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EVALUATION OF BETA-FUNCTION B-SPLINES, III: GLOBAL MONOMIAL BASES

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Abstract: This is the third and last paper in a sequence of three papers on the evaluation of *Beta-function B-splines* (BFBS), the first and second paper in the sequence being [4] and [5], respectively. This sequence of papers studies explicit representations of BFBS yielding computationally efficient explicit formulae for evaluation of BFBS in terms of polynomial bases used in data interpolation, data fitting and geometric modelling, as well as in the design of multilevel constructions such as, e.g., multiwavelets. While in [4] an *interpolatory representation* of BFBS was developed in terms of *local monomial bases*, and in [5] a *Bezier-type representation* was derived in *local Bernstein bases*, in the present paper a representation of BFBS in *global monomial bases* is obtained, suitable for use, e.g., in relevance to computing Fourier, Laplace and other transforms in operational calculus.

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1. Introduction

Generalized expo-rational B-splines (GERBS) were introduced in [2], [6] as a generalization of ERBS [1], [8] including the polynomial simplified modifications of ERBS: Euler Beta-function B-splines (BFBS) [2], [6]. The article [3] contains a justification of the definition of BFBS and an exposition of the basic properties of BFBS.

Definition 1. (see [3]) Let $t_k \in \mathbb{R}$ and $t_k < t_{k+1}$ for $k = 0, 1, 2, ..., n + 1$. Consider the strictly increasing knot-vector $\{t_k\}_{k=0}^{n+1}$. A Beta-function B-spline (BFBS), associated with three strictly increasing knots t_{k-1}, t_k and t_{k+1}, $B_k(t) = B_k(i_{k-1}, i_k, i_{k+1}; t)$, $k = 1, \ldots, n$, is defined by

$$B_k(t) = \begin{cases} S_{k-1} \int\limits_{t_{k-1}}^{t} \psi_{k-1}(s)ds, & \text{if} \quad t \in (t_{k-1}, t_k), \\ S_k \int\limits_{t}^{t_{k+1}} \psi_k(s)ds, & \text{if} \quad t \in (t_k, t_{k+1}), \\ 1, & \text{if} \quad t = t_k, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

with

$$S_k = \left[\int\limits_{t_k}^{t_{k+1}} \psi_k(t)dt \right]^{-1}, \tag{2}$$

and

$$\psi_k(t) = C_k \frac{(t - t_k)^{i_k} (t_{k+1} - t)^{i_{k+1}}}{(t_{k+1} - t_k)^{i_k + i_{k+1}}}, \quad t \in [t_k, t_{k+1}], \tag{3}$$

where

$$C_k = \binom{i_k + i_{k+1}}{i_k}, \tag{4}$$

and

$$i_l > 0, \quad l = k - 1, k, k + 1. \tag{5}$$

Remark 1. The definition can be extended to B_k, $k = 0, n + 1$, with obvious modifications.

As already discussed in [4], it is of considerable interest, both theoretical and computational, to find explicit representations of BFBS between the adjacent knots in terms of polynomial bases that are typically used in approximation theory, Computer-Aided Geometric Design (CAGD) and operational calculus.

The article [4] was the first one in a sequence of three papers dedicated to the evaluation of BFBS between the knots in terms of expansions in polynomial bases. The second paper in the sequence was [5]; the present paper is the third, and last, paper in this sequence.

In [4] an interpolatory representation of BFBS was obtained in terms of the *local monomial* bases

$$1,\ t-t_k,\ (t-t_k)^2,\ \ldots,\ (t-t_k)^{i_{k-1}+i_k},\quad t\in(t_{k-1},t_k), \tag{6}$$

and

$$1,\ t-t_k,\ (t-t_k)^2,\ \ldots,\ (t-t_k)^{i_k+i_{k+1}},\quad t\in(t_k,t_{k+1}). \tag{7}$$

which coincide, modulo normalization, with the bases in the Taylor interpolation polynomials at the the central knot t_k of degree $i_{k-1}+i_k$ and i_k+i_{k+1}, respectively [4, Remark 1].

The objective of the second paper [5] of the sequence was to derive a different, Bezier-type, representation of BFBS in terms of the local Bernstein polynomials, shifted and scaled onto $t\in[t_k,t_{k+1}]$:

$$b_{m,i}(t_k,t_{k+1};t)=\frac{1}{(t_{k+1}-t_k)^n}\binom{n}{i}(t-t_k)^i(t_{k+1}-t)^{n-i},\quad i=0,\ldots,m, \tag{8}$$

for appropriate value of m, depending on i_k and i_{k+1}, $k=0,\ldots,n$. In this way, the BFBS is being computed as a *polynomial Bezier curve* between each pair of neighbouring knots.

The present paper addresses the derivation of a representation of BFBS in terms of *global monomial bases*, which can be useful in the context of using Fourier, Laplace, and some other integral transforms. Since the local monomial and Bernstein bases in [4, 5] are closely related to the structure of BFBS as incomplete Euler Beta-functions, while the global monomial bases are not so closely related with this specific type of special functions, the proof of the main result in the present paper is, somehow, more technical.

In the next Section 2 we shall formulate explicitly an auxiliary result, proved in [4, Lemma 1], which is used in the proofs of the main results in all three papers of the present sequence. The main result is formulated and proved in Section 3. (In order to make it possible for the reader to compare the three types of representation of BFBS in a self-contained form, in Section 2 we provide also a brief exposition of the main results in [4, 5].) Finally, Section 4 contains some concluding remarks, including orientation about ongoing and future work on the topic of BFBS.

2. Preliminaries

The following technical result will be used in the proofs in Section 3.

Lemma 1. *([4, Lemma 1], [5, Lemma 1].) Formula (1) is equivalent to*

$$B_k(t) = \begin{cases} S_{k-1} d_{k-1} \int\limits_{t_{k-1}}^{t} \varphi_{k-1}(\tau) d\tau, , & \text{if} \quad t \in (t_{k-1}, t_k), \\ S_k d_k \int\limits_{t}^{t_{k+1}} \varphi_k(\tau) d\tau, , & \text{if} \quad t \in (t_k, t_{k+1}), \\ 1, & \text{if} \quad t = t_k, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

where

$$d_{k-1} = \frac{(i_{k-1} + i_k)!}{i_{k-1}! i_k!} \frac{1}{(t_k - t_{k-1})^{i_{k-1}+i_k}}, \tag{10}$$

$$S_{k-1} = \frac{i_{k-1} + i_k + 1}{t_k - t_{k-1}}, \tag{11}$$

$$d_k = \frac{(i_k + i_{k+1})!}{i_k! i_{k+1}!} \frac{1}{(t_{k+1} - t_k)^{i_k+i_{k+1}}}, \tag{12}$$

$$S_k = \frac{i_k + i_{k+1} + 1}{t_{k+1} - t_k}, \tag{13}$$

$k = 0, \dots, n$.

For the purpose of comparison between the main results in [4, 5] and the present paper (see Section 3), we provide here a concise exposition of the main results in [4, 5], as follows.

Theorem 1. (see [4, Theorem 1]) *Under the conditions of Definition 1, let $k = 1, \dots, n$.*

(i) If $t \in (t_{k-1}, t_k)$, *then,*

$$B_k(t) = S_{k-1}d_{k-1} \sum_{l=i_k}^{i_{k-1}+i_k} \frac{1}{l+1} \left[(t_k - t_{k-1})^{l+1} - (t - t_k)^{l+1} \right] \times \left[\binom{i_{k-1}}{l - i_k} (t_k - t_{k-1})^{i_{k-1}+i_k-l} (-1)^{i_k} \right], \quad (14)$$

where

$$S_{k-1}d_{k-1} = \binom{i_{k-1}+i_k}{i_{k-1}} \frac{i_{k-1}+i_k+1}{(t_k - t_{k-1})^{i_{k-1}+i_k+1}}. \quad (15)$$

(ii) If $t \in (t_{k-1}, t_k)$, *then,*

$$B_k(t) = S_k d_k \sum_{l=i_k}^{i_k+i_{k+1}} \frac{1}{l+1} \left[(t_{k+1} - t_k)^{l+1} - (t - t_k)^{l+1} \right] \times \left[\binom{i_{k+1}}{l - i_k} (t_{k+1} - t_k)^{i_k+i_{k+1}-l} (-1)^{l-i_k} \right], \quad (16)$$

where

$$S_k d_k = \binom{i_k + i_{k+1}}{i_k} \frac{i_k + i_{k+1} + 1}{(t_{k+1} - t_k)^{i_k+i_{k+1}+1}}. \quad (17)$$

Theorem 2. (see [5, Theorem 1]) *Under the conditions of Definition 1, let* $k = 1, \ldots, n$.

(i) If $t \in (t_{k-1}, t_k)$, *then,*

$$B_k(t) = \sum_{l=0}^{i_k} \binom{i_{k-1}+i_k+1}{l} \frac{(t_k - t)^l (t - t_{k-1})^{i_{k-1}+i_k+1-l}}{(t_k - t_{k-1})^{i_{k-1}+i_k+1}} = \quad (18)$$

$$= \sum_{l=0}^{i_k} b_{i_{k-1}+i_k+1,l}(t_{k-1}, t_k; t). \quad (19)$$

(ii) If $t \in (t_k, t_{k+1})$, *then,*

$$B_k(t) = \sum_{l=0}^{i_k} \binom{i_k+i_{k+1}+1}{l} \frac{(t - t_k)^l (t_{k+1} - t)^{i_k+i_{k+1}+1-l}}{(t_{k+1} - t_k)^{i_k+i_{k+1}+1}} = \quad (20)$$

$$= \sum_{l=0}^{i_k} b_{i_k+i_{k+1}+1,l}(t_k, t_{k+1}; t). \quad (21)$$

3. BFBS Evaluation in Global Monomial Bases

The main result of the present paper, Theorem 3, is an explicit computation of BFBS between neighbouring knots t_k and t_{k+1} in terms of the monomial bases $1,\ t,\ t^2,\ \ldots,\ m$, where m depends on k.

Theorem 3. *Under the conditions of Definition 1, let $k = 1, \ldots, n$.*

(i) If $t \in (t_{k-1}, t_k)$, then,

$$B_k(t) = \binom{i_{k-1}+i_k}{i_{k-1}} \frac{i_{k-1}+i_k+1}{(t_k - t_{k-1})^{i_{k-1}+i_k+1}} \times \left(\sum_{l=0}^{i_{k-1}+i_k+1} \frac{1}{l+1} t^{l+1} \beta_{k-1,l} - \sum_{l=1}^{i_{k-1}+i_k+1} \frac{1}{l} \beta_{k-1,l-1} t_{k-1}^{l} \right) \tag{22}$$

where $\beta_{k-1,l}$ is given by

$$\beta_{k-1,l} = (-1)^{i_{k-1}-l} \left(\sum_{\mu=\max\{0,\,l-i_k\}}^{\min\{l,\,i_{k-1}\}} \binom{i_{k-1}}{\mu} \binom{i_k}{l-\mu} t_{k-1}^{i_{k-1}-\mu} t_k^{i_k-l+\mu} \right) \tag{23}$$

$$= (-1)^{i_k-l} \left(\sum_{\nu=\max\{0,\,l-i_{k-1}\}}^{\min\{l,\,i_k\}} \binom{i_{k-1}}{l-\nu} \binom{i_k}{\nu} t_{k-1}^{i_{k-1}-l+\nu} t_k^{i_k-\nu} \right). \tag{24}$$

(ii) If $t \in (t_k, t_{k+1})$, then,

$$B_k(t) = \binom{i_k+i_{k+1}}{i_k} \frac{i_k+i_{k+1}+1}{(t_{k+1} - t_k)^{i_k+i_{k+1}+1}} \times \left(\sum_{l=0}^{i_k+i_{k+1}+1} \frac{1}{l+1} t_{k+1}^{l+1} \alpha_{k,l} - \sum_{l=1}^{i_k+i_{k+1}+1} \frac{1}{l} \alpha_{k,l-1} t^{l} \right) \tag{25}$$

where $\alpha_{k,l}$ is given by

$$\alpha_{k,l} = (-1)^{i_k-l} \left(\sum_{\mu=\max\{0,\,l-i_{k+1}\}}^{\min\{l,\,i_k\}} \binom{i_k}{\mu} \binom{i_{k+1}}{l-\mu} t_k^{i_k-\mu} t_{k+1}^{i_{k+1}-l+\mu} \right) \tag{26}$$

$$= (-1)^{i_{k+1}-l} \left(\sum_{\nu=\max\{0,\,l-i_k\}}^{\min\{l,\,i_{k+1}\}} \binom{i_k}{l-\nu} \binom{i_{k+1}}{\nu} t_k^{i_k-l+\nu} t_{k+1}^{i_{k+1}-\nu} \right). \tag{27}$$

Proof. Case (ii): $t \in (t_k, t_{k+1})$. Applying Lemma 1,

$$B_k(t) = S_k d_k \int_t^{t_{k+1}} \varphi_k(\tau) d\tau.$$

Evaluation of $\varphi_k(\tau)$ using the Binomial theorem yields

$$\begin{aligned}
\varphi_k(\tau) &= (\tau - t_k)^{i_k}(t_{k+1} - \tau)^{i_{k+1}} \\
&= (-1)^{i_{k+1}}(\tau - t_k)^{i_k}(\tau - t_{k+1})^{i_{k+1}} \\
&= (-1)^{i_{k+1}} \left(\sum_{\mu=0}^{i_k} \binom{i_k}{\mu} (-1)^{i_k-\mu} t_k^{i_k-\mu} \tau^{\mu} \right) \\
&\quad \times \left(\sum_{\nu=0}^{i_{k+1}} \binom{i_{k+1}}{\nu} (-1)^{i_{k+1}-\nu} t_{k+1}^{i_{k+1}-\nu} \tau^{\nu} \right) \\
&= (-1)^{i_{k+1}} \sum_{\mu=0}^{i_k} \sum_{\nu=0}^{i_{k+1}} \binom{i_k}{\mu} \binom{i_{k+1}}{\nu} (-1)^{i_k+i_{k+1}-\mu-\nu} t_k^{i_k-\mu} t_{k+1}^{i_{k+1}-\nu} \tau^{\mu+\nu}.
\end{aligned}$$

In the inner sum, change indices, as follows

$$\begin{aligned}
&\mu + \nu = l, \qquad l = \mu + 0, \ldots, \mu + i_{k+1}, \quad \mu = 0, \ldots, i_k, \\
&\nu = l - \mu,
\end{aligned} \tag{28}$$

to obtain

$$\begin{aligned}
\varphi_k(\tau) &= (-1)^{i_{k+1}} \sum_{\mu=0}^{i_k} \sum_{l=\mu}^{\mu+i_{k+1}} \binom{i_k}{\mu} \binom{i_{k+1}}{l-\mu} (-1)^{i_k+i_{k+1}-l} t_k^{i_k-\mu} t_{k+1}^{i_{k+1}-l+\mu} \tau^l \\
&= (-1)^{i_{k+1}} \sum_{\mu=0}^{i_k} \sum_{l=\mu}^{\mu+i_{k+1}} C_{k;l,\mu} \tau^l \\
&= (-1)^{i_{k+1}} \sum_{l=0}^{i_k+i_{k+1}} \left(\sum_{\mu=0}^{i_k} C_{k;l,\mu} \right) \tau^l,
\end{aligned}$$

where, completing the definition of $C_{k;l,\mu}$ where appropriate, we arrive at

$$C_{k;l,\mu} = \begin{cases} 0, & \text{if} \quad 0 \le l < \mu, \\ \binom{i_k}{\mu}\binom{i_{k+1}}{l-\mu}(-1)^{i_k+i_{k+1}-l} t_k^{i_k-\mu} t_{k+1}^{i_{k+1}-l+\mu}, & \text{if} \quad \mu \le l \\ & \quad \le \mu + i_{k+1}, \\ 0, & \text{if} \quad \mu + i_{k+1} \\ & \quad < l \le i_k + i_{k+1}. \end{cases} \tag{29}$$

Re-organizing the information in (29) by re-ordering with respect to the index μ yields

$$g(l) \le \mu \le f(l), \tag{30}$$

where

$$f(l) = \begin{cases} l, & \text{if} \quad 0 \le l < i_k, \\ i_k, & \text{if} \quad i_k \le l \le i_k + i_{k+1}. \end{cases} \tag{31}$$

$$g(l) = \begin{cases} 0, & \text{if} \quad 0 \le l \le i_{k+1}, \\ l - i_{k+1}, & \text{if} \quad i_{k+1} \le l \le i_k + i_{k+1}. \end{cases} \tag{32}$$

Next, observe that formulae (30) and (32) can be written in the equivalent and more concise form

$$\begin{aligned} f(l) &= \min\{l,\, i_k\}, \\ g(l) &= \max\{0,\, l - i_{k+1}\}, \end{aligned} \tag{33}$$

respectively – see Figure 1; alltogether, we get

$$\begin{aligned} &\varphi_k(\tau) \\ &= (-1)^{i_{k+1}} \sum_{l=0}^{i_k+i_{k+1}} \tau^l \left(\sum_{\mu=\max\{0,\, l-i_{k+1}\}}^{\min\{l,\, i_k\}} \binom{i_k}{\mu}\binom{i_{k+1}}{l-\mu}(-1)^{i_k+i_{k+1}-l} t_k^{i_k-\mu} t_{k+1}^{i_{k+1}-l+\mu} \right) \\ &= (-1)^{i_{k+1}} \sum_{l=0}^{i_k+i_{k+1}} \tau^l (-1)^{i_k+i_{k+1}-l} \left(\sum_{\mu=\max\{0,\, l-i_{k+1}\}}^{\min\{l,\, i_k\}} \binom{i_k}{\mu}\binom{i_{k+1}}{l-\mu} t_k^{i_k-\mu} t_{k+1}^{i_{k+1}-l+\mu} \right) \\ &= \sum_{l=0}^{i_k+i_{k+1}} \tau^l (-1)^{i_k-l} \left(\sum_{\mu=\max\{0,\, l-i_{k+1}\}}^{\min\{l,\, i_k\}} \binom{i_k}{\mu}\binom{i_{k+1}}{l-\mu} t_k^{i_k-\mu} t_{k+1}^{i_{k+1}-l+\mu} \right). \end{aligned}$$

Thus, we obtain

$$\varphi_k(\tau) = \sum_{l=0}^{i_k+i_{k+1}} \tau^l \alpha_{k,l} = R_{k;1}(\tau), \ \tau \in (t_k, t_{k+1}), \tag{34}$$

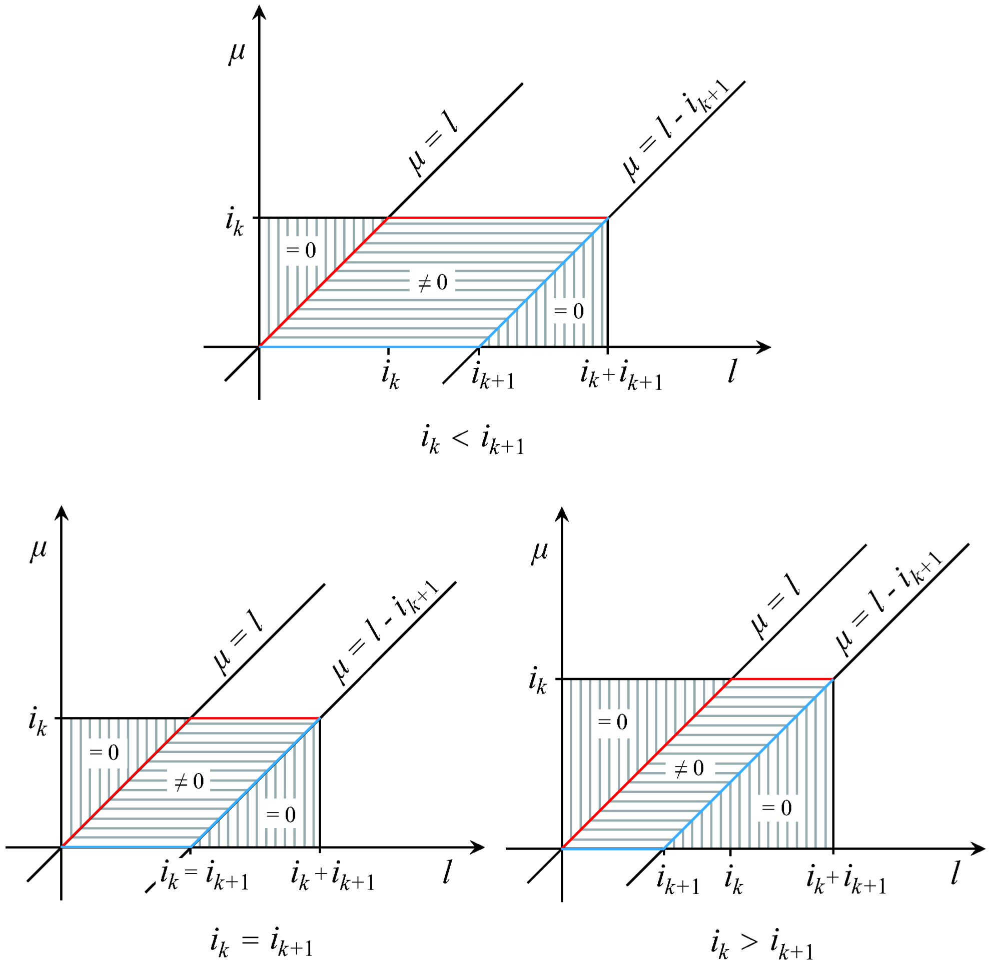

Figure 1: Distribution of the zero and non-zero values of the coefficient $C_{k;l,\mu}$, ((29),(30),(31) and (32)). $f(l)$ is plotted in red, and $g(l)$ is plotted in blue (on a gray-scale image, the graph of $f(l)$ consists of a 'slanted part' which is always left of the respective 'slanted part' of the graph of $g(l)$, and of a 'horizontal part' which is always above the respective 'horizontal part' of the graph of $g(l)$).

where $\alpha_{k,l}$ is given by

$$\alpha_{k,l} = (-1)^{i_k - l} \left(\sum_{\mu=\max\{0,\, l-i_{k+1}\}}^{\min\{l,\, i_k\}} \binom{i_k}{\mu} \binom{i_{k+1}}{l-\mu} t_k^{i_k-\mu} t_{k+1}^{i_{k+1}-l+\mu} \right),$$

i.e., (26) holds.

Taking in consideration the symmetry in the definition of φ_k, we write φ_k as

$$\varphi_k(\tau) = (-1)^{i_k}(t_k - \tau)^{i_k}(t_{k+1} - \tau)^{i_{k+1}},$$

and choosing a different change of indices in (28):

$$\mu + \nu = \lambda, \qquad \lambda = 0 + \nu, \dots, i_k + \nu, \quad \nu = 0, \dots, i_{k+1},$$
$$\mu = \lambda - \nu,$$

we further obtain a symmetrical final expression for $\varphi_k(\tau)$:

$$\begin{aligned}\varphi_k(\tau) &= (-1)^{i_k} \sum_{\lambda=0}^{i_k+i_{k+1}} \tau^\lambda (-1)^{i_k+i_{k+1}-\lambda} \\ &\quad \times \left(\sum_{\nu=\max\{0,\,\lambda-i_k\}}^{\min\{\lambda,\,i_{k+1}\}} \binom{i_k}{\lambda-\nu} \binom{i_{k+1}}{\nu} t_k^{i_k-\lambda+\nu} t_{k+1}^{i_{k+1}-\nu} \right) \\ &= \sum_{\lambda=0}^{i_k+i_{k+1}} \tau^\lambda (-1)^{i_{k+1}-\lambda} \left(\sum_{\nu=\max\{0,\,\lambda-i_k\}}^{\min\{\lambda,\,i_{k+1}\}} \binom{i_k}{\lambda-\nu} \binom{i_{k+1}}{\nu} t_k^{i_k-\lambda+\nu} t_{k+1}^{i_{k+1}-\nu} \right).\end{aligned}$$

Hence,

$$\varphi_k(\tau) = \sum_{\lambda=0}^{i_k+i_{k+1}} \tau^\lambda \beta_{k,\lambda} = R_{k;2}(\tau),\ \tau \in (t_k, t_{k+1}), \tag{35}$$

where

$$\beta_{k,\lambda} = (-1)^{i_{k+1}-\lambda} \left(\sum_{\nu=\max\{0,\,\lambda-i_k\}}^{\min\{\lambda,\,i_{k+1}\}} \binom{i_k}{\lambda-\nu} \binom{i_{k+1}}{\nu} t_k^{i_k-\lambda+\nu} t_{k+1}^{i_{k+1}-\nu} \right), \tag{36}$$

i.e., (27) holds. Obviously, $R_{k;1} = R_{k;2}$ on (t_k, t_{k+1}), which, in view of the fact that the monomials are linearly independent on (t_k, t_{k+1}), together with (36) provides the two different expressions (26, 27) for the coefficients of τ^l.

Now integrate from t to t_{k+1}, to get the following chain of equalities

$$B_k(t) = S_k d_k \int_t^{t_{k+1}} \varphi_k(\tau) d\tau$$

$$= S_k d_k \int_t^{t_{k+1}} \sum_{l=0}^{i_k+i_{k+1}} \tau^l \alpha_{k,l} d\tau$$

$$= S_k d_k \sum_{l=0}^{i_k+i_{k+1}} \frac{1}{l+1} (\tau^{l+1} \alpha_{k,l}) \Big|_t^{t_{k+1}}$$

$$= S_k d_k \left(\sum_{l=0}^{i_k+i_{k+1}} \frac{1}{l+1} t_{k+1}^{l+1} \alpha_{k,l} - \sum_{l=0}^{i_k+i_{k+1}} \frac{1}{l+1} t^{l+1} \alpha_{k,l} \right)$$

$$= S_k d_k \left(\sum_{l=0}^{i_k+i_{k+1}+1} \frac{1}{l+1} t_{k+1}^{l+1} \alpha_{k,l} - \sum_{\lambda=1}^{i_k+i_{k+1}+1} \frac{1}{\lambda} \alpha_{k,\lambda-1} t^{\lambda} \right)$$

Hence, as a final expression for $B_k(t)$ on the interval $t \in (t_k, t_{k+1})$, we obtain

$$B_k(t) = \binom{i_k + i_{k+1}}{i_k} \frac{i_k + i_{k+1} + 1}{(t_{k+1} - t_k)^{i_k+i_{k+1}+1}} \times \left(\sum_{l=0}^{i_k+i_{k+1}+1} \frac{1}{l+1} t_{k+1}^{l+1} \alpha_{k,l} - \sum_{\lambda=1}^{i_k+i_{k+1}+1} \frac{1}{\lambda} \alpha_{k,\lambda-1} t^{\lambda} \right),$$

i.e., (25) holds, where $\alpha_{k,l} = \beta_{k,l}$ are given by (26) and (27).

Case (i): $t \in (t_{k-1}, t_k)$. The proof is analogous to the one of Case (ii) (see also Remark 3).

Applying Lemma 1 again,

$$B_k(t) = S_{k-1} d_{k-1} \int_{t_{k-1}}^{t} \varphi_{k-1}(\tau) d\tau$$

Analogously to the calculation for $\varphi_k(\tau)$ in (t_k, t_{k+1}), ((34), (26), (35) and (27)), for $\varphi_{k-1}(\tau)$ in (t_{k-1}, t_k) we have

$$\varphi_{k-1}(\tau) = \sum_{l=0}^{i_{k-1}+i_k} \tau^l \alpha_{k-1,l} = R_{k-1;1}(\tau), \ \tau \in (t_{k-1}, t_k), \tag{37}$$

where

$$\alpha_{k-1,l} = (-1)^{i_{k-1}-l} \left(\sum_{\mu=\max\{0,\, l-i_k\}}^{\min\{l,\, i_{k-1}\}} \binom{i_{k-1}}{\mu} \binom{i_k}{l-\mu} t_{k-1}^{i_{k-1}-\mu} t_k^{i_k-l+\mu} \right). \tag{38}$$

Symmetrically,

$$\varphi_{k-1}(\tau) = \sum_{\lambda=0}^{i_{k-1}+i_k} \tau^{\lambda}\beta_{k-1,\lambda} = R_{k-1;2}(\tau),\ \tau \in (t_{k-1}, t_k), \tag{39}$$

where $\beta_{k-1,\lambda}$ is given by

$$\beta_{k-1,\lambda} = (-1)^{i_k-\lambda} \left(\sum_{\nu=\max\{0,\,\lambda-i_{k-1}\}}^{\min\{\lambda,\,i_k\}} \binom{i_{k-1}}{\lambda-\nu}\binom{i_k}{\nu} t_{k-1}^{i_{k-1}-\lambda+\nu} t_k^{i_k-\nu} \right). \tag{40}$$

Obviously, $R_{k-1;1} = R_{k-1;2}$ on (t_{k-1}, t_k), which, in view of the linear independence of the monomials on (t_{k-1}, t_k), implies the two different expressions (38, 40) and, hence, the validity of (23, 24), for the coefficients of τ^l.

Now, integrate from t_{k-1} to t:

$$\begin{aligned}
B_k(t) &= S_{k-1}d_{k-1} \int_{t_{k-1}}^{t} \varphi_{k-1}(\tau)d\tau \\
&= S_{k-1}d_{k-1} \int_{t_{k-1}}^{t} \sum_{l=0}^{i_{k-1}+i_k} \tau^l \beta_{k-1,l} d\tau \\
&= S_{k-1}d_{k-1} \sum_{l=0}^{i_{k-1}+i_k} \frac{1}{l+1}(\tau^{l+1}\beta_{k-1,l}) \Big|_{t_{k-1}}^{t} \\
&= S_{k-1}d_{k-1} \left(\sum_{l=0}^{i_{k-1}+i_k} \frac{1}{l+1} t^{l+1}\beta_{k-1,l} - \sum_{l=0}^{i_{k-1}+i_k} \frac{1}{l+1} t_{k-1}^{l+1}\beta_{k-1,l} \right) \\
&= S_{k-1}d_{k-1} \left(\sum_{l=0}^{i_{k-1}+i_k+1} \frac{1}{l+1} t^{l+1}\beta_{k-1,l} - \sum_{\lambda=1}^{i_{k-1}+i_k+1} \frac{1}{\lambda}\beta_{k-1,\lambda-1} t_{k-1}^{\lambda} \right)
\end{aligned}$$

As a final expression for $B_k(t)$ in Case (i) we obtain

$$\begin{aligned}
B_k(t) = \binom{i_{k-1}+i_k}{i_{k-1}} &\frac{i_{k-1}+i_k+1}{(t_k - t_{k-1})^{i_{k-1}+i_k+1}} \\
&\times \left(\sum_{l=0}^{i_{k-1}+i_k+1} \frac{1}{l+1} t^{l+1}\beta_{k-1,l} - \sum_{\lambda=1}^{i_{k-1}+i_k+1} \frac{1}{\lambda}\beta_{k-1,\lambda-1} t_{k-1}^{\lambda} \right),
\end{aligned}$$

i.e., (22) holds, where $\beta_{k-1,l} = \alpha_{k-1,l}$ are given by (23) and (24). □

4. Concluding Remarks

Remark 2. (The global monomial form of BFBS and operational calculi.) Strictly speaking, the interpolatory representation in terms of local monomial bases from [4] is already a good tool when computing Fourier and Laplace transforms. Due to the shifts in the local monomial bases, however, it is necessary to repeat the same computations involving these shifts every time when a Fourier/Laplace transform of an expression containing the BFBS is being computed. When using the representation in terms of global monomial bases developed in the present paper, the above-said computations have to be made only once, as part of the computations in the proof of Theorem 3. Since monomials of any number d of variables are easy to integrate in respective d-fold integrals, this remark remains valid, with respective modifications, also for other integral transforms (e.g., Riemann-Liouville fractional anti-derivatives and Marchaud fractional derivatives), for any number of variables $d = 1, 2, 3, \ldots$.

Remark 3. (Providing analogies for the upgrades of BFBS to the multivariate case.) In every paper of the sequence ([4], [5], and the present one) there was a separation of the proof of the respective main results ([4, Theorem 1], [4, Theorem 1] and Theorem 3 here) into two cases: (i) $t \in [t_{k-1}, t_k]$, and (ii) $t \in [t_k, t_{k+1}]$, respectively, for all relevant k. These cases exhibited certain symmetries which could have been used to shorten the proof of one of these cases by symmetry arguments. Unquestionably, such symmetries exist also in the study of the upgrades of these results in the multivariate cases, but in dimensions higher than 1 these symmetries are becoming increasingly complex to trace and expose rigorously in sufficient detail. (In particular, this is true for the upgrade of BFBS on triangulations (simplectifications) in $\mathbb{R}^d$, $d = 2, 3, \ldots$.) This is why in all of the afore-mentioned proofs we have opted for a sufficiently detailed consideration of *each* of the cases, thus providing preparation for tracing the analogues of these cases when studying the multivariate upgrades of BFBS.

Remark 4. (Geometric and algebraic properties of BFBS, see also [5, Remarks 3, 4].) As already mentioned in [4] and [5], by starting from an expansion of BFBS in any given polynomial basis it is, of course, possible to obtain a local representation of BFBS between neighbouring knots in terms of any other polynomial basis spanning the polynomials which are of the same degree, using the transformation matrix for change between the two bases. In particular, it is possible, starting from any one of the three polynomial representations developed in [4], [5] and the present paper, and using the respective transformation matrix for basis change, to obtain any of the other two representations. The chains of equalities leading to each of the three representations provide some insight into linear-algebraic properties of the transformation matrix between the Bernstein bases and the monomial bases (as well as of the inverse transformation matrix between these bases) which can be described in terms of BFBS and the underlying incomplete Euler Beta-function. We intend to return on exploring this topic in subsequent research, after having studied also the definition of BFBS in the multivariate case, especially, in the case of BFBS on triangulations. Note that the construction of BFBS on simplices in higher dimensions is no longer related to the classical Euler's Beta-function but to a more general class of special functions which can still be described, at least, partially, in terms of the Gamma-function.

Remark 5. (Background of BFBS, and ongoing and future work on BFBS.) Expo-rational B-splines (ERBS) were introduced in [1] and their propertis were studied in detail in [8]. However, as a construction, BFBS have appeared before in a different, probabilistic, context: they have been used to evaluate a cumulative distribution function (see, e.g., [6] of a random variable from a binomial distribution. In this setting, BFBS in the part of its support where it increases from 0 to 1 has been referred to as *regularized incomplete Euler Beta-function.*

Applications of ERBS to CAGD were considered in [1, 8, 12, 9, 10, 11, 7]. In forthcoming work on BFBS we intend to address:

- Applications of BFBS in the same fields where ERBS have now been tested, such as
  - geometric modelling of *parametric curves, surfaces and volume deformations*;
  - *finite/boundary element analysis* and *finite volume analysis*;
  - multilevel techniques, notably, in the construction of GERBS-based *multiwavelets* (where BFBS are expected to outperform ERBS from practical point of view);
- *comparison of the properties* of BFBS
  - versus classical polynomial Schoenberg B-splines,
  - versus ERBS,
  - between BFBS with different values of the parameters in their definition;

- *comparison of the performance* of BFBS versus that of ERBS on theoretical and computational benchmark examples;

- *upgrading the definition* of BFBS in the *multivariate case* (the most important upgrade being for *BFBS on triangulations (simplectifications, in higher dimensions)*, where the newly defined BFBS finite elements / B-splines on triangulations are expected to have very considerable impact on the theory and practical applications of the emerging new field of *isogeometric analysis*.

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