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SUBSETS OF A CURVE $X \subset \mathbb{P}^n$ COMPUTING INFINITELY MANY X-RANKS

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Abstract: Fix integers $k \ge 2$ and $n \ge 2k-1$. Fix a $(k-2)$-dimensional linear subspace $V \subset \mathbb{P}^n$. Here we construct smooth curves $X \subset \mathbb{P}^n$ such that for a general $P \in V$ there is $S \subset X$ computing the X-rank of P and containing V. For fixed X and V we also get infinitely many such sets S.

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Let $X \subseteq \mathbb{P}^n$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle\ \rangle$ denote the linear span (see [3], [2], [1]). Let $\mathcal{S}(X,P)$ be the set of all $S \subset X$ computing $r_X(P)$, i.e. the set of all subsets $S \subset X$ such that $\sharp(S) = r_X(P)$ and $P \in \langle S \rangle$. Notice that any $S \in \mathcal{S}(X,P)$ is linearly independent and $P \notin \langle S' \rangle$ for any $S' \subsetneq S$. Quite often there are $P, P' \in \mathbb{P}^n$ such that $P \ne P'$ and $\mathcal{S}(X,P) = \mathcal{S}(X,P')$, but usually this phenomenon arises for when $\sharp(\mathcal{S}(X,P)) = 1$. If $\dim(X) \ge 2$, there is also the possibility of a linear space $A \subset \mathbb{P}^n$ such that $A \cap X$ has positive dimension and there are infinitely many $S \subset X \cap A$ spanning A and computing $r_X(P)$ for a general $P \in A$ (here $r_X(P) = \dim(A) + 1$. Here we construct different types of examples (see Theorem 1 and Example 1) and show in what sense these examples are extremal.

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Theorem 1. *Fix integers $k \geq 2$ and $n \geq 2k-1$. Fix a $(k-2)$-dimensional linear subspace $V \subset \mathbb{P}^n$. There is a smooth, connected and non-degenerate curve $X \subset \mathbb{P}^n$, a non-empty open subset U of V and subsets $S_P \subset X$, $P \in U$, such that $r_X(P) = k$, $S_P \in \mathcal{S}(X, P)$ and $V \subset \langle S_P \rangle$ for all $P \in U$.*

The following example gives a constructive proof of Theorem 1.

Example 1. Fix integers $k \geq 2$ and $n \geq 2k-1$. Fix a $(k-2)$-dimensional linear subspace $V \subset \mathbb{P}^n$ and an $(n-k+1)$-dimensional linear subspace $M \subset \mathbb{P}^n$. Let $Y \subseteq M$ be a rational normal curve of M. Let $F \subset \mathbb{P}^n$ be the cone with base Y and V as a vertex. Since Y spans M and $\langle V \cup M \rangle = \mathbb{P}^n$, F is an integral and non-degenerate subvariety of $\mathbb{P}^n$ and $\dim(F) = k$. Fix $k-1$ integers $a_i \geq 2$, $1 \leq i \leq k-1$. Let X be the intersection of F with $k-1$ general hypersurfaces of degree $a_1, \dots, a_s$. Since $\dim(V) = k-2$ and the $k-1$ hypersurfaces are general, we have $X \cap V = \emptyset$. Bertini's Theorem gives that X is a smooth, connected curve spanning $\mathbb{P}^n$. For each $Q \in Y$ set $V_Q := \langle V \cup \{Q\} \rangle$. Since $M \cap V = \emptyset$, each V_Q is an $(k-1)$-dimensional linear subspace of $\mathbb{P}^n$ contained in F. Let J be the set of all $Q \in Y$ such that $V_Q \cap X$ is a reduced set of $a_1 \cdots a_{k-1}$ points spanning V_Q. By Bertini's Theorem J contains a non-empty open subset of Y. Bertini's Theorem implies $V \cap X = \emptyset$. Thus the linear projection from V induces a finite morphism $f : X \to Y$. We have $\deg(f) = a_1 \cdots a_{k-1}$. A monodromy argument gives that for general $Q \in Y$ (say for Q in a subset T of J) the set $V_Q \cap X$ is in linearly general in V_Q, i.e. any k of its points span V_Q. Since $\dim(V) = k-1$ and $V \cap X = \emptyset$, we have $r_X(V) \geq k$. Since $V \subset \langle S \rangle$ for every $S \subset V_Q \cap X$ such that $Q \in J$ and $\sharp(S) = k$. We claim that $r_X(P) = k$ for a general $P \in V$. The claim would proved that this example prove Theorem 1. Assume that the claim is false and take the only integer $r \geq k-1$ such that $r_X(P) = r$ for a general $P \in V$. Fix $P \in V$ with $r_X(P) = r$ and $S \subset X$ such that $\sharp(S) = r$ and $P \in \langle S \rangle$. Let $\ell_V : \mathbb{P}^n \setminus V \to M$ denote the linear projection from V. Recall that $X \cap V = \emptyset$. Hence $V \cap S = \emptyset$. Since $\langle S \rangle \cap V \neq \emptyset$, $\ell_V(S)$ spans a linear subspace of M of dimension at most $r-2$. Since Y is a rational normal curve of $M \cong \mathbb{P}^{n-k+1}$, any subset of Y with cardinality at most $n-k+2$ is linearly independent. Since $r \leq k-1 \leq n-k+2$, we get $\sharp(\ell_V(S)) < \sharp(S)$. Call $m_1, \dots, m_x$ the cardinality of the fibers of $\ell_V|S$ with cardinality at least 2. Since $\dim(Y) = 1$, for general $P \in V$ we may assume that S contains only elements of $V_Q \cap X$ with $Q \in JT$. Thus we get $\dim(V \cap \langle S \rangle) = m_1 + \cdots + m_x - 1$. Since S computes $r_X(P)$, we have $P \notin \langle S' \rangle$ for any $S' \subsetneq S$. Thus $m_1 + \cdots + m_x = r$. Since $r \leq k-1$ and $\dim(Y) = 1$, the set of all such $\langle S \rangle \cap V$ does not cover a non-empty open subset of V.

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