International Journal of Pure and Applied Mathematics
Volume 66 No. 4 2011, 389-391

SUBSCHEMES OF A PROJECTIVE SUBVARIETY $X \subset \mathbb{P}^n$ MINIMALLY SPANNING A GIVEN $P \in \mathbb{P}^n$

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let $X \subset \mathbb{P}^n$ be an integral non-degenerate subvariety. Fix $P \in \mathbb{P}^n$ and an integer $k > 0$. Let $\mathcal{Z}(X, P, k)$ (resp. $\mathcal{S}(X, P, k)$) be the set of all zero-dimensional subscheme (resp. zero-dimensional and reduced) $Z \subset X$ such that $\deg(Z) = k$, P is in the linear span $\langle Z \rangle$ of Z but $P \notin \langle Z' \rangle$ for all $Z' \subsetneq Z$. We study these sets when X is a linearly normal curve with low genus with respect to n.

AMS Subject Classification: 14N05
Key Words: ranks, X-rank, zero-dimensional scheme, linearly normal curve

1. Introduction

Let $X \subseteq \mathbb{P}^n$ be an integral and non-degenerate variety defined over an algebraically closed field. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \ \rangle$ denote the linear span. Let $z_X(P)$ be the minimal integer $\deg(Z)$ for some zero-dimensional scheme $Z \subset X$ such that $P \in \langle Z \rangle$. Let $\mathcal{Z}(X, P)$ denote the set of all zero-dimensional subschemes of X computing $z_X(P)$. Notice that if Z computes $z_X(P)$, then $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$. For every integer $k > 0$ let $\mathcal{S}(X, P, k)$ be the set of all subsets $S \subset X$ such that $\sharp(S) = k$, $P \in \langle S \rangle$ and $P \notin \langle S' \rangle$ for any $S' \subsetneq S$. Take any $Z \in \mathcal{Z}(X, P, k)$ and any $S \in \mathcal{S}(X, P, k)$. The latter condition in the definition of these sets implies $\dim(\langle Z \rangle) = \dim(\langle S \rangle) = k - 1$, i.e. Z and S are linearly independent. Thus $\mathcal{Z}(X, P, k) = \mathcal{S}(X, P, k) = \emptyset$ for

Received: August 2, 2010

all $k \geq n+2$. Obviously $\mathcal{Z}(X,P,k) = \emptyset$ if $k < z_X(P)$, $\mathcal{Z}(X,P,z_X(P)) \neq \emptyset$, $\mathcal{S}(X,P,k) = \emptyset$ if $k < r_X(P)$, $\mathcal{S}(X,P,r_X(P)) \neq \emptyset$ and $\mathcal{S}(X,P,k) \subseteq \mathcal{Z}(X,P,k)$ for all k. Obviously both $\mathcal{Z}(X,P,n+1)$ and $\mathcal{S}(X,P,n+1)$ contain a non-empty open subset of the symmetric product of $n+1$ copies of X. Thus $\dim(\mathcal{Z}(X,P,n+1)) \geq (n+1)m$, $\dim(\mathcal{S}(X,P,n+1)) = (n+1)m$ and every subset of X with cardinality $n+1$ is a limit of a family of elements of $\mathcal{S}(X,P,n+1)$.

Here we prove the following results.

Proposition 1. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate subvariety. Let $\alpha(X)$ be the maximal integer $t > 0$ such that every subset $A \subset X$ with $\sharp(A) = t$ is linearly independent. Let $\beta(X)$ be the maximal integer $t > 0$ such that every zero-dimensional subscheme $Z \subset X$ with $\deg(Z) = t$ is linearly independent. Fix $P \in \mathbb{P}^n$ and an integer $k > 0$.*

(a) If $z_X(P) + 1 \leq k \leq \beta(X) - k$, then $\mathcal{Z}(X,P,k) = \mathcal{S}(X,P,k) = \emptyset$.

(b) Assume $\beta(X) = 2k+1$ and $\mathcal{Z}(X,P,k) \neq \emptyset$. Fix any $A \subset X$ computing $z_X(P)$ and any $B \in \mathcal{Z}(X,P,k)$. Then $A \cap B = \emptyset$.

(c) If $r_X(P) + 1 \leq k \leq \alpha(X) - k$, then $\mathcal{S}(X,P,k) = \emptyset$.

(d) Assume $\alpha(X) = 2k+1$ and $\mathcal{S}(X,P,k) \neq \emptyset$. Fix any $A \subset X$ computing $r_X(P)$ and any $B \in \mathcal{S}(X,P,k)$. Then $A \cap B = \emptyset$.

Corollary 1. *Let $X \subset \mathbb{P}^n$ be an integral and linearly normal curve with arithmetic genus $g \geq 0$. Fix $P \in \mathbb{P}^n$. Fix an integer k such that $z_X(P) + 1 \leq k \leq n - 3g + 1 - k$. Then $\mathcal{Z}(X,P,k) = \mathcal{S}(X,P,k) = \emptyset$.*

Taking $g = 0$ and $k = 1$ in Corollary 1 we get the following result.

Corollary 2. *Let $X \subset \mathbb{P}^n$, $n \geq 3$, be a rational normal curve. Then $\mathcal{Z}(X,P,k) = \emptyset$ for all $P \in X$ and all $k \in \{2,\dots,n\}$.*

2. The Proofs

We lift from [1] the following lemma and its proof.

Lemma 1. *Fix any $P \in \mathbb{P}^n$ and two zero-dimensional subschemes A, B of X such that $A \neq B$, $P \in \langle A \rangle$, $P \in \langle B \rangle$, $P \notin \langle A' \rangle$ for any $A' \subsetneqq A$ and $P \notin \langle B' \rangle$ for any $B' \subsetneqq B$. Then $h^1(\mathbb{P}^n, \mathcal{I}_{A \cup B}(1)) > 0$.*

Proof. Since A and B are zero-dimensional, we have the inequality

$$h^1(\mathbb{P}^n, \mathcal{I}_{A \cup B}(1)) \geq \max\{h^1(\mathbb{P}^n, \mathcal{I}_A(1)), h^1(\mathbb{P}^n, \mathcal{I}_B(1))\}.$$

Thus we may assume $h^1(\mathbb{P}^n, \mathcal{I}_A(1)) = h^1(\mathbb{P}^n, \mathcal{I}_B(1)) = 0$, i.e. $\dim(\langle A\rangle) = \deg(A) - 1$ and $\dim(\langle B\rangle) = \deg(B) - 1$. Set $D := A \cap B$ (scheme-theoretic intersection). Thus $\deg(A \cup B) = \deg(A) + \deg(B) - \deg(D)$. Since $D \subseteq A$ and A is linearly independent, we have $\dim(\langle D\rangle) = \deg(D) - 1$. Since $h^1(\mathbb{P}^n, \mathcal{I}_{A\cup B}(1)) > 0$ if and only if $\dim(\langle A \cup B\rangle) \leq \deg(A \cup B) - 1$, we get $h^1(\mathbb{P}^n, \mathcal{I}_{A\cup B}(1)) > 0$ if and only if $\langle D\rangle \subsetneqq \langle A\rangle \cap \langle B\rangle$. Since $A \neq B$, $D \subsetneqq A$. Hence $P \notin \langle D\rangle$. Since $P \in \langle A\rangle \cap \langle B\rangle$, we are done. □

Proof of Proposition 1. We only check parts (a) and (b), since the proofs of parts (c) and (d) are similar (just take as A and B reduced subschemes). Assume $z_X(P) + 1 \leq k \leq \beta(X) - k$. Since $\mathcal{S}(X, P, k) \subseteq \mathcal{Z}(X, P, k)$, it is sufficient to prove $\mathcal{Z}(X, P, k) = \emptyset$. Fix $A \subset X$ computing $z_X(P)$ and assume the existence of $B \in \mathcal{Z}(X, P, k)$. Lemma 1 and the definition of $\beta(X)$ gives $\deg(A \cup B) \geq \beta(X) + 1$, contradiction. In the set-up of part (b), i.e. if $\beta(X) = 2k + 1$, then Lemma 1 gives $\deg(A \cup B) \geq \deg(A) + \deg(B)$, i.e. $A \cap B = \emptyset$. □

Proof of Corollary 1. Since $n \geq g+1$ and X is non-degenerate, $h^1(X, \mathcal{O}_X(1)) = 0$. Hence the linear normality of X gives $\deg(X) = n - g$. Since $h^1(X, F) = 0$ for every torsion free sheaf F on X such that $\deg(F) \geq 2g - 1$, we have $h^1(X, \mathcal{I}_{Z,X}(1)) = 0$ for every zero-dimensional scheme $Z \subset X$ such that $\deg(Z) \leq n - 3g + 1$. Since X is linearly normal, if $h^1(X, \mathcal{I}_{Z,X}(1)) = 0$, then $h^1(\mathbb{P}^n, \mathcal{I}_Z(1)) = 0$. Thus $\beta(X) \geq n - 3g + 1$. Apply part (a) of Proposition 1. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, Subsets of the variety $X \subset \mathbb{P}^n$ computing the X-rank of a point of $\mathbb{P}^n$, *Preprint*.

392