International Journal of Pure and Applied Mathematics
Volume 66 No. 4 2011, 365-371

ON GENERALIZED FUNCTIONS CONNECTED WITH THE FUNCTION $\mathcal{P}(2, x)$

Rubén Alejandro Cerutti
Faculty of Exact Sciences
National University of Nordeste
Avda. Libertad 5540, Corrientes, 3400, ARGENTINA
e-mail: rcerutti@exa.unne.edu.ar

Abstract: A new family of generalized functions associated with the function $\mathcal{P}(2, x) = \left(\sum_{l=1}^{p} x_l^2\right)^2 - \left(\sum_{l=p+1}^{p+q} x_l^2\right)^2$ is introduced. The functions of this family can be considered as a generalization of the distributions $(P \pm i0)^\lambda$ and others introduced by Gelfand and Shilov (cf. [1]), where P is the non degenerate quadratic form $P = \sum_{l=1}^{p} x_l^2 - \sum_{l=p+1}^{p+q} x_l^2$.

The Fourier transform is obtained and elementary properties are studied.

AMS Subject Classification: 46F10, 16F12
Key Words: distribution theory, generalized function, quadratic forms

1. Introduction

The well known distribution $(P \pm i0)^\lambda$ introduced by Gelfand and Shilov (cf. [1]) are defined by the following limit

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \left(P \pm i\varepsilon |x|^2\right)^\lambda, \tag{1.1}$$

where λ is a complex number, $\varepsilon > 0$, $|x|^2 = x_1^2 + \cdots + x_n^2$, and

$$P = P(x) = \sum_{l=1}^{p} x_l^2 - \sum_{l=p+1}^{p+q} x_l^2, \tag{1.2}$$

where $p+q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$. These distributions $(P \pm i0)^\lambda$ are analytic as functions of λ everywhere except at $\lambda = -\frac{n}{2} - k$, $k = 0, 1, 2, \cdots$, where they have simple poles (cf. [1], p. 275).

Received: March 8, 2010

The Fourier transform is

$$\mathfrak{F}\left[(P \pm i0)^{\lambda}\right] = \mathcal{C}(\lambda, n, q)(Q \mp i0)^{-\lambda-\frac{n}{2}}, \tag{1.3}$$

where

$$\mathcal{C}(\lambda, n, q) = \frac{e^{\pm i\frac{\pi}{2}q} 2^{2\lambda+n} \pi^{\frac{n}{2}} \Gamma\left(\lambda + \frac{n}{2}\right)}{(2\pi)^{\frac{n}{2}} \Gamma(-\lambda)}, \tag{1.4}$$

and $Q = Q(y) = \sum_{i=1}^{p} y_i^2 - \sum_{i=p+1}^{p+q} y_i^2$.

Analogously, Gelfand defined the distributions $(m^2 + P \pm i0)^{\lambda}$, where m is a real non negative number as the limit

$$(m^2 + P \pm i0)^{\lambda} = \lim_{\varepsilon \to 0} (m^2 + P \pm i\varepsilon|x|^2)^{\lambda} \quad \text{(cf [1], p. 289)}. \tag{1.5}$$

When m is different to zero these distributions are entire distributional functions of λ.

Fourier transform is

$$\mathfrak{F}[(m^2 + P \pm i0)^{\lambda}] = \frac{e^{\pm i\frac{\pi}{2}q} 2^{2\lambda+1} (\sqrt{2\pi})^n (m)^{\frac{n}{2}+\lambda}}{\Gamma(-\lambda)} \frac{K_{\frac{n}{2}+\lambda}\left[m(Q \mp i0)^{\frac{1}{2}}\right]}{(Q \mp i0)^{\frac{1}{2}\left(\frac{n}{2}+\lambda\right)}}, \tag{1.6}$$

where $K_{\gamma}(z)$ denotes the modified Bessel function of third kind (cf. [3], p. 78),

$$K_{\nu}(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_{\nu}(z)}{\sin \pi\nu}, \tag{1.7}$$

$$I_{\nu}(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{2m+\nu}}{m!\Gamma(m + \nu + 1)}. \tag{1.8}$$

Frequently, these generalized functions may be expressed in terms of others distributions, also connected with the quadratic form P.

So, we have

$$(P \pm i0)^{\lambda} = P_+^{\lambda} + e^{\pm i\pi\lambda} P_-^{\lambda}, \tag{1.9}$$

where

$$P_+^{\lambda} = \begin{cases} P^{\lambda} & \text{if } P \geq 0, \\ 0 & \text{if } P < 0, \end{cases}$$

$$P_-^{\lambda} = \begin{cases} 0 & \text{if } P > 0, \\ (-P)^{\lambda} & \text{if } P \leq 0. \end{cases} \tag{1.10}$$

Similarly, we have

$$(m^2 + Q \pm i0)^{\lambda} = (m^2 + P)_+^{\lambda} + e^{\pm i\pi\lambda}(m^2 + P)_-^{\lambda}. \tag{1.11}$$

Kananthai and Nonlaopon (cf. [2]) introduced the generalized function

$$\mathcal{P}^{\lambda} = \mathcal{P}^{\lambda}(l, x) = \left(\left(\sum_{i=1}^{p} x_i^2 \right)^l - \left(\sum_{i=p+1}^{p+q} x_i^2 \right)^l \right)^{\lambda}, \tag{1.12}$$

where λ is a complex number, and $p + q = n$, $l = 1, 2, \ldots$.

They found that $\mathcal{P}^{\lambda}$ has two sets of singularities, the first one at $\lambda = -1, -2, \cdots, -k, \cdots$, and the second at $\lambda = -\frac{n}{2l}, -\frac{n}{2l} - 1, \cdots, -\frac{n}{2l} - s \cdots$, where $s = 0, 1, 2, \cdots$.

In this paper we will define some generalized function formally analogue to $(P \pm i0)^{\lambda}$ based on $\mathcal{P}^{\lambda}$. We will introduce them as limit of others generalized functions, and we will evaluate Fourier transforms.

2. Main Results

Let $x = (x_1, x_2, \cdots, x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$ and let $P = P(x)$ be the quadratic form in n variables given by

$$P = P(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2, \tag{2.1}$$

where p is the number of positive terms, q is the number of negative terms of the form P, and $p + q = n$, the dimension of the space.

Let us consider the function $\mathcal{P}^{\lambda}(l, x)$ given by (1.12) in the particular case when $l = 2$. In this case we have

$$\mathcal{P}^{\lambda} = \mathcal{P}^{\lambda}(2, x) = \left(\left(\sum_{i=1}^{p} x_i^2 \right)^2 - \left(\sum_{i=p+1}^{p+q} x_i^2 \right)^2 \right)^{\lambda}. \tag{2.2}$$

From the definition (2.2), the function $\mathcal{P}(x)$ may be written as the product

$$\mathcal{P}^{\lambda} = \mathcal{P}^{\lambda}(2, x) = \left(\sum_{i=1}^{n} x_i^2 \right)^{\lambda} \left[\left(\sum_{i=1}^{p} x_i^2 \right) - \left(\sum_{i=p+1}^{p+q} x_i^2 \right) \right]^{\lambda} = (r^2 P)^{\lambda}, \tag{2.3}$$

where we had writed $r^2 = \sum_{i=1}^{n} x_i^2$, and $P = P(x)$ is given by (2.1).

This last expression of $\mathcal{P}^{\lambda}(2, x)$, that shows the closely relationship with the quadratic form P given by (2.1) we will use to obtain its singular points and to define a generalized function analogue to the $(P \pm i0)^{\lambda}$ one.

In finding the singularities we begin by considering the generalized function $\mathcal{P}_+^\lambda$ defined by

$$
\begin{aligned}
(\mathcal{P}_+^\lambda, \varphi) &= \int_{\mathcal{P}>0} \mathcal{P}^\lambda(x)\varphi(x)\,dx & (2.4)\\
&= \int_{\mathcal{P}>0} (r^2 P)^\lambda \varphi(x)\,dx, & (2.5)
\end{aligned}
$$

where $r^2 = \sum_{i=1}^n x_i^2$, and P is given by (2.2).

We introduce bipolar coordinates

$$x_1 = rw_1,\; x_2 = rw_2,\; \cdots,\; x_p = rw_p,\; x_{p+1} = sw_{p+1},\; \cdots,\; x_{p+q} = sw_{p+q}.$$

Then (2.5) becomes

$$(\mathcal{P}_+^\lambda, \varphi) = \int_{P>0} \left(r^2\left(r^2 - s^2\right)\right)^\lambda \varphi r^{p-1} s^{q-1} d\Omega^{(p)}\, d\Omega^{(q)}\, dr\, ds. \qquad (2.6)$$

Writing

$$\psi(r,s) = \int \varphi d\Omega^{(p)}\, \Omega^{(q)}, \qquad (2.7)$$

we have

$$(\mathcal{P}_+^\lambda, \varphi) = \int_0^\infty \int_0^r (r^2)^\lambda \left(r^2 - s^2\right)^\lambda \psi(r,s) r^{p-1} s^{q-1}\, dr\, ds. \qquad (2.8)$$

We known that φ is a function differentiable infinitely many times with compact support, then $\psi(r,s)$, also it is as function of r^2 and s^2.

Making the change of variables $u = r^2$; $v = s^2$, and writing

$$\psi(r,s) = \psi_1(u,v) \qquad (2.9)$$

we obtain

$$(\mathcal{P}_+^\lambda, \varphi) = \frac{1}{4}\int_0^\infty \int_0^u (u-v)^\lambda u^\lambda \psi_1(r,v) r^{\frac{1}{2}(p-1)-\frac{1}{2}} v^{\frac{1}{2}(q-1)-\frac{1}{2}}\, du\, dv. \qquad (2.10)$$

Writing $v = ut$, we have

$$(\mathcal{P}_+^\lambda, \varphi) = \frac{1}{4}\int_0^\infty u^{2\lambda+\frac{n}{2}-2}\, du \int_0^1 (1-t)^\lambda t^{\frac{1}{2}(q-1)-\frac{1}{2}} \psi_1(u,tu)\, dt. \qquad (2.11)$$

According the procedure by Gelfand and Shilov the equation (2.11) shows that $\left(\mathcal{P}_+^\lambda, \varphi\right)$ has two set of poles, the first one at $\lambda = -1,\, -2,\, \cdots,\, -k\cdots$, and the second at $\lambda = -\frac{n}{4};\, -\frac{n}{4} - 1;\, \cdots,\, -\frac{n}{4} - k,\, \cdots$, k a non negative integer.

This sets of poles has also been found by Kananthai in a different way (cf. [2]).

Notice that by change of variables the function $\mathcal{P}(2,x)$ and also $\mathcal{P}(l,x)$ can be expressed as a quadratic form in two variables. We can obtain new generalized functions by applying the ideas of Gelfand and Shilov.

Then we have the following

Definition 1. Let λ be a complex number, $\mathcal{P} = \mathcal{P}(2,x)$ given by (2.6). We define the generalized function $(G \pm i0)^\lambda$ as the following limit

$$\begin{aligned}(G \pm i0)^\lambda &= \lim_{\varepsilon \to 0} \left(r^2 P \pm i\varepsilon r^4\right)^\lambda \\ &= \left(r^2 P \pm i0\right)^\lambda . \end{aligned} \tag{2.12}$$

Also we may writte

$$(G \pm i0)^\lambda = \left(r^2(P \pm i0)\right)^\lambda . \tag{2.13}$$

The existence of these distributions is justified at the same mode that by Gelfand and Shilov.

Taking into account that the generalized function $(P \pm i0)^\lambda$ when the number of negative terms q is equal to zero reduces at $(r^2)^\lambda$, we can obtain the Fourier transform of $(G \pm i0)^\lambda$ and then we have

$$\mathfrak{F}\left[(G \pm i0)^\lambda\right] = \mathfrak{F}\left[\left(r^2 P \pm i0\right)^\lambda\right] = C(\lambda, n)\left(\overline{r}^2 \overline{P} \mp i0\right)^{-\lambda - \frac{n}{4}}, \tag{2.14}$$

where the constant $C(\lambda, n)$ is given by

$$C(\lambda, n) = \frac{e^{\pm i \frac{\pi}{2} q} 2^{2\lambda + n} \pi^{\frac{n}{2}} \Gamma\left(2\lambda + \frac{n}{2}\right)}{\Gamma(-2\lambda)} \tag{2.15}$$

and $\overline{r}^2 \overline{P}$ denote the dual of the function $\mathcal{P}(2,x)$

$$\overline{r}^2 \overline{P} = \left(\sum_{i=1}^{n} \xi_i^2\right)\left(\sum_{i=1}^{p} \xi_i^2 - \sum_{i=p+1}^{p+q} \xi_i^2\right) = \overline{Q} \tag{2.16}$$

The new generalized functions $(G \pm i0)^\lambda$ may by expressed in terms of the G_+^λ and G_-^λ obtained from (2.9)

$$(G + i0)^\lambda = G_+^\lambda + e^{i\pi\lambda} G_-^\lambda \tag{2.17}$$

$$(G - i0)^\lambda = G_+^\lambda + e^{-i\pi\lambda} G_-^\lambda, \tag{2.18}$$

where

$$G_+^\lambda = (r^2 P)_+^\lambda = r^{2\lambda} P_+^\lambda, \tag{2.19}$$

$$G_-^\lambda = (r^2 P)_-^\lambda = r^{2\lambda} P_-^\lambda, \tag{2.20}$$

and

$$G_+^\lambda = \begin{cases} r^{2\lambda} P_+^\lambda & \text{if} P > 0, \\ 0 & \text{if} P \leq 0, \end{cases} \tag{2.21}$$

$$G_-^\lambda = \begin{cases} 0 & \text{if} P \geq 0, \\ r^{2\lambda}(-P)^\lambda & \text{if} P < 0. \end{cases} \tag{2.22}$$

When λ is a nonnegative integer we have that $(G + i0)^\lambda$, $(G - i0)^\lambda$ and G^λ coincide. Form (2.13), (2.15), (2.17) and (2.18) we may obtain the Fourier transform of $G_\pm^\lambda$ and of G_-^λ. In fact, after some elementary operations, we obtain

$$\mathfrak{F}[G_+^\lambda] = C(\lambda, n) \frac{1}{2i} \left\{ e^{-\pi\left(2\lambda + \frac{q}{2}\right)i} (\overline{Q} - i0)^{-\lambda - \frac{n}{4}} - e^{\frac{\pi}{2} qi} (\overline{Q} + i0)^{-\lambda - \frac{n}{4}} \right\} \tag{2.23}$$

and

$$\mathfrak{F}[G_+^\lambda] = C(\lambda, n) \frac{1}{2i} \left\{ e^{\frac{\pi}{2} qi} (\overline{Q} - i0)^{-\lambda - \frac{n}{4}} - e^{\frac{\pi}{2} qi} (\overline{Q} + i0)^{-\lambda - \frac{n}{4}} \right\}, \tag{2.24}$$

where

$$C(\lambda, n) = 2^{4\lambda + n} \pi^{\frac{n}{2} - 1} \Gamma(2\lambda + 1) \Gamma(2\lambda + \frac{n}{2}) \tag{2.25}$$

3. Remark

To show the deppendence of the parameter l we write the definitory formula (2.12) in the following mode

$$(G \pm i0)^\lambda = \lim_{\varepsilon \to 0} \left(\mathcal{P}(l, x) \pm i e |x|^{2l} \right)^\lambda \tag{3.1}$$

for $l = 1$ or $l = 2$, we may have the Fourier transform writing

$$\mathfrak{F}\left[(G \pm i0)^\lambda \right] = C(\lambda, l, n) (\overline{G}(l, x) \mp i0)^{-\lambda - \frac{n}{2l}}, \tag{3.2}$$

where

$$\overline{G}(l, x) = \left(\sum_{i=1}^{p} \xi_i^2 \right)^l - \left(\sum_{i=p+1}^{p+q} \xi_i^2 \right)^l \tag{3.3}$$

and the constant $C(\lambda, l, n)$ is

$$C(\lambda, l, n) = \frac{e^{\pm i\frac{\pi}{2}q} 2^{2l\lambda+n} \pi^{\frac{n}{2}} \Gamma\left(l\lambda + \frac{n}{2}\right)}{\Gamma(-l\lambda)}. \tag{3.4}$$

As particular case we have:

1. If in (3.1) we put $l = 1$, and q the number of negative terms of $\mathcal{P}(l, x)$ is equal to zero, we obtain the generalized function r^λ (cf. [1]).

2. If in (3.1) we put $l = 1$ and $q \neq 0$ it results the well known distribution $(P \pm i0)^\lambda$.

3. If in (3.1) we put $l = 2$ and $q \neq 0$, we have (2.12).

References

[1] I.M. Gelfand, G.E. Shilov, *Generalized Functions*, Volume I, Academic Press, New York (1964).

[2] A. Kananthai, K. Nonlaopon, On the residue of the generalized function $\mathcal{P}^\lambda$, *The Journal of Math.*, **2** (2003), 49-57.

[3] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Second Edition, Cambridge University Press (1994).

372