International Journal of Pure and Applied Mathematics
Volume 65 No. 3 2010, 381-388

ON THE ERGODICITY OF AN ALTERNATING PRODUCT EXPANSION

Chrysoula Ganatsiou[1 §], Perakis Konstantinos[2]

[1]Department of Civil Engineering
University of Thessaly
Pedion Areos, Volos, 38334, GREECE
e-mail: ganatsio@otenet.gr

[2]Department of Planning and Regional Development
University of Thessaly
Pedion Areos, Volos, 38334, GREECE

Abstract: The ergodic behaviour of the transformation which produces an alternating product expansion for any number in (0,1) introduced by A. and J. Knopfmacher is investigated.

AMS Subject Classification: 11K55, 37A25, 37A50
Key Words: alternating product expansion, basic operator, ergodic behaviour, Lebesgue measure

1. Introduction

If x is any real number in the interval (1,2) then regarding alternating product representations for real numbers introduced by A. Knopfmacher and J. Knopfmacher (see [3], [6]), x has a unique product expansion of the form

$$x = \left(1 + \frac{1}{\alpha_1}\right) \cdot \left(1 - \frac{1}{\alpha_2}\right) \dots \left(1 + \frac{(-1)^{n-1}}{\alpha_n}\right) \dots \qquad (1)$$

where

Received: October 25, 2010

§Correspondence author

$$\alpha_{i+1} \geq \begin{cases} (\alpha_i+1)^2\,, \text{ if } i \text{ is odd,} \\ \alpha_i^2-1, \text{ if } i \text{ is even,} \end{cases}$$

$$\alpha_i > 1, \text{for every } i \in \aleph^*=\{1,2,\dots\}. \quad (1.2)$$

This generalizes G. Cantor's representations by positive products given in 1869 (see A. and J. Knopfmacher [5]).

We focus on any number $x*$ in the interval (0,1). In this case we may have that

$$x* = \frac{1}{\alpha_1} - \left(1+\frac{1}{\alpha_1}\right)\cdot\frac{1}{\alpha_2} + \left(1+\frac{1}{\alpha_1}\right)\cdot\left(1-\frac{1}{\alpha_2}\right)\cdot\frac{1}{\alpha_3} + \dots$$
$$+ (-1)^{n+1}\cdot\left(1+\frac{1}{\alpha_1}\right)\cdot\left(1-\frac{1}{\alpha_2}\right)\dots\left(1+\frac{(-1)^{n-2}}{\alpha_{n-1}}\right)\cdot\left(1+\frac{(-1)^{n-1}}{\alpha_n}\right)$$
$$\cdot\left(1+\frac{(-1)^{n}}{\alpha_{n+1}}\right)\cdot\frac{1}{\alpha_{n+2}} + \dots. \quad (1.3)$$

or

$$x* = \frac{1}{\alpha_1} - \left(1+\frac{1}{\alpha_1}\right)\cdot\frac{1}{\alpha_2} + \left(1+\frac{1}{\alpha_1}\right)\cdot\left(1-\frac{1}{\alpha_2}\right)\cdot\frac{1}{\alpha_3} + \dots$$
$$+(-1)^n\cdot\left(1+\frac{1}{\alpha_1}\right)\cdot\left(1-\frac{1}{\alpha_2}\right)\cdots\left(1+\frac{(-1)^{n-2}}{\alpha_{n-1}}\right)\cdot\left(1+\frac{(-1)^{n-1}}{\alpha_n}\right)\cdot r_n(x*),$$

where

$$r_n(x*) = \frac{1}{\alpha_{n+1}} - \left(1+\frac{(-1)^n}{\alpha_{n+1}}\right)\cdot r_{n+1}(x*)\ , \quad n \in \aleph = \{0,1,2,...\},$$

or

$$r_n(x*) = \frac{1}{\alpha_{n+1}} - \left(1+\frac{1}{\alpha_{n+1}}\right)\cdot r_{n+1}(x*) \text{ with } 0 < r_n(x*) < \frac{1}{\alpha_n-1},$$

if n is an even number and

$$r_n(x*) = \frac{1}{\alpha_{n+1}} - \left(1-\frac{1}{\alpha_{n+1}}\right)\cdot r_{n+1}(x*) \text{ with } 0 < r_n(x*) < \frac{1}{\alpha_n+1},$$

if n is an odd number respectively.

The abovementioned relation (1.3) may be taken as follows. Let $T:(0,1)\to(0,1)$ be the transformation defined by

$$T(x*) = \left(\frac{1}{\left[\frac{1}{x*}\right]} - x*\right)\quad\left(1+\frac{1}{\left[\frac{1}{x*}\right]}\right). \quad (2)$$

In the case that $\alpha_1 \equiv \alpha_1(x*) = \left[\frac{1}{x*}\right]$ and $\alpha_{n+1} \equiv \alpha_{n+1}(x*) = \left[\frac{1}{T^n(x*)}\right]$, for $T^n(x*) \neq 0$, where

$$T^n(x*) = \left(\frac{1}{\left[\frac{1}{T^{n-1}(x*)}\right]} - T^{n-1}(x*)\right) \left(1 + \frac{(-1)^{n-1}}{\left[\frac{1}{T^{n-1}(x*)}\right]}\right), \quad n \in \aleph*,$$

with $T^0(x*) \equiv x*$, we conclude that the transformation T generates the unique finite or infinite representation (1.3), for every $x* \in (0,1)$.

The present paper arises as an attempt to study the ergodic behaviour of the transformation T defined by (2) which generates an alternating product expansion for any number in (0,1) defined by (1.3). In particular we prove that the transformation T is ergodic with respect to the Lebesgue measure λ under a special condition. Our approach is given in the context of the application of ergodic theory to representations for real numbers. For a more detailed study of the application of ergodic theory to problems and other interesting aspects of number theory we refer the reader to (Ch. Ganatsiou [2], F. Schweiger [8]) and others.

2. Preliminaries

Let

$$K_n \equiv K_n(i_1, i_2, ..., i_n) = \{x* \in (0,1)/\alpha_1(x*) = i_1, \alpha_2(x*) = i_2, ..., \alpha_n(x*) = i_n,\ \alpha_{n+m}(x*) \geq 1, m \in \aleph^*\}, \quad (2.1)$$

for any $i_1, i_2, \ldots, i_n \geq 1$, be the set of all $x^* \in (0,1)$ having a unique expansion of the form (1.3) such that the digits $\alpha_1(x^*), \alpha_2(x^*), \ldots, \alpha_n(x^*)$ have the concrete values $i_1, i_2, \ldots, i_n$ respectively. For n=0 we may take $K_o \equiv (0,1)$.

Regarding the bounds of the set (2.1) we obtain

Proposition 2.1. *The set $K_n \equiv K_n(i_1, i_2, \ldots, i_n)$ is bounded with bounds given by the relations*

$M_n \equiv \sup K_n(i_1, i_2, \ldots, i_n)$

$$= \frac{1}{i_1} - \left(1 + \frac{1}{i_1}\right) \cdot \frac{1}{i_2} + \left(1 + \frac{1}{i_1}\right) \cdot \left(1 - \frac{1}{i_2}\right) \cdot \frac{1}{i_3} + \ldots +$$

$$+ (-1)^{n-1} \cdot \left(1 + \frac{1}{i_1}\right) \cdot \left(1 - \frac{1}{i_2}\right) \ldots \left(1 + \frac{(-1)^{n-2}}{i_{n-1}}\right) \cdot \frac{1}{i_n} +$$

$$+(-1)^n\cdot\left(1+\frac{1}{i_1}\right)\cdot\left(1-\frac{1}{i_2}\right)\ldots\left(1+\frac{(-1)^{n-2}}{i_{n-1}}\right)\cdot\left(1+\frac{(-1)^{n-1}}{i_n}\right)\frac{1}{i_n-1},$$

$m_n \equiv inf\, K_n(i_1, i_2,\ldots, i_n)$

$$=\frac{1}{i_1}-\left(1+\frac{1}{i_1}\right)\frac{1}{i_2}+\left(1+\frac{1}{i_1}\right)\cdot\left(1-\frac{1}{i_2}\right)\cdot\frac{1}{i_3}+\ldots+$$

$$+(-1)^{n-1}\cdot\left(1+\frac{1}{i_1}\right)\cdot\left(1-\frac{1}{i_2}\right)\ldots\left(1+\frac{(-1)^{n-2}}{i_{n-1}}\right)\cdot\frac{1}{i_n},$$

if n is an even number.

In the case that n is an odd number we have

$M_n \equiv sup\, K_n(i_1, i_2,\ldots, i_n)$

$$=\frac{1}{i_1}-\left(1+\frac{1}{i_1}\right)\cdot\frac{1}{i_2}+\left(1+\frac{1}{i_1}\right)\cdot\left(1-\frac{1}{i_2}\right)\cdot\frac{1}{i_3}+\ldots+$$

$$+(-1)^{n-1}\cdot\left(1+\frac{1}{i_1}\right)\cdot\left(1-\frac{1}{i_2}\right)\ldots\left(1+\frac{(-1)^{n-2}}{i_{n-1}}\right)\cdot\frac{1}{i_n},$$

$m_n \equiv inf\, K_n(i_1, i_2,\ldots, i_n)$

$$=\frac{1}{i_1}-\left(1+\frac{1}{i_1}\right)\frac{1}{i_2}+\left(1+\frac{1}{i_1}\right)\cdot\left(1-\frac{1}{i_2}\right)\cdot\frac{1}{i_3}+\ldots+$$

$$+(-1)^{n-1}\cdot\left(1+\frac{1}{i_1}\right)\cdot\left(1-\frac{1}{i_2}\right)\ldots\left(1+\frac{(-1)^{n-2}}{i_{n-1}}\right)\cdot\frac{1}{i_n}+$$

$$+(-1)^n\cdot\left(1+\frac{1}{i_1}\right)\cdot\left(1-\frac{1}{i_2}\right)\ldots\left(1+\frac{(-1)^{n-2}}{i_{n-1}}\right)\cdot\left(1+\frac{(-1)^{n-1}}{i_n}\right)\frac{1}{i_n+1}.$$

Furthermore its Lebesgue measure is given by

$\lambda\left(K_n\left(i_1, i_2, ..., i_n\right)\right)=\prod_{r=1}^{n}\left(1+\frac{(-1)^{r-1}}{i_r}\right)\frac{1}{i_n+(-1)^{n-1}}$, *for any* $n\in\aleph*$.

For the proof of the Proposition 2.1 we follow an analogous proof used in Ch. Ganatsiou [1].

3. Ergodicity of the Transformation T

We are now able to investigate the ergodic behaviour of the transformation T defined by (2) and to prove the following

Theorem 3.1. *The alternating product expansion basic operator T is ergodic with respect to the Lebesgue measure λ regarding only for a family of Borel subsets of (0,1).*

Proof. Let h_n: (0,1) $\rightarrow$ K_n be a linear map defined by the relation

$$h_n(t) = \sum_{j=1}^{n} \frac{(-1)^{j-1} \cdot \lambda(K_{j-1})}{i_j}[i_{j-1}+(-1)^{j-2}]+(-1)^n \cdot t \cdot \lambda(K_n)[i_n+(-1)^{n-1}]$$
$$= \sum_{j=1}^{n} (-1)^{j-1} \cdot \prod_{r=1}^{j-1} \left(1 + \frac{(-1)^{r-1}}{i_r}\right) \cdot \frac{1}{i_j} + (-1)^n \cdot t \cdot \prod_{r=1}^{n} \left(1 + \frac{(-1)^{r-1}}{i_r}\right).$$

For every $x^* \in K_n$ we have that

$$x* = \sum_{j=1}^{\infty} (-1)^{j-1} \cdot \left(1 + \frac{1}{\alpha_1}\right) \cdot \left(1 - \frac{1}{\alpha_2}\right) \dots \left(1 + \frac{(-1)^{j-2}}{\alpha_{j-1}}\right) \cdot \frac{1}{\alpha_j}$$
$$= \sum_{j=1}^{n} (-1)^{j-1} \cdot \lambda(K_{j-1}) \cdot [i_{j-1} + (-1)^{j-2}] \frac{1}{i_j} +$$
$$+ \lambda(K_n) \cdot [i_n + (-1)^{n-1}] \sum_{j=n+1}^{\infty} (-1)^{j-1} \cdot \left(1 + \frac{(-1)^n}{\alpha_{n+1}}\right) \dots \left(1 + \frac{(-1)^{j-2}}{\alpha_{j-1}}\right) \cdot \frac{1}{\alpha_j}$$
$$= h_n(T^n(x*)).$$

This means that $h_n^{-1} = T^n$. Moreover

$M_n \equiv \lim\limits_{t \to \frac{1}{i_n - 1}^-} h_n(t)$, $\quad m_n \equiv \lim\limits_{t \to 0^+} h_n(t)$, for any $n \in \aleph^*$,

where

$M_n \equiv \lim\limits_{t \to \frac{1}{i_n - 1}^-} h_n(t) = \sum_{j=1}^{n} \frac{(-1)^{j-1} \cdot \lambda(K_{j-1})}{i_j}[i_{j-1} + (-1)^{j-2}] + (-1)^n \lambda(K_n)$,

$m_n \equiv \lim\limits_{t \to 0^+} h_n(t) = \sum_{j=1}^{n} \frac{(-1)^{j-1} \cdot \lambda(K_{j-1})}{i_j}[i_{j-1} + (-1)^{j-2}]$,

for any n even number. For any n odd number we have

$M_n \equiv \lim\limits_{t \to 0^+} h_n(t) = \sum_{j=1}^{n} \frac{(-1)^{j-1} \cdot \lambda(K_{j-1})}{i_j}[i_{j-1} + (-1)^{j-2}]$,

$$\mathrm{m}_n \equiv \lim_{t\to \frac{1}{i_n+1}^-} h_n(t) = \sum_{j=1}^{n} \frac{(-1)^{j-1}\cdot\lambda(K_{j-1})}{i_j}[i_{j-1}+(-1)^{j-2}] + (-1)^n\lambda(K_n).$$

Hence for any interval $(\beta_1, \beta_2) \subseteq$ (0,1) with $0 < \lambda(\beta_1, \beta_2) < \frac{1}{[i_n+(-1)^{n-1}]\lambda(K_n)}$ we take

$$\lambda\left(T^{-n}(\beta_1,\beta_2)\cap K_n\right) = \lambda\left(h_n(\beta_1,\beta_2)\cap K_n\right) = \left|h_n\left(\beta_2\right) - h_n\left(\beta_1\right)\right|$$
$$= (\beta_2-\beta_1)\cdot[i_n+(-1)^{n-1}]\lambda\left(K_n\right) = \lambda(\beta_1,\beta_2)\cdot[i_n+(-1)^{n-1}]\lambda\left(K_n\right).$$

Equivalently

$$\lambda\left(T^{-n}B\cap K_n\right) = \lambda\left(B\right)\cdot\lambda\left(K_n\right)[i_n+(-1)^{n-1}], \tag{3}$$

for any set B in the Boolean ring R of all finite disjoint unions of intervals $(\beta_1, \beta_2) \subseteq (0, 1)$ satisfied the relation 0< $\lambda(B) < \frac{1}{[i_n+(-1)^{n-1}]\lambda(K_n)}$. By using standard measure theory as in H. Jager & C. de Vroedt [4], we obtain a same equation which holds for any Borel set B in (0,1) satisfied the above relation.

If we take as B a Borel set in (0,1) satisfied the above relation such that T^{-1}B=B, where T is the measurable, nonsingular alternating product expansion basic operator, then we have that T^{-n} B=B, for any n∈ ℵ∗. Moreover by using relation (3) we obtain that

$$\lambda\left(B\cap K_n\right) = \lambda\left(B\right)\cdot\lambda\left(K_n\right)[i_n+(-1)^{n-1}]$$

or

$$\lambda\left(B\cap K_n\right) = c\cdot\lambda\left(K_n\right),$$

where c=λ (B) $[i_n+(-1)^{n-1}] > 0$, for any n∈ ℵ∗.

Let us define by C the collection of all the cylinders K_n, n∈ ℵ∗, where the digits α_i, for any i∈ ℵ∗, satisfy the relation (1.2). Then any open subinterval of (0,1) is an at most denumerable (or countable) disjoint union of elements of C (λ $\alpha.s.$)As a consequence

$\lambda\left(B\cap C\right) = c\cdot\lambda\left(C\right)$, where c=λ (B) $[i_n+(-1)^{n-1}] > 0$, (3.2)

for any set C (a countable disjoint union of the fundamental intervals K_n). By using Definition 4.3, Theorem 4.5 and relation (3.2) the proof is complete?

References

[1] Ch. Ganatsiou, On the stochastic behaviour of the digits in the modified Engel-type alternating series representations for real numbers, In: *Proceedings of the Workshop on Differential Geometry, Global Analysis and Lie Algebras*, Aristotle University of Thessaloniki, Balkan Press (1997).

[2] Ch. Ganatsiou, On the application of ergodic theory to alternating Engel series, *International Journal of Mathematics and Mathematical Sciences*, **25**, No. 12 (2001), 811-817.

[3] K.H. Indlekofer, A. Knopfmacher, J. Knopfmacher, Alternating Balkema-Oppenheim expansions of real numbers, *Bull. Soc. Math. Belgique B*, **44** (1992), 17-28.

[4] H. Jager, C. de Vroedt, Lüroth series and their ergodic properties, *Nederl. Akad. Wetensch. Proc.* Ser. A, **31** (1969), 31-42.

[5] A. Knopfmacher, J. Knopfmacher, A new construction of the real numbers (via infinite products), *Nicuw. Arch. Wisk.*, **25**, No. 1 (1987), 19-31.

[6] A. Knopfmacher, J. Knopfmacher, New series and product representations for real numbers, *Manuscript* (1990).

[7] K. Knopp, Mengentheorestische Behandlung einiger probleme der diophantischen approximation und der transfiniten wahrscheinlichkeiten, *Math. Ann.*, **95** (1926), 409-426, In German.

[8] F. Schweiger, Ergodische Theorie der Engelschen und Sylvesterschen Reihen, *Czechoslovak Math. J.*, **20**, No. 95 (1970), 243-245.

4. Appendix

Definition 4.1. Let (Ω_1, F_1, P_1) and (Ω_2, F_2, P_2) be two probability spaces.

(i) A map T: $\Omega_1 \to \Omega_2$ is called a transformation.

(ii) A transformation $T : \Omega_1 \to \Omega_2$ is said to be measurable if for every $\mathrm{E} \in \mathrm{F}_2$, $T^{-1}E \in \mathrm{F}_1$.

(iii) A measurable transformation $T : \Omega_1 \to \Omega_2$ is said to be non-singular if for every
$E \in \mathrm{F}_2$ with P_2 (E) $= 0$, P_1 (T^{-1} E) $= 0$.

Definition 4.2. (i) A transformation T: $\Omega_1 \to \Omega_2$ is called measure-preserving (or a homomorphism) if it is measurable and for any $E \in \mathrm{F}_2$, P_2 (E) $= P_1$ (T^{-1} E).

(ii) A homomorphism is called an isomorphism if $T$is an one-to-one map of Ω, onto Ω_2 and if T^{-1} is also a homomorphism.

Now let the two above-mentioned probability spaces be identical.

Definition 4.3. A measurable nonsingular transformation T is called ergodic if the relation $T^{-1}E = E$, for $E \in$ F implies $P(E) = 0$ or $P(E) = 1$. If the weaker assumption
T^{-1} E $\subset$ E, for $E \in$ F already implies $P(E) = 0$ or 1, then T is called strongly ergodic.

Definition 4.4. If P_1 and P_2 are two probability measures on the probability space (Ω,F), then P_1 is said to be absolutely continuous with respect to P_2 if P_2 $(E) = 0$ implies $P_1(E) = 0$ which we denote by $P_1 \ll \mathrm{P}_2$. In the case that $P_1 \ll \mathrm{P}_2 \ll \mathrm{P}_1$, then P_1 and P_2 are said to be equivalent.

Under the above definitions we take the following useful classical criterion for ergodicity of K. Knopp [7].

Theorem 4.5. *Let E be a Lebesque-measurable subset of (0,1) with P (E)>0. Assume that there is a collection J of subintervals of (0,1) with the following properties:*

1. *every open subinterval of (0,1) is at most a denumerable union of disjoint elements of J (P a.s.) and*
2. *for every B $\in$ J, P (EB) $\geq$ cP (B) with a constant c $>$ 0.*

Then P (E) = 1.