International Journal of Pure and Applied Mathematics
Volume 66 No. 4 2011, 373-384

CAUSAL RIESZ POTENTIAL AND CAUSAL POISSON OPERATORS

Rubén Alejandro Cerutti

Faculty of Exact Sciences
National University of Nordeste
Avda. Libertad 5540, Corrientes, 3400, ARGENTINA
e-mail: rcerutti@exa.unne.edu.ar

Abstract: Relationship between Riesz potentials and the Poisson kernel in the radial case has been studied by Samko (cf. [9]) and Rubin (cf. [8]) among others authors. The last author gives a theorem that establish that the Riesz potential may be expressed as an unidimensional integral whose integrand is a Poisson's integral. In this paper analogues relations between ultrahyperbolic causal Riesz potential and causal Poisson's integral is obtained.

AMS Subject Classification: 46F10
Key Words: causal Riesz potential, causal Poisson integrals

1. Preliminaries

Let $x = (x_1, x_2, .., x_n)$ be a point of the Euclidean space $\mathbb{R}^n$. We write

$$P = P(x) = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2; \tag{1.1}$$

a non degenerate quadratic form, where $p + q = n$.

The generalized function $(P \pm i0)^\lambda$ was introduced by Gelfand and Shilov (cf. [1]) as the following limit

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \left(P \pm i\varepsilon|x^2|\right)^\lambda, \tag{1.2}$$

where λ as a complex number, $\varepsilon > 0$ and $|x|^2 = x_1^2 + ... + x_n^2$.

Received: June 18, 2010

Let $\{h_\alpha(P \pm i0, n)\}\ \alpha \in \mathbb{C}$ be the family of the causal (anticausal) distributions introduced by Trione (cf. [10]) defined by

$$H_\alpha(P \pm i0, n) = \frac{e^{i\frac{\pi}{2}\alpha} e^{i\frac{\pi}{2}q} \Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})} (P \pm i0)^{\frac{\alpha-n}{2}}, \tag{1.3}$$

where Γ is the Euler Gamma function.

The Fourier transform of $H_\alpha(P \pm i0, n)$ is given by

$$\begin{aligned} \mathcal{F}[H_\alpha(P \pm io, n)](\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{i\langle x, \xi\rangle} H_\alpha(P \pm io, n) dx \\ &= \frac{(Q \mp io)^{-\frac{\alpha}{2}}}{(2\pi)^{\frac{n}{2}}}, \end{aligned} \tag{1.4}$$

where Q is the quadratics form

$$Q = \xi_1^2 + ... + \xi_p^2 - \xi_{p+1}^2 - ... - \xi_{p+q}^2;$$

and $p + q = n$.

By $S(\mathbb{R}^n) = S$ we denote the Schwartz space infinitely differentiable rapidly decreasing funtions in $\mathbb{R}^n$.

Definition 1. Let φ be a function belongs to S. The causal (anticausal) Riesz potential of order $\alpha > 0$ is defined by the convolution

$$R^\alpha \varphi = H_\alpha(P \pm i0, n) * \varphi \tag{1.5}$$

(cf. [1]).

This definition in analogue to the one due to Samko (cf. [9, p.556]).

The integral in (1.5) results convergente in the case $\alpha > n-2$; and it admits an analytical continuation respect to α for $\alpha \leq n - 2$.

If the number q, the negative terms of the form P, is equal to zero, P reduce to $|x|^2$ and then (1.5) reduce to the integral operator

$$(I^\alpha \varphi) = \frac{1}{C(n, \alpha, q = 0)} \int_{\mathbb{R}^n} \frac{\varphi(y)}{\|x - y|^n} dy \tag{1.6}$$

$$C(n, \alpha, q = 0) = \pi^{\frac{n}{2}} 2^\alpha \Gamma(\frac{\alpha}{2}) \Gamma(\frac{n - \alpha}{2})$$

called the elliptic Riesz potential of order α (cf. [7], also [8]).

As it is known the distributional functions $R^\alpha\varphi$ are causal (anticausal) analogues to the ultrahyperbolic potential introduced by Nozaki (cf. [6]) defined by the convolution

$$\mathcal{U}^\alpha f = \Phi_\alpha * f, \tag{1.7}$$

where Φ_α is given by

$$\Phi = \frac{r_+^{\alpha-n}}{C_n(\alpha)} \tag{1.8}$$

$r_+^{\alpha-n} = P^{\frac{\alpha-n}{2}}$, with $x_1 > 0$, P given by (1.1), and the constant $C_n(\alpha)$, is

$$C_n = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{2-\alpha-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma\left(\alpha\right)}{\Gamma\left(\frac{2+\alpha-p}{2}\right)\Gamma\left(\frac{p-\alpha}{2}\right)}. \tag{1.9}$$

It can be observed that if $p = 1$ is considered, from (1.8) and (1.9) we obtain

$$M_\alpha(u) = \begin{cases} \dfrac{u^{\frac{\alpha-n}{2}}}{N_n(\alpha)} & \text{if } x \in \Gamma_+, \\ 0 & \text{if } x \notin \Gamma_+, \end{cases} \tag{1.10}$$

where $u = x_1^2 - x_2^2 - ... - x_n^2$ denotes the cone $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0, u > 0\}$ and the constant $N_n(\alpha) = \pi^{\frac{n-2}{2}}2^{\alpha-1}\Gamma\left(\frac{\alpha-n+2}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)$.

$M_\alpha(u)$ is the hyperbolic Riesz kernel (cf. [7], also[11]).

By putting $n = 1$ in (1.10) and taking into account the Legendre duplication formula for the Gamma function

$$\Gamma(z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right)$$

we obtain

$$I_\alpha(x) = \begin{cases} \dfrac{x^{\alpha-1}}{\Gamma(\alpha)} & \text{if } \ x > 0, \\ 0 & \text{if } \ x \leq 0, \end{cases} \tag{1.11}$$

or equivalently $I_\alpha = \frac{x_+^{\alpha-1}}{\Gamma(\alpha)}$, where $x_+^{\alpha-1}$ is the generalized function

$$x_+^\lambda = \begin{cases} x^\lambda & \text{if } \ x > 0, \\ 0 & \text{if } \ x \leq 0 \end{cases} \tag{1.12}$$

(cf. [3]) for $\lambda = \alpha - 1$. $I_{\alpha(x)}$ is precisely the singular Riemann-Liouville kernel (cf. [7]).

Formally analogues to the $(P \pm i0)^\lambda$ generalized functions are the $(t^2 + P \pm i0)^\lambda$ distributions defined by the limit

$$(t^2 + P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \left(t^2 + P \pm i\varepsilon |x|^2\right)^\lambda, \tag{1.13}$$

where t is a positive real number, $\varepsilon > 0$, $|x|^2 = x_1^2 + ... + x_n^2$ and $P = P(x)$ as in (1.1).

It may be observed that $(t^2 + P \pm i0)^\lambda$ are entire distributional functions of λ. (cf. [10])

They will be used to introduce a kernel that may be considered a generalization of the Poisson kernel.

To invert causal Riesz potentials we have used the causal hypersingular integrals technique defining the causal Riesz derivative of order α, (cf. [1]) as

$$D^\alpha \varphi = \frac{1}{d_{n,l}(\alpha)} T_l^\alpha \varphi,$$

where $T_l^\alpha \varphi$ is the causal hypersingular integral given by the following

$$T_l^\alpha \varphi(x) = \int_{\mathbb{R}^n} \frac{\sum_{k=0}^{l} \binom{l}{k} (-1)^k \varphi(x - kt)}{(P + i0)^{\frac{n+\alpha}{2}}} dt.$$

The normalizing coefficient $\frac{1}{d_{n,l}(\alpha)}$ is chosen that the Fourier transform is given by

$$\mathcal{F}[D^\alpha \varphi] = (Q - i0)^{-\frac{\alpha}{2}} \mathcal{F}[\varphi].$$

Then, if $\varphi = R^\alpha f$, $f \in S$, we have that

$$D^\alpha \varphi = f.$$

In this paper firstly we will introduce a kernel that generalize the Poisson's one and then, by using the so called causal Poisson integrals we invert causal Riesz potential by means a one dimensional integral.

We follows the ideas by Rubin (cf. [8]) expossed to invert elliptic Riesz potentials.

2. The Causal Poisson Kernel and the Causal Poisson Integral

In this section we introduce a generalization of the radial Poisson kernel by means of distributions connected whit the quadratic form P.

Definition 2. Let (x,t) be point of $\mathbb{R}^n \times \mathbb{R}_+$,$x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$. For $t \in \mathbb{R}_+$ consider the following generalized function

$$\mathcal{P}(P+i0,t) = \frac{e^{i\frac{\pi}{2}q}\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{t}{(t^2+P+i0)^{\frac{n+1}{2}}}, \tag{2.1}$$

where $(t^2+P+i0)^{\frac{n+1}{2}}$ es given by (1.13) for $\lambda = \frac{n+1}{2}$.

We may observe that when $q=0$, (2.1) reduces at the Poisson kernel

$$\mathcal{P}(x,t) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{t}{\left(t^2+\|x\|^2\right)^{\frac{n+1}{2}}} \tag{2.2}$$

(cf. [9, p.457], also [8, p.217]).

Taking into account the Fourier transform of the $(t^2+P+i0)^{\lambda}$ distribution, the partial Fourier transform in x of (2.1) is

$$\mathcal{F}_x\left[\mathcal{P}(P+i0,t)\right](\xi) = \frac{e^{i\frac{\pi}{2}}2^{-\frac{n+1}{2}+1}(2\pi)^{\frac{n}{2}}t^{-\frac{1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}(Q-i0)^{\frac{1}{4}}\frac{\pi}{2}K_{-\frac{1}{2}}\left(t(Q-i0)^{\frac{1}{2}}\right), \tag{2.3}$$

From the well-known relationship (cf. [4, f.8.469.3] and also [5, p. 112])

$$K_{\pm\frac{1}{2}(z)} = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \tag{2.4}$$

From (2.3) and (2.4) it results

$$\mathcal{F}_x\left[\mathcal{P}(P+i0,t)\right](\xi) = \frac{e^{i\frac{\pi}{2}q}\pi^{\frac{n+1}{2}}}{2^{\frac{1}{2}}\Gamma\left(\frac{n+1}{2}\right)}e^{-t(Q-i0)^{\frac{1}{2}}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \tag{2.5}$$

$$= e^{i\frac{\pi}{2}q}e^{-t(Q-i0)^{\frac{1}{2}}}. \tag{2.6}$$

If in (2.5) $q=0$ is considered, we have

$$\mathcal{F}_x\left[\mathcal{P}(x,t)\right](\xi) = e^{-t|x|} \tag{2.7}$$

that is the Fourier transform of the Poisson elliptic kernel (cf. [9, p.498]).

2.1. Causal Poisson Integral

Definition 3. Let φ be function belongs to S, the Schwartzian space of functions.

The causal Poisson integral is defined by the following convolution

$$(P_t\varphi)(x) = \int_{\mathbb{R}^n} \mathcal{P}(y,t)\varphi(x-y)dy, \text{ for} t > 0. \tag{2.8}$$

Some elementary properties of these integrals are pointed in the following.

Lemma 1. *1. For any $t > 0$ the partial Fourier transform*

$$\mathcal{F}_x\left[P_t(\varphi)\right](\xi) = e^{-t(Q-i0)^{\frac{1}{2}}}\mathcal{F}_x(\varphi)(\xi)$$

2. For $\varphi \in S$,

$$\lim_{t\to 0}(P\varphi)(x,t) = \varphi(x)$$

For the proof of this lemma we refer to [2].

Now we will present another expression of the $(P \pm i0)^\lambda$ distribution which will be used in further discussions.

Lemma 2. *For α a real number, $0 < \alpha < n$, we have the following integral representation of the $(Q-i0)^{\frac{\alpha-n}{2}}$ generalized function*

$$(Q-i0)^{\frac{\alpha-n}{2}} = \frac{2}{B\left(\frac{\alpha+1}{2},\frac{n-\alpha}{2}\right)}\int_0^\infty \frac{t^\alpha}{(t^2+P+i0)^{\frac{n+1}{2}}}dt. \tag{2.9}$$

Proof. Consider the following well known generalized function

$$G(Q-i0) = (Q-i0)^{\frac{\alpha-n}{2}}\frac{1}{2}B\left(\frac{\alpha+1}{2},\frac{n-\alpha}{2}\right), \tag{2.10}$$

where $B(x,y)$ is the Beta function $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

According to a Trione's theorem that allows us computing the Fourier transform of a generalized function depending on $(Q \pm i0)$ distribution as if is a radial distribution (cf. [10]), i.e. making the formal substitution $(Q \pm i0) \to |y|^2$, we have

$$\mathcal{F}\left[G(Q-i0)\right] = \mathcal{F}\left[G\left(|y|^2\right)\right]_{|\xi|^2 \to P+i0}. \tag{2.11}$$

The symbol on the right hand member of (2.11) has the following meaning, we evaluate the Fourier transforms as it was a radial function and then replace $|\xi|^2$ by $(P+i0)$.

Then

$$\mathcal{F}\left[G\left(y\right)\right]\left(\xi\right)=\mathcal{F}\left[|y|^{\alpha-n}\frac{1}{2}\right]B\left(\frac{\alpha+1}{2},\frac{n-\alpha}{2}\right)\left(\xi\right) \tag{2.12}$$

Taking into account formula 3.251.2 from [4] and putting $\alpha=\mu-1$, and $\frac{n+1}{2}=1-\mu$, we have

$$G\left(|y|\right)=|y|^{\alpha-n}\int_0^\infty \frac{x^\alpha}{\left(1+x^2\right)^{\frac{n+1}{2}}}dx=\int_0^\infty \frac{\left(|y|x\right)^\alpha}{\left(|y|^2\right)^{\frac{n+1}{2}}\left(1+x^2\right)^{\frac{n+1}{2}}}|y|dx. \tag{2.13}$$

Making the substitution $|y|x=t$, we have

$$G(|y|)=\int_0^\infty \frac{t^\alpha}{\left(t^2+|y|^2\right)^{\frac{n+1}{2}}}|y|dt \tag{2.14}$$

Then from (2.12) and (2.14), we may rewrite

$$(Q-i0)^{\frac{\alpha-n}{2}}\frac{1}{2}B\left(\frac{\alpha+1}{2},\frac{n-\alpha}{2}\right)=\int_0^\infty \frac{t^2}{\left(t^2+Q+i0\right)^{\frac{n+1}{2}}}dt \tag{2.15}$$

that is the thesis of Lemma.

(2.15) and other analogue expressions will be used in the next discussions and we will deal with in the same mode.

Notice that the left-hand side of (1.13) when α is replaced by $-\alpha$, for $\alpha>0$, exists in the distributional sense, thus in this case the right-hand side must be understood in the same mode. □

3. Representation of Causal Riesz Potentials by Means of Causal Poisson Integrals

In the section we given an integral representation of the causal Riesz potential by means of a causal Poisson integral.

Theorem 1. *Let α be a complex number that $Re(\alpha)<n$ and let φ be a function belonging to S. Then the causal Riesz potential of order α, $R^\alpha\varphi$ admits a representation given by the following integral:*

$$(R^\alpha\varphi)(x)=\frac{e^{i\frac{\pi}{2}q}}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}(\mathcal{P}_t\varphi)(t)dt. \tag{3.1}$$

Proof. By considering the left hand member of (3.1) we have:

$$\begin{aligned} I &= \frac{e^{i\frac{\pi}{2}q}}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}(\mathcal{P}_t\varphi)(t)dt \\ &= \frac{e^{i\frac{\pi}{2}q}}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\left(\int_{\mathbb{R}^n}\frac{C_n\varphi(x-y)}{(t^2+P+i0)^{\frac{n+1}{2}}}tdy\right)dt \\ &= \frac{C_n e^{i\frac{\pi}{2}q}}{\Gamma(\alpha)}\int_{\mathbb{R}^n}\varphi(x-y)dy\int_0^\infty\frac{t^\alpha}{(t^2+P+i0)^{\frac{n+1}{2}}}dt. \end{aligned} \tag{3.2}$$

Taking into account (2.15), it results

$$I = \frac{C_n\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)}{2\Gamma(\alpha)\Gamma\left(\frac{n+1}{2}\right)}\int_{\mathbb{R}^n}\varphi(x-y)(P+i0)^{\frac{\alpha-n}{2}}dy, \tag{3.3}$$

where

$$C_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \tag{3.4}$$

We conclude that

$$I = \frac{e^{i\frac{\pi}{2}q}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)2^\alpha\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}{2\Gamma(\alpha)\pi^{\frac{n-1}{2}}e^{i\frac{\pi}{2}q}e^{i\pi\alpha}\Gamma\left(\frac{n-\alpha}{2}\right)}R^\alpha\varphi = \frac{2^{\alpha-1}\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}{2\Gamma(\alpha)\pi^{\frac{1}{2}}}R^\alpha\varphi \tag{3.5}$$

Applying formula 1.2.3, p.3 from [5], we have

$$\Gamma\left(\frac{\alpha+1}{2}\right) = \frac{\sqrt{\pi}\Gamma(\alpha)}{2^{\alpha-1}\Gamma\left(\frac{\alpha}{2}\right)}. \tag{3.6}$$

Then, from (3.5) and (3.6), it results

$$I = R^\alpha\varphi \tag{3.7}$$

which is the thesis of Theorema 1. □

Theorem 1 allows us to invert causal Riesz potential by using unidimensional integral that involve causal Poisson integral. In fact, we have the following.

Theorem 2. *Let α by a positive real number, $\alpha \neq 0, 1, 2, ...$, and let φ be a function belonging to S, an let $D^\alpha\varphi(x)$ the causal Riesz derivative of order α.*

Then $D^\alpha\varphi(x)$ admits a representation given by the following integral

$$D^\alpha\varphi(x) = \frac{e^{i\frac{\pi}{2}q}}{\Gamma(-\alpha)}\int_0^\infty\frac{1}{t^{\alpha+1}}\left(\sum_{k=0}^{l}\begin{pmatrix} l \\ k \end{pmatrix}(-1)^k(P_{kt}\varphi)(x)\right)dt. \tag{3.8}$$

Proof. By definition we have that the causal Riesz derivative is given by the following n dimensional integral (cf. [1])

$$\begin{aligned} D^\alpha \varphi(x) &= \frac{(T_l^\alpha \varphi)(x)}{d_{n,l}(\alpha)} \\ &= \frac{1}{d_{n,l}} \int_{\mathbb{R}^n} \sum_{k=0}^{l} \left(\begin{array}{c} l \\ k \end{array} \right) (-1)^k (P+i0)^{-\frac{n+\alpha}{2}} \varphi(x-kt)dt, \quad (3.9) \end{aligned}$$

where

$$d_{n,l}(\alpha) = \frac{\pi^{\frac{n}{2}+1} e^{i\frac{\pi}{2}q} \mathcal{A}_l(\alpha)}{2^\alpha \Gamma\left(1+\frac{\alpha}{2}\right) \Gamma\left(\frac{n-\alpha}{2}\right) \sin\frac{\pi}{2}\alpha}. \quad (3.10)$$

and $\mathcal{A}_l(\alpha)$ is the function of the parameter α given by the following expression

$$\mathcal{A}_l(\alpha) = \sum_{k=0}^{l} \binom{l}{k} (-1)^k k^\alpha, \quad a > 0 \quad \text{(cf. [3], also [1])}.$$

Let us show that $(T_l^\alpha \varphi)(x)$ may be written as an integral which integrand depends on the ultrahyperbolic Poisson's kernel.

In fact

$$(T_l^\alpha \varphi)(x) = \frac{1}{d_{n,l}} \sum_{k=0}^{l} \left(\begin{array}{c} l \\ k \end{array} \right) (-1)^k \int_{\mathbb{R}^n} (P+i0)^{-\frac{n+\alpha}{2}} \varphi(x-kt)dt. \quad (3.11)$$

Making the substitution $y = kt$, it result

$$\begin{aligned} \frac{(T_l^\alpha \varphi)(x)}{d_{n,l}(\alpha)} &= \frac{1}{d_{n,l}} \sum_{k=0}^{l} \left(\begin{array}{c} l \\ k \end{array} \right) (-1)^k k^\alpha \int_{\mathbb{R}^n} (P+i0)^{-\frac{n+\alpha}{2}} \varphi(x-kt)dt. \\ &= \frac{1}{d_{n,l}} \sum_{k=0}^{l} \left(\begin{array}{c} l \\ k \end{array} \right) (-1)^k k^\alpha \frac{2^{-\alpha} \pi^{\frac{n}{2}} \Gamma\left(-\frac{\alpha}{2}\right)}{e^{-i\frac{\pi}{2}q} \Gamma\left(\frac{n+\alpha}{2}\right)} R^{-\alpha} \varphi(x) \\ &= \frac{1}{d_{n,l}(\alpha)} \mathcal{A}_t(\alpha) \frac{2^{-\alpha} \pi^{\frac{n}{2}} \Gamma\left(-\frac{\alpha}{2}\right)}{e^{-i\frac{\pi}{2}q} \Gamma\left(\frac{n+\alpha}{2}\right)} R^{-\alpha} \varphi(x). \quad (3.12) \end{aligned}$$

From (3.10), (1.3) and (3.12) we have

$$\begin{aligned} \frac{(T_l^\alpha \varphi)(x)}{d_{n,l}(\alpha)} &= \frac{2^\alpha \Gamma\left(1+\frac{\alpha}{2}\right) \Gamma\left(\frac{n-\alpha}{2}\right) \sin\frac{\pi}{2}\alpha}{\pi^{\frac{n}{2}} \pi e^{i\frac{\pi}{2}q} \mathcal{A}_t(\alpha)} \frac{\mathcal{A}_t(\alpha) e^{i\frac{\pi}{2}\alpha} \pi^{\frac{n}{2}} \Gamma\left(-\frac{\alpha}{2}\right)}{e^{i\frac{\pi}{2}q} \Gamma\left(\frac{n+\alpha}{2}\right)} R^{-\alpha} \varphi(x) \\ &= R^{-\alpha} \varphi(x). \quad (3.13) \end{aligned}$$

Let us consider the integral in the right hand side of (3.7). Formally we have

$$I(\varphi) = \frac{1}{C(\alpha,l)} \int_0^{\infty} \left(\sum_{k=0}^{l} \binom{l}{k} (-1)^k (P_{kt}(x)) \right) \frac{dt}{t^{\alpha+1}}, \tag{3.14}$$

where

$$C(\alpha,l) = \Gamma(-\alpha) \sum_{k=0}^{l} \binom{l}{k} (-1)^k k^{\alpha}. \tag{3.15}$$

Applying the definition of the causal Poisson kernel, we have

$$I(\varphi) = \frac{C_n}{C(\alpha,l)} \int_0^{\infty} \sum_{k=0}^{l} \binom{l}{k} (-1)^k \left(\int_{\mathbb{R}^n} \frac{kt}{(P+i0+(kt)^2)^{\frac{n+1}{2}}} \varphi(x-y)dy \right) \frac{dt}{t^{\alpha+1}}$$

changing the order of integration

$$I(\varphi) = \frac{C_n}{C(\alpha,l)} \int_{\mathbb{R}^n} \sum_{k=0}^{l} \binom{l}{k} (-1)^k k^{\alpha} \left(\int_0^{\infty} \frac{(z)^{-\alpha}}{(P+i0+(z)^2)^{\frac{n+1}{2}}} dt \right) \varphi(x-y)dy$$

Making the change $kt = z$, we obtain

$$I(\varphi) = \frac{C_n}{C(\alpha,l)} \times \int_{\mathbb{R}^n} \sum_{k=0}^{l} \binom{l}{k} (-1)^k k^{\alpha+1} \left(\int_0^{\infty} \frac{(kt)^{-\alpha}}{(P+i0+(kt)^2)^{\frac{n+1}{2}}} dt \right) \varphi(x-y)dy$$

Applying Lemma 2, we have

$$I(\varphi) = \frac{C_n}{C(\alpha,l)} \sum_{k=0}^{l} \binom{l}{k} (-1)^k k^{\alpha} \frac{1}{2} B\left(\frac{-\alpha+1}{2}, \frac{n+\alpha}{2} \right) \int_{\mathbb{R}^n} (P+i0)^{-(\frac{\alpha+n}{2})} \varphi(x-y)dy.$$

From (3.4) and the definition of the Beta function we have

$$I(\varphi) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{2} \frac{\Gamma(-\frac{\alpha}{2}+\frac{1}{2})\Gamma(\frac{n+\alpha}{2})}{\Gamma(\frac{n+1}{2})} \frac{1}{\Gamma(-\alpha)} \int_{\mathbb{R}^n} (p+i0)^{-(\frac{\alpha+n}{2})} \varphi(x-y)dy.$$

From formula (1.2.3) of [5] we have that

$$2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}\,\Gamma(2z),$$

for $z = -\frac{\alpha}{2}$, we obtain

$$I(\varphi) = \frac{\Gamma(-\alpha)\Gamma(\frac{n+\alpha}{2})}{\Gamma(-\alpha)2^{-\alpha}\Gamma(-\frac{\alpha}{2})\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} (p+i0)^{-(\frac{\alpha+n}{2})}\varphi(x-y)dy, \tag{3.16}$$

$$I(\varphi) = H_{-\alpha}(P+i0) * \varphi = R^{-\alpha}\varphi \tag{3.17}$$

Then, from (3.8), (3.12), (3.14) and (3.17) we obtain an integral representation of the causal Riesz derivative in terms of causal Poisson integral given by

$$D^{\alpha}\varphi = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} \left(\sum_{k=0}^{l} \binom{l}{k} (-1)^k (P_{kt}\varphi)(x) \right) \frac{dt}{t^{\alpha+1}}$$

wich is the thesis of Theorem 2. □

References

[1] R.A. Cerutti, *On the Inversion of Causal Riez Potentials*, Serie de Trabajos de Matemtica, No. 248, Instituto Argentino de Matemática y Facultad de Ciencias Exactas, Universidad de Buenos Aires (1995).

[2] R.A. Cerutti, Una generalizacion del núcleo de Poisson, *Revista del Instituto de Matemática*, Facultad de Ingeniera, Universidad de Nacional del Nordeste, N-s: 5-3 (2007), 67-71.

[3] I. Gerlfand, G. Shilov, *Generalized Functions*, Volume I, Academic Press, New York (1964).

[4] I. Gradshteyn, I. Ryshik, *Tables of Integrals, Series and Products*, Academic Press (1965).

[5] N. Lebedev, *Special Functions and their Applications*, Dover Publications (1972).

[6] Y. Nozaki, On Riemann-Liouville integral of ultrahyperbolic type, *Kodai Math. Seminar Reports*, **6**, No. 2 (2964), 69-82.

[7] M. Riesz, L'Integrale de Riemann-Liouville et de problème de Cauchy, *Acta Mathematica*, **81** (1949), 1-223.

[8] B. Rudin, *Fractional Integrals and Potentials*, Pitman Monographs, Logman (1996).

[9] S. Samko, *Hypersingular Integrals and their Application*, Taylor and Francis, New York (2002).

[10] S.E. Trione, *Distributional Products*, Cursos de Matemática, No. 3, Instituto Argentino de Matemtica, IAM-CONICET, Buenos Aires (1980).

[11] S.E. Trione, On Marcel Riesz's ultrahyperbolic kernel, *Studies in Applied Math.*, **79** (1988), 185-191.