty}\lambda\sum_{k=1}^{n}c_kp_{1,n-k}(x)\mathrm{d}x\\
&+\|(p_0,p_1,p_2)\|\\
&\times\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\left\{-\frac{\partial p_{2,n}(x,y)}{\partial y}-(\lambda+b_2(y))p_{2,n}(x,y)\right\}\mathrm{d}y\mathrm{d}x\\
&+\|(p_0,p_1,p_2)\|\sum_{n=1}^{\infty}\int_0^{\infty}\int_0^{\infty}\lambda\sum_{k=1}^{n}c_kp_{2,n-k}(x,y)\mathrm{d}y\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \|(p_0,p_1,p_2)\| \sum_{n=0}^{\infty} \left\{ -\lambda p_{0,n} + \int_0^\infty p_{1,n}(x) b_1(x) \mathrm{d}x \right\} \\
&\quad - \|(p_0,p_1,p_2)\| \alpha \sum_{n=1}^{\infty} p_{0,n} - \|(p_0,p_1,p_2)\| \sum_{n=0}^{\infty} \int_0^\infty \frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x} \mathrm{d}x \\
&\quad - \|(p_0,p_1,p_2)\| \sum_{n=0}^{\infty} \int_0^\infty (\lambda + v + b_1(x)) p_{1,n}(x) \mathrm{d}x \\
&\quad + \|(p_0,p_1,p_2)\| \sum_{n=0}^{\infty} \int_0^\infty \int_0^\infty p_{2,n}(x,y) b_2(y) \mathrm{d}y \mathrm{d}x \\
&\quad + \lambda \|(p_0,p_1,p_2)\| \sum_{k=1}^{\infty} c_k \sum_{n=0}^{\infty} \int_0^\infty p_{1,n}(x) \mathrm{d}x \\
&\quad - \|(p_0,p_1,p_2)\| \sum_{n=0}^{\infty} \int_0^\infty \int_0^\infty \frac{\partial p_{2,n}(x,y)}{\partial y} \mathrm{d}y \mathrm{d}x \\
&\quad - \|(p_0,p_1,p_2)\| \sum_{n=0}^{\infty} \int_0^\infty \int_0^\infty (\lambda + b_2(y)) p_{2,n}(x,y) \mathrm{d}y \mathrm{d}x \\
&\quad + \lambda \|(p_0,p_1,p_2)\| \sum_{k=1}^{\infty} c_k \sum_{n=0}^{\infty} \int_0^\infty \int_0^\infty p_{2,n}(x,y) \mathrm{d}y \mathrm{d}x \\
&= -\lambda \|(p_0,p_1,p_2)\| \sum_{n=0}^{\infty} p_{0,n} - \|(p_0,p_1,p_2)\| \alpha \sum_{n=1}^{\infty} p_{0,n} \\
&\quad + \|(p_0,p_1,p_2)\| \sum_{n=0}^{\infty} p_{1,n}(0) \\
&\quad - \|(p_0,p_1,p_2)\| \sum_{n=0}^{\infty} \int_0^\infty (\lambda + v) p_{1,n}(x) \mathrm{d}x \\
&\quad + \lambda \|(p_0,p_1,p_2)\| \sum_{n=0}^{\infty} \int_0^\infty p_{1,n}(x) \mathrm{d}x \\
&\quad + \|(p_0,p_1,p_2)\| \sum_{n=0}^{\infty} \int_0^\infty p_{2,n}(x,0) \mathrm{d}x \\
&\quad - \lambda \|(p_0,p_1,p_2)\| \sum_{n=0}^{\infty} \int_0^\infty \int_0^\infty p_{2,n}(x,y) \mathrm{d}y \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda\|(p_0,p_1,p_2)\|\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}p_{2,n}(x,y)\mathrm{d}y\mathrm{d}x\\
=&-\lambda\|(p_0,p_1,p_2)\|\sum_{n=0}^{\infty}p_{0,n}-\|(p_0,p_1,p_2)\|\alpha\sum_{n=1}^{\infty}p_{0,n}\\
&+\|(p_0,p_1,p_2)\|\sum_{n=0}^{\infty}\left\{\lambda\sum_{k=1}^{n+1}c_kp_{0,n+1-k}+\alpha p_{0,n+1}\right\}\\
&-\|(p_0,p_1,p_2)\|\sum_{n=0}^{\infty}\int_0^{\infty}vp_{1,n}(x)dx+\|(p_0,p_1,p_2)\|\sum_{n=0}^{\infty}\int_0^{\infty}vp_{1,n}(x)dx\\
=&-\lambda\|(p_0,p_1,p_2)\|\sum_{n=0}^{\infty}p_{0,n}-\|(p_0,p_1,p_2)\|\alpha\sum_{n=1}^{\infty}p_{0,n}\\
&+\lambda\|(p_0,p_1,p_2)\|\sum_{k=1}^{\infty}c_k\sum_{n=0}^{\infty}p_{0,n}+\|(p_0,p_1,p_2)\|\alpha\sum_{n=1}^{\infty}p_{0,n}\\
=&\ 0.
\end{aligned}
$$

Which shows that $A+U+E$ is a conservative operator. Since the initial value $(p_0,p_1,p_2)(0)\in D(A^2)\cap S$, by using the Fattorini Theorem, see [3], we obtain the following result.

Theorem 2.2. *$T(t)$ is isometric for the initial value of the system (1.11), that is,*

$$\|T(t)(p_0,p_1,p_2)(0)\|=\|(p_0,p_1,p_2)(0)\|,\quad \forall t\in[0,\infty). \tag{2.26}$$

From Theorem 2.1 and Theorem 2.2 we obtain the desired result in this paper.

Theorem 2.3. *If $b_1=\sup_{x\in[0,\infty)}b_1(x)<\infty$ and $b_2=\sup_{y\in[0,\infty)}b_2(y)<\infty$, then the system (1.11) has a unique nonnegative time-dependent solution $(p_0(t),p_1(x,t),p_2(x,y,t))$ which satisfies*

$$\|(p_0(t),p_1(\cdot,t),p_2(\cdot,\cdot,t))\|=1,\quad \forall t\in[0,\infty).$$

Proof. Since $(p_0,p_1,p_2)(0)\in D(A^2)\cap S$, by Theorem 2.1 and Gupur et al [5] we know that the system (1.11) has a unique nonnegative time- dependent solution $(p_0(t),p_1(x,t),p_2(x,y,t))$ which can be expressed as

$$(p_0(t),p_1(x,t),p_2(x,y,t))=T(t)(p_0,p_1,p_2)(0),\quad t\in[0,\infty). \tag{2.27}$$

From which together with Theorem 2.2 (i.e., (2.26)) we have

$$\|(p_0(t),p_1(\cdot,t),p_2(\cdot,\cdot,t))\|=\|T(t)(p_0,p_1,p_2)(0)\|$$

$$= \|(p_0, p_1, p_2)(0)\| = 1, \quad \forall t \in [0, \infty). \tag{2.28}$$

□

(2.28) just reflects the physical background of the problem.

If we can know the spectrum of $A + U + E$ on the imaginary axis, then by Theorem 2.1 and Theorem 14 in Gupur et al [5] we obtain the asymptotic behavior of the time-dependent solution of the system (1.11), which describes the above hypothesis. It is our next research work.

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