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ON THE REAL PART OF SECANT VARIETIES OF REAL VARIETIES EMBEDDED IN A PROJECTIVE SPACE

E. Ballico

Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let $X \subset \mathbb{P}^n$ a smooth variety defined over $\mathbb{R}$. Here we describe a large part of the real part of the secant varieties $\sigma_k(X) \subsetneq \mathbb{P}^n$ in term of the connected components of $X(\mathbb{R})$ and $X(\mathbb{C}) \setminus X(\mathbb{R})$ when X is not weakly $(k-1)$-defective.

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In this paper all schemes are defined over $\mathbb{C}$ and some of them are defined over $\mathbb{R}$. Thus if Y is reduced, then Y means $Y(\mathbb{C})$.

For any closed subscheme $Z \subset \mathbb{P}^n$ let $\langle Z \rangle$ denote its linear span. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate projective variety. For every $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ is the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$. For each integer $t \geq 1$ let $\sigma_t(X)$ denote the t-secant variety of X, i.e. the closure in $\mathbb{P}^n$ of the union of all $(t-1)$-dimensional linear subspaces of $\mathbb{P}^n$ spanned by t points of X. Notice that our convention gives $\sigma_1(X) = X$. Let $z_X(P)$ the minimal degree of a zero-dimensional scheme $Z \subset X$ such that $P \in \langle Z \rangle$. Let $\mathcal{S}(X, P)$ be the set of all $S \subset X$ computing $r_X(P)$, i.e. the set of all reduced $S \subset X$ such that $\sharp(S) = r_X(P)$ and $P \in \langle S \rangle$. Let $\mathcal{Z}(X, P)$ be the set of all zero-dimensional schemes $Z \subset X$ computing $z_X(P)$, i.e. the set of all zero-dimensional subschemes $Z \subset X$ such that $\deg(Z) = r_X(P)$ and $P \in \langle Z \rangle$.

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Notice that every element $Z \in \mathcal{Z}(X,P) \cup \mathcal{S}(X,P)$ is linearly independent, i.e. $\dim(\langle Z \rangle) = \deg(Z) - 1$. Now assume that X is defined over $\mathbb{R}$, that the prescribed inclusion $X \subset \mathbb{P}^n$ is defined over $\mathbb{R}$. Fix $P \in \mathbb{P}^n(\mathbb{R})$. Recall that $\mathbb{P}^n$ means also the topological space $\mathbb{P}^n(\mathbb{C})$ with the Euclidean topology. Call $\sigma : \mathbb{P}^n \to \mathbb{P}^n$ the complex conjugation. Thus $\mathbb{P}^n(\mathbb{R}) = \{P \in \mathbb{P}^n : \sigma(P) = P\}$. We will use σ to denote the complex conjugation on each subscheme of $\mathbb{P}^n$. Thus $\sigma_t(X)(\mathbb{R}) = \sigma_t(X) \cap \mathbb{P}^n(\mathbb{R})$. The complex conjugation σ acts on each family of subschemes. Hence for all $P \in \mathbb{P}^n$ we have $\sigma(\mathcal{S}(X,P)) = \mathcal{S}(X,P)$ and $\sigma(\mathcal{Z}(X,P)) = \mathcal{Z}(X,P)$. These relations are more interesting when $\sharp(\mathcal{S}(X,P)) = 1$ (resp. $\sharp(\mathcal{Z}(X,P)) = 1$), because it says that the only object computing $r_X(P)$ (resp. $z_X(P)$) is defined over $\mathbb{R}$. We found in the literature one case in which $\mathcal{Z}(X,P)$ is unique, say $\mathcal{Z}(X,P) = \{Z\}$, and reduced (non $(k-1)$-weakly subvarieties in the sense of [1]). Thus if $P \in \mathbb{P}^n(\mathbb{R})$, then $\sigma(Z) = Z$, i.e. Z is defined over $\mathbb{R}$. In this case we show that everything a priori possible is possible (concerning the number of the connected components of Z which are defined over $\mathbb{R}$ and the pairs of complex conjugation one). Set $m := \dim(X)$. We assume X smooth. Since X is smooth and integral, $X(\mathbb{C})$ is a connected compact complex manifold of dimension m and a compact and connected C^∞ manifold of dimension $2m$. The set $X(\mathbb{C})$ is the disjoint union of some compact and connected manifolds of dimension m. We always assume $X(\mathbb{R}) \neq \emptyset$ (if $X(\mathbb{R}) = \emptyset$, then Theorem 1 below is trivially true taking $y = w = 0$ and $z = 1$). Since X is smooth, the last assumption is equivalent to assuming that the set $X(\mathbb{R})$ is Zariski dense in X. Let $D_1, \dots, D_y$ denote the connected components of $X(\mathbb{R})$. Let $U_1, \dots, U_z, A_1, B_1, \dots, A_w, B_w$, $y \ge 0$, $w \ge 0$, $(y,w) \neq (0,0)$, be the connected components of $X \setminus X(\mathbb{R})$ with $\sigma(U_i) = U_i$ for all i and $\sigma(A_j) = B_j$ for all j. We fix the integer y, z, w and a labeling of $D_1, \dots, D_y, U_1, \dots, U_z, A_1, B_1, \dots, A_w, B_w$ as above (hence $\sigma(B_j) = A_j$ for all j). For every integer $k > 0$ let $I(y,z,x,k)$ be the set of all quadruples (t, f, h_1, h_2) with the following properties:

(a) t is an integer such that $0 \le 2t \le k$,

(b) f is a function $f : \{1, \dots, y\} \to \mathbb{N}$ such that $\sum_{i=1}^{y} f(i) = k - 2t$;

(c) $h_1 : \{1, \dots, z\} \to \mathbb{N}$ and $h_2 : \{1, \dots, w\} \to \mathbb{N}$ are functions such that $\sum_{i=1}^{z} h_1(i) + \sum_{j=1}^{w} h_2(j) = t$.

Theorem 1. *Fix an integer k such that $k(x+1) \le n-2$. Assume that X is not $(k-1)$-weakly defective in the sense of [1]. Then there exists a non-empty and σ-invariant open subset U of $\sigma_k(X)$ such that $U(\mathbb{R})$ is the disjoint union of real smooth manifolds M_i, $i \in I(y,z,z,k)$, each of them of real dimension*

$k(x+1)-1$ such that $r_X(Q)=k$ and $\sharp(\mathcal{Z}(X,Q))=1$ for every $Q\in U$, and for each $P\in M_i$, say $i=(t,f,h)$, the only set $S\subset X$ computing $r_X(P)$ has $k-2t$ elements in $X(\mathbb{R})$, exactly $f(i)$ of them being in D_i, $1\le i\le y$, and t pairs of complex conjugation points in $X\setminus X(\mathbb{R})$. Among these pairs exactly $h_1(i)$ of them are in U_i, $1\le i\le z$, and $h_2(j)$ of them are in $A_j\cup B_j$, $1\le j\le w$. The set $\sigma_k(X)(\mathbb{R})\setminus U(\mathbb{R})$ has Hausdorff dimension at most $k(x+1)-2$.

We recall that curves are not $(k-1)$-weakly defective for any k (except the line if $n=1$) (see [1]). We recall that if X is a smooth curve, then every connected component D_i of $X(\mathbb{R})$ is diffeomorphic to a circle and that for a fixed genus g all possible triples (y,z,w) (see [3], Proposition 3.1); either $(y,w)=(0,1)$ (disconnecting case) or $(y,w)=(1,0)$ (non-disconnecting case); we have $0\le y\le g+1$, $(y,w)=(1,0)$ if $y=0$, $(y,w)=(0,1)$ if $y=g+1$ and $y\equiv g+1 \pmod 2$ if $(y,w)=(0,1)$.

Proof of Theorem 1. Let $X^{\langle t\rangle}$ denote the set of all subsets of X with cardinality t. By [2], Proposition 1.5, there is a non-empty Zariski open subset V of $\sigma_t(X)_{reg}\setminus\sigma_{t-1}(X)$ (with the convention $\sigma_0(X):=\emptyset$) such that $\sharp(\mathcal{Z}(X,Q))=1$ and $r_X(Q)=k$ for all $Q\in V$. Hence the unique $Z\in\mathcal{Z}(X,Q)$ is reduced. Set $U:=V\cap\sigma(U)$. Thus U is σ-invariant. Take any $P\in U(\mathbb{R})$ and call $S\subset X$ the unique element of $\mathcal{S}(X,P)$. Since $\sigma(P)=P$, the uniqueness of S implies $\sigma(S)=S$. We have $\sharp(S)=k$. The cardinality of $S':=S\cap(X\setminus X(\mathbb{R}))$ is even, say $\sharp(S')=t$, and S' may be decomposed into pairs of complex conjugation points. Set $h_1(i):=\sharp(S'\cap U_i)$, $1\le y\le z$, and $h_2(j):=\sharp(S'\cap(A_j\cup B_j))$, $1\le j\le w$. We have $S\setminus S'=S\cap X(\mathbb{R})$. Set $f(i):=\sharp((S\setminus S')\cap D_i$, $1\le i\le y$. To conclude we need to prove the converse, i.e. we need to prove that all elements of $I(y,z,w,k)$ are realized, i.e. that $U(\mathbb{R})$ has $\sharp(I(y,z,w,k))$ connected components, each of them being a k-dimensional real topological manifold. Fix $(t,f,h_1,h_2)\in I(y,z,w,k)$. Notice that the set of all $S\in X^{\langle t\rangle}$ such that $\sigma(S)=S$, $\sharp(S\cap D_i)=f(i)$ for all $i\in\{1,\dots,y\}$, $\sharp(S\cap U_i)=h_1(i)$ for all $i\in\{1,\dots,z\}$ and $\sharp(S\cap(A_i\cup B_i))=h_2(i)$ for all $i\in\{1,\dots,w\}$ is Zariski dense in $X^{\langle t\rangle}$. $\square$

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