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# BETA-FUNCTION B-SPLINES: DEFINITION AND BASIC PROPERTIES

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**Abstract:** Beta-function B-splines (BFBS) were introduced by the author in 2006 and announced in [3] as a particular example of smooth generalized expo-rational B-splines (GERBS) which are not true expo-rational B-splines (ERBS). The practical justification of the introduction of BFBS was that they offer a tradeoff between the good geometric-modelling properties of ERBS and the explicitness and simplicity of computation of other GERBS which are less smooth than ERBS.

The objectives of the present paper are:

- To provide rigorous definition of BFBS.
- To study the basic properties of BFBS and to explore the analogy of these properties to respective 'superproperties' of ERBS. The organization of this study is, therefore, similar to the study of the basic properties of ERBS in [4].
- To provide appropriate notation and numbering in relevance to the definition and properties of BFBS which can be used for fast, yet sufficiently precise, references to the respective definitions and properties in subsequent research relevant to BFBS.

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## 1. Introduction

Expo-rational B-splines (ERBS) were introduced by the author in 2003. Their definition and the results of a study of their properties were announced in 2004 in [1]. A detailed exposition of the results in [1] with graphical visualization of the properties of ERBS was published in [4].

**Definition 1.** (see [4]) Let $t_k \in \mathbb{R}$ and $t_k < t_{k+1}$ for $k = 0, 1, 2, ..., n+1$. Consider the strictly increasing knot-vector $\{t_k\}_{k=0}^{n+1}$. The ERBS $B_k(t)$, $k = 0, \dots, n+1$, associated with this knot-vector are defined, as follows.

$$B_k(t) = \begin{cases} \int\limits_{t_{k-1}}^{t} \varphi_{k-1}(s)ds, & t_{k-1} < t \leq t_k \\ 1 - \int\limits_{t_{k-1}}^{t} \varphi_k(s)ds, & t_k < t < t_{k+1} \\ 0, & \text{otherwise} \end{cases}, \tag{1}$$

with

$$\varphi_k(t) = \frac{e^{-\beta_k \frac{[t-((1-\lambda_k)t_k+\lambda_k t_{k+1})]^{2\sigma_k}}{((t-t_k)(t_{k+1}-t)^{\gamma_k})^{\alpha_k}}}}{\int_{t_k}^{t_{k+1}} e^{-\beta_k \frac{[s-((1-\lambda_k)t_k+\lambda_k t_{k+1})]^{2\sigma_k}}{((s-t_k)(t_{k+1}-s)^{\gamma_k})^{\alpha_k}}} ds}, \tag{2}$$

where

$$\alpha_k > 0, \quad \beta_k > 0, \quad \gamma_k > 0, \quad 0 \leq \lambda_k \leq 1, \quad \sigma_k \geq 0,$$

are the intrinsic parameters, defaulting to: $\alpha_k = \beta_k = \gamma_k = \sigma_k = 1, \lambda_k = \frac{1}{2}$.

**Remark 1.** Note that in this setting of the definition the 'initial' and the 'final' ERBS, corresponding to $k = 0$ and $k = n+1$, respectively, are non-zero over only one interval of monotonicity each: $[t_0, t_1]$ for $B_0$, and $[t_n, t_{n+1}]$ for $B_{n+1}$.

ERBS are $C^\infty$-smooth and exhibit a diversity of 'superproperties' [4] compared to usual polynomial B-splines, but the integrals in their definition can not be solved in elementary functions, and, to compute ERBS values, fast-converging Romberg quadrature processes have been proposed in [4]. Although these quadrature processes have proven to be quite efficient in view of the $C^\infty$-smoothness of the integrand in (2), it was of considerable interest to try to find modifications of ERBS for which the density $\varphi_k$ in (2) is simpler and/or more efficient to be integrated; in particular, it was of interest to find appropriate sufficiently smooth $\varphi_k$ in (2) which can be integrated in terms of elementary functions. This was one among several motivations to conduct research on appropriate generalizations of ERBS which are simpler to compute, and yet retain essential part of the 'superproperties' of ERBS. The results of this research were first announced by the author in [2] and later published in [3]. One of the most important questions which needed to be answered in this study was to determine the maximal meaningful class of generalized ERBS (GERBS) which retain the most basic properties of ERBS. (For example, this large class should contain both the $C^\infty$-smooth ERBS and the $C^0$-smooth (continuous) piecewise affine B-splines generated by the given knot-vector.) In [3], the answer to this question was given by Definition 3, as follows.

**Definition 2.** (see [3]) Consider the system $\{F_i\}_{i=1}^{n+1}$ of *cumulative distribution functions* (see, e.g., [5, 6]) (CDF, for short) such that $F_i$ is supported on the interval span $[t_{i-1}, t_i]$, i.e.,

1. the left-hand limit $F(t_{i-1}+) = F(t_{i-1}) = 0$,
2. the left-hand limit $F(t_i+) = F(t_i) = 1$,
3. $F(t) = 0$ for $t \in (-\infty, t_{i-1}]$,
4. $F(t) = 1$ for $t \in [t_i, +\infty)$, and $F(t)$ is monotonously increasing, possibly discontinuous, but left-continuous for $t \in [t_{i-1}, t_i]$.
5. The $j$-th (g)eneralized (e)xpo-(r)ational (B)-(s)pline (GERBS), is defined, as follows.

$$G_j(t) = \begin{cases} F_j(t), & \text{if} \quad t \in (t_{j-1}, t_j], \\ 1 - F_{j+1}(t), & \text{if} \quad t \in (t_j, t_{j+1}), \\ 0, & \text{if} \quad t \in (-\infty, t_{j-1}] \cup [t_{j+1}, +\infty), \end{cases} \tag{3}$$

$j = 1, \dots, n$.

In [3] it was also noted that Definition 2 admits a more concise and intuitive (yet rigorous) reformulation, as follows.

**Definition 3.** (Equivalent to Definition 2, see [3, end of Section 3]). GERBS is any piecewise monotone reparametrization of a piecewise affine B-spline which preserves the intervals of monotonicity of the latter, as well as the range of the latter in each of these intervals.

Once the general class of GERBS was identified via the equivalent Definitions 2 and 3, in [3] we proceeded with a study of a hierarchy of increasingly specialized GERBS which were exhibiting an increasing set of useful properties simultaneously with the increase of their computational complexity. This chain of nested classes began with discontinuous and $C^0$-regular (continuous) classes containing the piecewise affine B-splines and, proceeding through classes with intermediate regularity, ended with the class of $C^\infty$-smooth GERBS containing (but not limited to) the ERBS class. More precisely, in [3] we considered the following chain of nested GERBS classes:

- general GERBS (defined via Definitions 2 and 3);
- possibly discontinuous GERBS (containing the classes of step functions (piecewise constant B-splines) and piecewise affine B-splines);
- continuous possibly non-smooth ($C^0$-regular) GERBS (containing the class of piecewise affine B-splines);
- absolutely continuous GERBS (in which case there exist densities $\varphi_k$ in (2), and these densities can be arbitrary non-negative Lebesgue-integrable functions, supported on the respective intervals $[t_k, t_{k+1}]$, $k = 0, \dots, n$;
- $C^m$-smooth GERBS, $m = 0, 1, \dots,$ (the case when the densities $\varphi_k$ in the previous item are $C^{m-1}$-smooth on $\mathbb{R}$): this case includes, but is not limited to, the $C^m$-smooth Beta-function B-splines (BFBS);
- $C^\infty$-smooth GERBS (including, but not limited to, ERBS).

In [3] BFBS were provided only as an example illustrating the construction of GERBS from the $C^m$-smooth class. In correspondence with this purpose, in [3] we only outlined the idea of the definition of BFBS: namely, their definition is the same as the one for ERBS, but the 'expo-rational' densities $\varphi_k$ in (2) are replaced by densities which are normalized Bernstein polynomials, shifted and scaled to the support segment $[t_k, t_{k+1}]$, $k = 0, \dots, n$.

The objectives of the present paper are:

1. To provide rigorous definition(s) of BFBS.

2. To study the basic properties of BFBS and to explore the analogy of these properties to respective 'superproperties' of ERBS. The organization of this study will, therefore, be similar to the study of the basic properties of ERBS in [4].

3. To provide appropriate notation and numbering in relevance to the definition and properties of BFBS which can be used for fast, yet sufficiently precise, references to the respective definitions and properties in subsequent research relevant to BFBS.

The organization of the remaining part of this paper is, as follows. The next Section 2 contains the definitions. Section 3 is dedicated to the basic properties of BFBS. Some concluding remarks can be found in Section 4, together with some visualization of graphs of BFBS and comparative visualization of BFBS and ERBS.

## 2. Definition of Beta-function B-splines (BFBS)

As in Section 1, consider a strictly increasing knot-vector $\{t_k\}_{k=0}^{n+1}$.

**Definition 4.** A Beta-function B-spline (BFBS), associated with three strictly increasing knots $t_{k-1}$, $t_k$ and $t_{k+1}$, $B_k(t) = B_k(i_{k-1}, i_k, i_{k+1}; t)$ is defined by

$$B_k(t) = \begin{cases} S_{k-1} \int\limits_{t_{k-1}}^{t} \psi_{k-1}(s)ds, & \text{if} \quad t \in (t_{k-1}, t_k), \\ S_k \int\limits_{t}^{t_{k+1}} \psi_k(s)ds, & \text{if} \quad t \in (t_k, t_{k+1}), \\ 1, & \text{if} \quad t = t_k, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

with

$$S_k = \left[ \int\limits_{t_k}^{t_{k+1}} \psi_k(t)dt \right]^{-1}, \tag{5}$$

and

$$\psi_k(t) = C_k \frac{(t-t_k)^{i_k}(t_{k+1}-t)^{i_{k+1}}}{(t_{k+1}-t_k)^{i_k+i_{k+1}}}, \quad t \in [t_k, t_{k+1}], \tag{6}$$

where

$$C_k = \binom{i_k + i_{k+1}}{i_k}, \tag{7}$$

and

$$i_l > 0, \quad l = k-1, k, k+1. \tag{8}$$

If we define $\psi_k$ over $(-\infty, +\infty)$, we have a similar definition:

**Definition 5.** A Beta-function B-spline (BFBS), associated with three strictly increasing knots $t_{k-1}$, $t_k$ and $t_{k+1}$, $B_k(t) = B_k(i_{k-1}, i_k, i_{k+1}; t)$ is defined by

$$B_k(t) = \begin{cases} S_{k-1} \displaystyle\int_{-\infty}^{t} \psi_{k-1}(s)ds, & \text{if} \quad t \in (-\infty, t_k), \\ S_k \displaystyle\int_{t}^{+\infty} \psi_k(s)ds, & \text{if} \quad t \in (t_k, +\infty), \end{cases} \tag{9}$$

with

$$S_k = \left[ \int_{-\infty}^{+\infty} \psi_k(t)dt \right]^{-1} = \left[ \int_{t_k}^{t_{k+1}} \psi_k(t)dt \right]^{-1}, \tag{10}$$

and

$$\psi_k(t) = \begin{cases} C_k \frac{(t-t_k)^{i_k}(t_{k+1}-t)^{i_{k+1}}}{(t_{k+1}-t_k)^{i_k+i_{k+1}}}, & \text{if} \quad t \in [t_k, t_{k+1}], \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

where $C_k$ and $i_l > 0$, $l = k-1, k, k+1$, are the same as in (7) and (8).

**Remark 2.** Remark 1 is also valid for Definitions 4 and 5, so $B_0$ and $B_{n+1}$ are associated with only two neighbouring knots each.

**Remark 3.** In principle, it is possible to have fractional values of $i_l$ in Definitions 4 and 5, but, since from practical point of view of geometric modelling and iso-geometric analysis we are interested in easily computable B-splines, in the sequel of this paper and in consequent BFBS-related research we shall be considering only the case $i_l \in \mathbb{N}$, unless a more general range for $i_l$ be explicitly specified. In the case of integer $i_l$'s the BFBS can be computed exactly, since in this case it is a polynomial between the knots. The integrals with variable limits in (9) are closely related with the incomplete Euler Beta-function, hence the name of the new B-spline. The case of integer $i_l$'s is also the case when the incomplete Beta-functions are polynomials; for a broader range of the $i_l$'s, the incomplete Beta-functions are special functions which can be computed via series expansion, by using specific approximate computation methods for other, more general classes of special functions. It is also possible to use numerical quadratures, as was the case with ERBS. When the latter computational approach is used and the $i_l$'s are integer, the numerical quadratures of respective order corresponding to the $i_l$'s are exact.

**Remark 4.** In the case of Definition 5 $\psi_{k-1}$ and $\psi_k$ are defined over $(-\infty, +\infty))$, $B_k(t_k) = 1$, which follows directly by continuity. This is a part of the next lemma.

**Remark 5.** Indeed, as specified in [3], $\psi_k$, $\forall k = 0, 1, \ldots, n$, are Bernstein polynomials for the respective interval $[t_k, t_{k+1}]$, up to a constant factor. Namely,

$$\text{if} \quad t_k = 0 \quad \text{and} \quad t_{k+1} = 1, \quad \psi_k(t) = C_k t^{i_k}(1-t)^{i_{k+1}}, \tag{12}$$

which coincides with the Bernstein polynomial

$$p(i_k, i_{k+1}, i_k + i_{k+1}; t) = \binom{i_k + i_{k+1}}{i_k} t^{i_k}(1-t)^{i_{k+1}} = \binom{i_k + i_{k+1}}{i_{k+1}} t^{i_k}(1-t)^{i_{k+1}} \tag{13}$$

of degree $i_k + i_{k+1}$, up to the factor $\binom{i_k + i_{k+1}}{i_k} = \binom{i_k + i_{k+1}}{i_{k+1}}$. For general interval $[t_k, t_{k+1}]$, $\psi_k$ is obtained by the respective linear change of variable, mapping $[0, 1]$ onto $[t_k, t_{k+1}]$.

**Lemma 1.** *Definitions 4 and 5 are equivalent.*

*Proof.* 'Definition 4 $\Rightarrow$ Definition 5' follows by restricting the definitions of $\psi_{k-1}$ and $\psi_k$ in Definition 5 to their respective definitions in Definition 4, without changing the values of the integrals. The values $B_k(t_k) = 1$ and $B_k(t) \equiv 0$, $t \in (-\infty, t_{k-1}] \bigcup [t_{k+1}, +\infty)$ can be directly computed in Definition 5, and the values correspond to their respective values in Definition 4.

'Definition 5 $\Rightarrow$ Definition 4' follows by extending the definitions of $\psi_{k-1}$ and $\psi_k$ in Definition 4 to their respective definitions in 5, without changing the values of the integrals. The values $B_k(t_k) = 1$ and $B_k(t) \equiv 0$, $t \in (-\infty, t_{k-1}] \bigcup [t_{k+1}, +\infty)$, postulated in Definition 4, are the same as the ones computed in Definition 5. □

In view of the equivalence of definitions 4 and 5, in the future we shall refer to any one of them as Definition 4.

**Lemma 2.** *A Beta-function B-spline, defined according to Definition 4, is continuous in the knots* $t_i$, $i = 0, 1, ..., n+1$.

*Proof.* Let us consider the interval $[t_{k-1}, t_{k+1}]$, and let us investigate first $\psi_{k-1}$ in $[t_{k-1}, t_k]$. We have

$$\begin{aligned} &\lim_{t \to t_{k-1}-} \psi_{k-1}(t) = 0 \quad \text{by Definition 4,} \\ &\psi_{k-1}(t) \quad \text{is a polynomial for} \quad t \in [t_{k-1}, t_k), \\ &\lim_{t \to t_{k-1}+} \psi_{k-1}(t) = \psi_{k-1}(t_{k-1}) = 0. \end{aligned} \tag{14}$$

$\Rightarrow \psi_{k-1}$ is continuous for $t = t_{k-1}$.
In the same way, with corresponding modifications, it can be proved that $\psi_{k-1}(t)$ is continuous for $t = t_k, t_{k+1}$.

$B_k(t)$ is an integral of a continuous function in $[t_{k-1}, t_k)$ and $(t_k, t_{k+1}]$
$\Rightarrow B_k(t)$ is continuously differentiable in these intervals
$\Rightarrow B_k(t)$ is continuous in these intervals
$\Rightarrow B_k(t)$ is continuous at $t_{k-1}$ and $t_{k+1}$.

Let us now see what happens at the knot $t_k$. We have:

$$\lim_{t \to t_k-} B_k(t) = \lim_{t \to t_k+} B_k(t) = B_k(t_k) = 1. \tag{15}$$

$\Rightarrow B_k(t)$ is continuous for $t = t_k$. □

## 3. Basic Properties

**Theorem 1.** *The BFBS* $B_k(t)$, $k = 0, ..., n+1$, *have the following properties.*

**Q1** *(Value and continuity at an internal knot* $t_k$ *and at boundary-of-support knots* $t_{k-1}$ *and* $t_{k+1}$*)*

*(a)* $B_k(t_k) = 1$;

*(b)* $\lim_{t \to t_k-} B_k(t) = \lim_{t \to t_k+} B_k(t) = B_k(t_k) = 1$;

*(c)* $B_k(t_{k-1}) = B_k(t_{k+1}) = 0$;

*(d)* $\lim_{t \to t_{k-1}-} B_k(t) = \lim_{t \to t_{k-1}+} B_k(t) = \lim_{t \to t_{k+1}-} B_k(t) = \lim_{t \to t_{k+1}+} B_k(t) = 0$.

**Q2** *(Non-negativity and range in $(0,1)$)*

$$B_k(t) \begin{cases} > 0, & \text{if} \quad t \in (t_{k-1}, t_{k+1}), \\ = 0, & \text{otherwise.} \end{cases}$$

*$B_k(t) < 1$ on $(t_{k-1}, t_k)$ and $(t_k, t_{k+1})$, and on each of these open intervals $B_k(t)$ takes all values between 0 and 1.*

**Q3** *(Affine partition of unity)*

*The set of basis functions forms an affine partition of unity on $[t_1, t_n]$: $\sum_{k=1}^{n} B_k(t) = 1$, $\forall t \in [t_1, t_n]$ holds.*

**Q4** *(Continuity and zero value of derivatives in the knots)*

*(a)* If $t \in (t_{k-1}, t_k)$, then

- *(a1)* $B_k \in C^\infty(t_{k-1}, t_k)$;
- *(a2)* $\lim_{t \to t_{k-1}+} D^j B_k(t) = 0$, $j = 0, 1, ..., i_{k-1}$;
- *(a3)* $\lim_{t \to t_k-} D^j B_k(t) = 0, \quad j = 1, 2, ..., i_k$;
- *(a4)* $\lim_{t \to t_{k-1}+} D^{i_{k-1}+1} B_k(t) \neq 0$;
- *(a5)* $\lim_{t \to t_k-} D^{i_k+1} B_k(t) \neq 0$.

*(b)* If $t \in (t_k, t_{k+1})$, then

- *(b1)* $B_k \in C^\infty(t_k, t_{k+1})$;
- *(b2)* $\lim_{t \to t_{k+1}-} D^j B_k(t) = 0$, $j = 0, 1, ..., i_{k+1}$;
- *(b3)* $\lim_{t \to t_k+} D^j B_k(t) = 0, \quad j = 1, 2, ..., i_k$;
- *(b4)* $\lim_{t \to t_{k+1}-} D^{i_{k+1}+1} B_k(t) \neq 0$;
- *(b5)* $\lim_{t \to t_k+} D^{i_k+1} B_k(t) \neq 0$.

*(c)* If $t \in (-\infty, t_{k-1})$ or $t \in (t_{k+1}, +\infty)$, then

$$D^j B_k(t) \equiv 0, \qquad j = 0, 1, ..., \text{ and}$$
$$D^j B_k(t_{k-1}-) = D^j B_k(t_{k+1}+) = 0, \quad j = 0, 1, ...$$

*(d) If $t = t_l$, $l = k-1, k, k+1$, then*

*(d1) $D^j B_k(t)$ is continuous in the respective knot, with the following additional defining:*

*((d1).1) $D^j B_k(t_k) = 0, \quad j = 1, 2, ..., i_k$,*

*((d1).2) $D^j B_k(t_{k-1}) = 0$, $j = 0, 1, ..., i_{k-1}$,*

*((d1).3) $D^j B_k(t_{k+1}) = 0$, $j = 0, 1, ..., i_{k+1}$;*

*(d2) $D^{i_l+1} B_k(t)$ is discontinuous (of first kind) at $t = t_l$, $l = k - 1, k, k+1$.*

**Q5** *(Global smoothness of the BFBS in $[t_1, t_n]$)*

*Every BFBS $B_j$, $j = 1, \ldots, n$, is in $C^s[t_1, t_n]$, where $s = \min\limits_{k=1,\ldots,n} i_k$.*

**Remark 6.** The derivatives of $B_k$ have the property $D^j B_k(t_l) = 0$, $l = k-1, k, k+1$, $j = 1, \ldots, i_l$, because $D^j B_k$ is a polynomial. In other cases, e.g., ERBS, the value of $D^j B_k(t_l)$, $l = k-1, k, k+1$, $j = 1, \ldots, i_l$, does not exist *per se* because the ERBS is computed (exactly) by a series which does not converge in the knots. In the case of ERBS, however, it is still true that $D^j B_k(t_l-) = D^j B_k(t_l+) = 0$, $l = k-1, k, k+1$, $j = 1, \ldots, i_l$, and, therefore, an additional defining of the values $D^j B_k(t_l)$ by 0 in the knot $t_l$ is possible (and is assumed to be done by default). With this convention, in item (d1) of **Q4** we give a formulation which is valid for both the case of a BFBS and the case of an ERBS .

*Proof. Proof of* **Q1**. Follows immediately from Lemma 2 and Definition 4.

*Proof of* **Q2**. We investigate $B_k(t)$ for $t \in [t_{k-1}, t_k]$ first.

We have:

$$
\begin{aligned}
&B_k(t_{k-1}) = 0 \quad \text{by } \mathbf{Q1}, \\
&B_k'(t) \equiv S_{k-1}\psi_{k-1}(t) > 0, \quad t \in (t_{k-1}, t_k), \\
&B_k(t_k) = 1 \quad \text{by } \mathbf{Q1}.
\end{aligned} \tag{16}
$$

Therefore, by (16), $B_k$ is strictly increasing on $[t_{k-1}, t_k]$, and

$$0 < B_k(t) < 1, \quad \forall t \in (t_{k-1}, t_k). \tag{17}$$

By Lemma 2, $\psi_{k-1}(t)$ is continuous at $t_{k-1}$ and $t_k$. Since $B_k(t)$ is an integral of $\psi_{k-1}(t)$, it is also continuous on $(t_{k-1}, t_k)$, (see Definition 4), that is, on $[t_{k-1}, t_k]$.

From this and (17), it follows that $B_k$ takes all values between 0 and 1.

The proof of **Q2** for $t \in [t_k, t_{k+1}]$ is analogous.

*Proof of* **Q3**. From Definition 4 it follows that for each $t \in [t_1, t_n]$ only one of the following cases holds:

(a) one B-spline $B_i(t_i) \neq 0$, $i = 1, \ldots, n$;

(b) two neighboring B-splines $B_k(t) \neq 0$ and $B_{k+1}(t) \neq 0$, if $k = 1, \ldots, n-1$, $t \neq t_i$, $i = 1, \ldots, n$.

Consider case (a). For $t = t_k$, $k = 1, \ldots, n-1$ we have:

$$\sum_{j=1}^{n} B_j(t_k) = B_k(t_k) = 1. \tag{18}$$

by Definition 4 or by Lemma 2.
Case *(a)* of **Q3** is proved.

For case (b), take the only $k$ such that $t \in (t_k, t_{k+1})$.
We have:

$$\sum_{j=1}^{n} B_j(t) = B_k(t) + B_{k+1}(t),$$

$$B_k(t) + B_{k+1}(t) = S_k \int_{t}^{t_{k+1}} \psi_k(s)ds + S_k \int_{t_k}^{t} \psi_k(s)ds$$

$$= S_k \int_{t_k}^{t_{k+1}} \psi_k(s)ds, \quad t \in (t_k, t_{k+1}),$$

$$S_k = \left[ \int_{t_k}^{t_{k+1}} \psi_k(s)ds \right]^{-1}.$$

$\Rightarrow B_k(t) + B_{k+1}(t) = 1$, $t \in (t_k, t_{k+1})$.
Case *(b)* of **Q3** is proved.

*Proof of* **Q4**. $B_k$ and its derivatives are polynomials in $(t_{k-1}, t_k)$, therefore, $B_k \in C^\infty(t_{k-1}, t_k) \Rightarrow$ *(a1)*.
To prove *(a2)* and *(a3)* let us denote:

$$(t - t_{k-1})^{i_{k-1}} = \alpha_{k-1}(t), \text{ and } (t_k - t)^{i_k} = \beta_{k-1}(t),$$

so we have:

$$\psi_{k-1}(t) = d_{k-1}\alpha_{k-1}(t)\beta_{k-1}(t),$$

where $d_{k-1}$ is the constant $C_{k-1}$ multiplied with the denominator of $\psi_{k-1}(t)$. Here we will use the Leibniz rule:

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} \cdot g^{(n-k)},$$

or, in equivalent form,

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{n-k} f^{(n-k)} \cdot g^{(k)}.$$

In *(a2)* we consider $t \to t_{k-1}+$, so we have:

$$\psi_{k-1}(t) = d_{k-1} \sum_{l=0}^{\nu} \binom{\nu}{l} \alpha_{k-1}^{(l)}(t)\beta_{k-1}^{(\nu-l)}(t), \quad \nu = 0, 1, \ldots, i_{k-1} - 1.$$

The leading term of the expansion near $t_{k-1}$, is $\alpha_{k-1}^{(\nu)}(t)\beta_{k-1}^{(0)}(t)$, which tends to 0 when $t \to t_{k-1}+$ if $\nu = 0, 1, \ldots, i_{k-1} - 1$.

If $\nu = i_{k-1}$, we have $\alpha_{k-1}^{(\nu)}(t_{k-1}+)\beta_{k-1}^{(0)}(t_{k-1}+) = const \neq 0$.

$$\begin{aligned} B_k^{(\nu+1)}(t)\Big|_{t=t_{k-1}+} &= S_{k-1}d_{k-1}\ \psi_{k-1}^{(\nu)}(t)\Big|_{t=t_{k-1}+} \\ &= S_{k-1}d_{k-1} \left[ \sum_{l=0}^{\nu} \binom{\nu}{l} \alpha_{k-1}^{(\nu)}(t)\beta_{k-1}^{(\nu-l)}(t) \right]\Bigg|_{t=t_{k-1}+} \\ &= S_{k-1}d_{k-1} \left[ \alpha_{k-1}^{(\nu)}(t_{k-1}+)\beta_{k-1}^{(0)}(t_{k-1}+) + \bar{\bar{o}} \left( \alpha_{k-1}^{(\nu)}(t_{k-1}+)\beta_{k-1}^{(0)}(t_{k-1}+) \right) \right]. \end{aligned}$$

So, we see that

$$B_k^{(\nu+1)}(t)\big|_{t=t_{k-1}+} \begin{cases} = 0, & \text{if} \quad \nu = 0, 1, \ldots, i_{k-1} - 1, \\ \neq 0, & \text{if} \quad \nu = i_{k-1}. \end{cases}$$

Here we use the facts that

$$\begin{aligned} &\alpha_{k-1}^{(i_{k-1}-1)}(t_{k-1}+) = \alpha_{k-1}^{(i_{k-1}-2)}(t_{k-1}+) = \cdots = \alpha_{k-1}^{(0)}(t_{k-1}+) = 0, \text{ and} \\ &\alpha_{k-1}^{(i_{k-1})}(t_{k-1}+) \neq 0, \quad \text{while} \quad \beta_{k-1}^{(0)}(t_{k-1}+) \neq 0. \end{aligned}$$

We have proved *(a2)* for $j = 1, 2, \ldots, i_{k-1}$ and *(a4)* for $j = i_{k-1} + 1$. For $j = 0$ the proof of *(a2)* follows from **Q1**.
Cases *(a3)*, *(a5)*, *(b1)*, *(b2)*, *(b3)*, *(b4)* and *(b5)* can be proved analogously.

To prove *(c)*, it is enough to note that $B_k \equiv const = 0$ on $(-\infty, t_{k-1})$ and $(t_{k+1}, +\infty)$ and, therefore, the same is true for all derivatives of $B_k$. Any sequence $B_k(\tau_m)$, $m = 1, 2, \ldots$, $\tau_m \in (-\infty, t_{k-1})$, $\tau_m \to k_{k-1}-$, or $\tau_m \in (t_{k+1}, +\infty)$, $\tau_m \to k_{k-1}+$, will be the sequence $0, 0, \ldots$, therefore, it has a limit ($\lim\limits_{t \to t_{k-1}-}$, respectively, $\lim\limits_{t \to t_{k+1}+}$), which is equal to 0.

To prove *(d1)*, using *(a3)* and *(b3)*, we see that, after defining

$$D^j B_k(t)|_{t=t_l} := 0, \quad j = 1, \ldots, i_l, l = k-1, k, k+1,$$

the derivatives $D^j B_k$ are continuous at $t_l$, $l = k-1, k, k+1$ (see Remark 6). This proves *((d1).1)*, *((d1).2)* and *((d1).3)*, hence, *(d1)*.
To prove *(d2)*, it is enough to observe that the left and the right limits at $t_l$, $l = k-1, k, k+1$ are different (one of them is $= 0$, the other one is $\neq 0$). For $t_k$ the left and the right limits are both $\neq 0$, but in general, these non-zero values are different from each other. This proves *(d2)*, hence, *(d)* and **Q4**.

*Proof of* **Q5**. For any $k = 1, \ldots, n$,

$$B_k \in C^\infty(t_l, t_{l+1}) \quad \text{for all} \quad l = 0, \ldots, n.$$

Moreover at all knots $t_l$, $l = 1, \ldots, n$, we have:

$$D^j B_k(t_l-) = D^j B_k(t_l+) = D^j B_k(t_l) = 0, \quad j = 1, \ldots, s,$$

because $s \leq i_l$ for every $l = 1, \ldots, n$.
Finally, the same is true one-sidedly at $t_0$ and $t_{n+1}$:

$$D^j B_k(t_0+) = D^j B_k(t_0) = 0, \qquad j = 1, \ldots, s,$$
$$D^j B_k(t_{n+1}-) = D^j B_k(t_{n+1}) = 0, \quad j = 1, \ldots, s.$$

□

**Corollary 1.** *If in* **Q2** *instead of the open intervals* $(t_{k-1}, t_k)$ *and* $(t_k, t_{k+1})$*, we take the closed intervals* $[t_{k-1}, t_k]$ *and* $[t_k, t_{k+1}]$*, we have* $0 \leq B_k(t) \leq 1$.

*Proof.* Follows from **Q2** and **Q1**. □

**Corollary 2.** *(Convex partition of unity)*
*The affine combination*

$$\sum_{k=1}^{n} B_k(t) = 1, \forall t \in [t_1, t_n]$$

*is convex.*

*Proof.* Follows from **Q3** and **Q2**. □

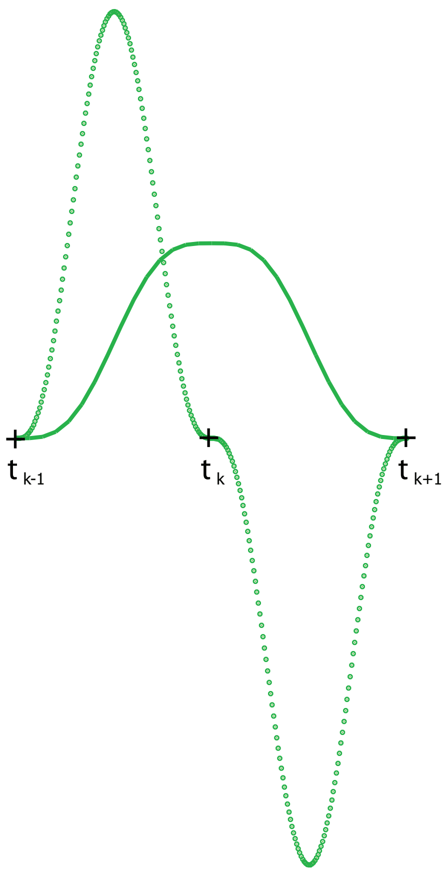


Figure 1: (see also [3, Figure 6]) The graph of a BFBS $G_k(t)$ (solid) and its first derivative (dotted).

## 4. Concluding Remarks

A graph of an Euler Beta-function B-spline basis function $G_k(t)$ (solid) and its first derivative (dotted) is given in Figure 1. The knots $t_{k-1}, t_k$ and $t_{k+1}$ are also marked on the plot.

Some comparative graphical analysis of ERBS and BFBS is provided in Figure 2. It is seen how increasing of the order of smoothness of the BFBS results in increasing the flatness around the knots of the B-spline and its derivative(s). The ERBS is $C^\infty$-smooth and, respectively, much flatter around the knots than any of the BFBS. At the same time, the 'peak' of the ERBS is better localized

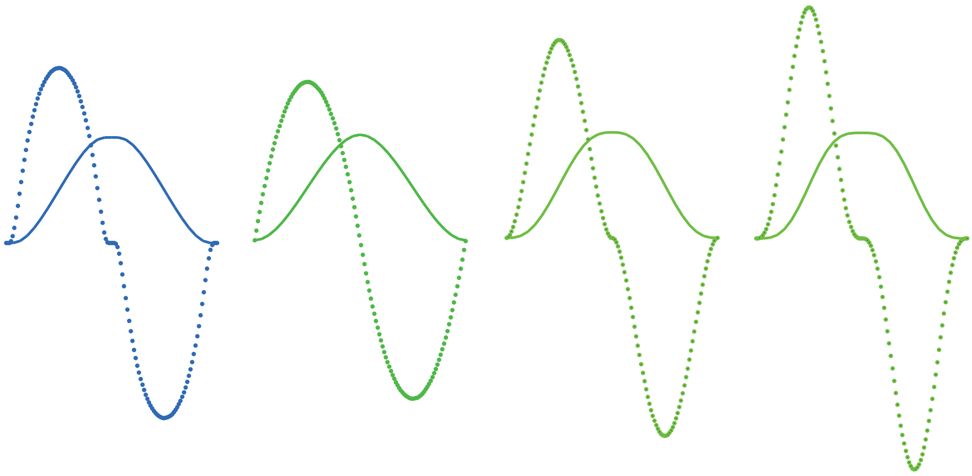

Figure 2: (see also [3, Figure 7]) ERBS (leftmost) and $C^1$-, $C^2$-, $C^3$-smooth BFBS. The BFBS given in Figure 1 is the rightmost in Figure 2.

than the 'peak' of a smooth BFBS (compare the ERBS on the leftmost picture with the BFBS on the rightmost picture) because graphs of exponential functions can 'bend' much faster than graphs of polynomial ones.

**Remark 7.** The formulation of Property Q5 can be extended to hold for $B_j$ for all $j = 0, \ldots, n+1$ rather than only for $j = 1, \ldots, n$, as Property Q5 is currently formulated, if the value $s = \min\limits_{k=1,\ldots,n} i_k$ be replaced by the (generally, smaller) value $s = \min\limits_{k=0,\ldots,n+1} i_k$. Moreover, the above values of $s$ are so small only when considering the uniform smoothness of $\{B_j\}$ as a whole function family; for every individual $B_j$ the value of the smoothness index $s$ increases to $s_j = \min\limits_{k=j-1,j,j+1} i_k$, $j = 1, \ldots, n$, with obvious modifications for $j = 0$ and $j = n+1$.

**Remark 8.** A comparison between the basic properties of BFBS in Section 3 and the basic properties of ERBS [4] shows that:

A. Basic properties related to geometric modelling, such as size of support, non-negativity, intervals of monotonicity, range of values, convex partition of unity, are all essentially the same with ERBS and with BFBS. (See also Remark 9, item 3.)

B. Basic properties related to regularity, such as overall smoothness, smoothness between the knots, smoothness at the knots, values of derivatives at the knots, are also essentially the same but, while for ERBS these properties hold without limitation on the order of the derivatives, with BFBS there are strict upper limits of the order of the derivatives, up to which these basic properties hold. These upper limits to the range of validity of the analogy with ERBS are determined by the parameters $i_l$, $l = 0, \ldots, n+1$. The bigger complexity of notation and branching of cases needed to describe the basic properties of BFBS are related to the need of description of the limits of the range of validity of the basic properties of BFBS analogous to those of ERBS.

**Remark 9.** Many of the basic properties, proved here for BFBS, are shared by the more general classes of GERBS [3] such as, e.g., the class of $C^m$-smooth GERBS, the class of absolutely continuous GERBS, and the general GERBS class. For instance,

(b) Virtually all the basic properties of BFBS proved here are shared by the more general class of $C^m$-smooth GERBS [3].

(b) The integral-based construction in (4) and (9) is shared by the more general class of absolutely continuous GERBS [3].

(b) Virtually all of the properties related to geometric modelling and explicitly mentioned in item A of Remark 8 are shared by the general GERBS class [3].

A next topic on the research on BFBS will be to provide explicit evaluation formulae for BFBS in terms of various polynomial bases.

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