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A NEW FRACTIONAL FOURIER TRANSFORM AND CONVOLUTION PRODUCTS

Luis Guillermo Romero[1], Rubén Alejandro Cerutti[2 §], Luciano Leonardo Luque[3]
[1,2,3]Faculty of Exact Sciences
National University of Nordeste
Avda. Libertad 5540, Corrientes, 3400, ARGENTINA
[2]e-mail: rcerutti@exa.unne.edu.ar

Abstract: In this short paper we introduce a new definition of the Fractional Fourier transform of order α, $0 < \alpha \leq 1$, of a function wich belongs to the Lizorkin space of functions. The simplicity of their definition allows us to deal with a novel convolution product due by Miana [5] and with the Weyl fractional calculus. We also present an inversion formulae.

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1. Introduction

In recent years several definitions of fractional Fourier transform have been introduced motivated by their application to obtain solutions of problems comming from optics, signal processing, quantum mechanics and others.

We know that there are various different definitions of fractional derivatives and integral, and they are not necessarily equivalent.

The existence of different fractional calculus conduct to an apparent paradoxe: there no exist one best definition of the fractional derivative and integrals or fractional Fourier transform.

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§Correspondence author

The aim of this paper is to introduce a new definition for the fractional Fourier transform of order α, $0 < \alpha \leq 1$, by a simple integral that contains as particular cases the conventional Fourier transform for $\alpha = 1$, and the one due to Luchko, Martinez and Trujillo in the case that the variable of the image is positive.

Moreover, we obtain an inversion formula and we deal the relation for this fractional Fourier transform and the classical convolution product, and the new one due to Miana.

The outline of this paper is as follows. In Section 2 we give some definitions needed for the next and introduce our definition of fractional Fourier transform. Also we study the action of this FFT on classical convolution product of functions and on the one due to Miana obtaining a fractional version of the Theorem of the interchange. In Section 3 the Riemann-Liouville operators are presented as convolution, classical convolution or the one introduced by Miana, with the singular kernel of Riemann-Liouville. We study the action of our FFT on them.

2. Preliminaires

We start by recalling some elementary definitions.

Definition 1. Let $u = u(t)$ be a function of the space $S(\mathbb{R})$ the Schwartzian space of functions that decay rapidly at infinity together with all derivatives.

The Fourier transform $\hat{u}(\omega)$ is given by the integral

$$\hat{u}(\omega) = \mathfrak{F}[u](\omega) = \int_{\mathbb{R}} u(t)e^{i\omega t}dt \tag{2.1}$$

and the inverse Fourier transform can by given as

$$\mathfrak{F}^{-1}[\hat{u}](t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}(\omega)e^{-i\omega t}d\omega \tag{2.2}$$

Definition 2. Let $V(\mathbb{R})$ be the set of functions

$$V(\mathbb{R}) = \left\{ v \,\epsilon\, S(\mathbb{R}) : v^{(n)}(0) = 0,\ n = 0, 1, 2, ... \right\}. \tag{2.3}$$

The Lizorkin space of functions $\phi(\mathbb{R})$ is defined as

$$\phi(\mathbb{R}) = \{\varphi \,\epsilon\, S(\mathbb{R}) : \mathfrak{F}[\varphi] \,\epsilon\, V(\mathbb{R})\}. \tag{2.4}$$

Taking into account the well known property of the Fourier transform of a derivative, we have

$$\begin{aligned}\int_{\mathbb{R}} x^n \varphi(x)dx &= \frac{1}{i^n}\int_{\mathbb{R}} i^n x^n e^{ix0}\varphi(x)dx \\ = \frac{1}{i^n}\int_{\mathbb{R}} (ix)^n e^{ix0}\varphi(x)dx &= \frac{1}{i^n}(\hat{\varphi})^{(n)}(0) = 0.\end{aligned} \tag{2.5}$$

From (2.5) the Lizorkin space may by characterized as the space of Schwartzian functions φ which are ortogonal to all polynomials.

Definition 3. Let f and g be functions belonging to $L^1(\mathbb{R}^+)$, the usual or classical convolution product is given by

$$(f * g)(t) = \int_0^t f(x-t)g(x)dx, \quad t > 0. \tag{2.6}$$

Definition 4. Let f and g be functions belonging to $L^1(\mathbb{R}^+)$. Miana in [5] introduce the convolution product $\circ$ as the integral

$$(f \circ g)(t) = \int_t^\infty f(x-t)g(x)dx, \quad t \geq 0. \tag{2.7}$$

2.1. Fractional Fourier Transform

Definition 5. Let u be a function belonging to $\phi(\mathbb{R})$.
The fractional Fourier transform (FFT) of order α, $0 < \alpha \leq 1$, is defined as

$$\hat{u}_\alpha(\omega) = \mathfrak{F}_\alpha[u](\omega) = \int_{\mathbb{R}} e^{i\omega^{1/\alpha}t} u(t)dt. \tag{2.8}$$

It may by observed that if $\alpha = 1$, (2.6)reduces at the conventionally Fourier transform given by (I.1), when $\omega > 0$ it reduce to the FFT introduced by LMT cf[3].

The connection between both conventionally and our fractional Fourier transformation is given by the following

Lemma. *Let u be a function of the space $\phi(\mathbb{R})$, let α be a real number, $0 < \alpha \leq 1$, then*

$$\mathfrak{F}_\alpha[u](\omega) = \mathfrak{F}[u](x), \text{ for } x = \omega^{1/\alpha} \tag{2.9}$$

Proof. Follows from the definitions (2.1) and (2.6). □

Now, we are able to find the inversion formula for the TFF. In fact. Let u a function belonging to $\phi(\mathbb{R})$. From (2.7) we may writte

$$\mathfrak{F}_\alpha[u](\omega) = \mathfrak{F}[u](x) = g_1(x), \text{ for } x = \omega^{1/\alpha}.$$

Then

$$u(t) = \mathfrak{F}_\alpha^{-1}(\mathfrak{F}_\alpha[u]) = \mathfrak{F}^{-1}(g_1(x))(t).$$

Taking into account (2.2) we have

$$\mathfrak{F}^{-1}(g_1(x))(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixt} g_1(x)dx = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixt}\hat{u}(x)dx. \tag{2.10}$$

By (1.7), $x = \omega^{1/\alpha}$, and $dx = \frac{1}{\omega}\ \omega^{\frac{1}{\alpha}-1}d\omega$, then

$$\begin{aligned}\mathfrak{F}^{-1}(g_1(x))(t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega^{1/\alpha}t}\hat{u}_\alpha(\omega)\ \frac{1}{\alpha}\ \omega^{\frac{1-\alpha}{\alpha}}d\omega \\ &= \frac{1}{2\pi\alpha}\int_{\mathbb{R}} e^{-i\omega^{1/\alpha}t}\hat{u}_\alpha(\omega)\omega^{\frac{1-\alpha}{\alpha}}d\omega\end{aligned} \tag{2.11}$$

Formula (1.9) allows as to define the inverse fractional Fourier transform of order α of a function $u \ \epsilon \ \phi(\mathbb{R})$.

Definition 6. Let u be a function of $\phi(\mathbb{R})$, $0 < \alpha \leq 1$.
The inverse fractional Fourier transform of order α is given by

$$\mathfrak{F}_\alpha^{-1}(\hat{u}_\alpha(\omega))(t) = \frac{1}{2\pi\alpha}\int_{\mathbb{R}} e^{-i\omega^{1/\alpha}t}\hat{u}_\alpha(\omega)\omega^{\frac{1-\alpha}{\alpha}}d\omega. \tag{2.12}$$

It may be observed that when $\alpha = 1$, (2.10) reduces to (2.2) the conventionally inverse Fourier transform.

Remark. Making the change of variable $x = \omega^{1/\alpha}$, and taking into account the formula that establish the relationship between the conventional and the fractional Fourier transform, easily we can prove that

$$\mathfrak{F}_\alpha^{-1}[\mathfrak{F}_\alpha[u]] = u, \tag{2.13}$$

i.e.

$$\mathfrak{F}_\alpha^{-1}[\mathfrak{F}_\alpha] = Id, \tag{2.14}$$

where Id denote the identity operator.

2.2. Properties of the Fractional Fourier Transform

Let u be a function belong of the Lizorkin's space $\phi(\mathbb{R})$, and $x \in \mathbb{R}$. Then, from (2.9) easily we may prove the validity of the following statements:

1. $\mathfrak{F}_\alpha \left[u(t-x)\right](\omega) = e^{i\omega^{1/\alpha}x}\ \mathfrak{F}_\alpha \left[u\right](\omega)$;
2. $\mathfrak{F}_\alpha \left[\frac{d}{dt}u(t)\right](\omega) = -i\omega^{1/\alpha}\mathfrak{F}_\alpha \left[u(t)\right](\omega)$;
3. $\mathfrak{F}_\alpha \left[\frac{d^n}{dt^n}u(t)\right](\omega) = (-i\omega^{1/\alpha})^n\mathfrak{F}_\alpha \left[u(t)\right](\omega)$;
4. $\frac{d}{d\omega}\mathfrak{F}_\alpha \left[u(t)\right](\omega) = \frac{\omega^{\frac{1-\alpha}{\alpha}}i}{\alpha}\mathfrak{F}_\alpha \left[t\ u(t)\right](\omega)$.

Proof.

1. From Definition 5 we have

$$\mathfrak{F}_\alpha \left[u(t-x)\right](\omega) = \int_{\mathbb{R}} e^{i\omega^{1/\alpha}t}u(t-x)dt,$$

and making the change of variable $v = t - x$, we obtain

$$\mathfrak{F}_\alpha \left[u(t-x)\right](\omega) = \int_{\mathbb{R}} e^{i\omega^{1/\alpha}(v+x)}u(v)dv$$
$$= \int_{\mathbb{R}} e^{i\omega^{1/\alpha}v}e^{i\omega^{1/\alpha}x}u(v)dv = e^{i\omega^{1/\alpha}x}\ \mathfrak{F}_\alpha \left[u\right](\omega)$$

2. The item 2 is a particular case of the item 3 for $n = 1$. We will prove the item 3

3. By Definition 5 we have

$$\mathfrak{F}_\alpha \left[\frac{d^n}{dt^n}u(t)\right](\omega) = \int_{\mathbb{R}} e^{i\omega^{1/\alpha}t}\frac{d^n}{dt^n}u(t)dt.$$

By integrating by parts we obtain

$$\mathfrak{F}_\alpha \left[\frac{d^n}{dt^n}u(t)\right](\omega) = \int_{\mathbb{R}} e^{i\omega^{1/\alpha}t}\frac{d^n}{dt^n}u(t)dt$$
$$= \left[e^{i\omega^{1/\alpha}t}\frac{d^{n-1}}{dt^{n-1}}u(t)\right]_{-\infty}^{\infty} - \int_{\mathbb{R}} i\omega^{1/\alpha}e^{i\omega^{1/\alpha}t}\frac{d^{n-1}}{dt^{n-1}}u(t)dt$$
$$= (-i\omega^{1/\alpha})\mathfrak{F}_\alpha \left[\frac{d^{n-1}}{dt^{n-1}}u(t)\right](\omega).$$

Remark. $\left[e^{i\omega^{1/\alpha}t}\frac{d^{n-1}}{dt^{n-1}}u(t)\right]_{-\infty}^{\infty} = 0$, because $\frac{d^{n-1}}{dt^{n-1}}u(t) \in \phi(\mathbb{R})$.

Integrating by parts n-times, we obtain

$$\mathfrak{F}_\alpha\left[\frac{d^n}{dt^n}u(t)\right](\omega) = (-i\omega^{1/\alpha})^n\ \mathfrak{F}_\alpha\left[u(t)\right](\omega).$$

4. Applying Definition 5 and differentiating under the integral sing we obtain

$$\frac{d}{d\omega}\mathfrak{F}_\alpha\left[u(t)\right](\omega) = \frac{d}{d\omega}\int_{\mathbb{R}} e^{i\omega^{1/\alpha}t}u(t)dt$$

$$= \int_{\mathbb{R}} \frac{d}{d\omega}e^{i\omega^{1/\alpha}t}u(t)dt = \int_{\mathbb{R}} (it)\ 1/\alpha\ \omega^{1/\alpha-1}e^{i\omega^{1/\alpha}t}u(t)dt$$

$$= \frac{\omega^{\frac{1-\alpha}{\alpha}}i}{\alpha}\int_{\mathbb{R}} te^{i\omega^{1/\alpha}t}u(t)dt = \frac{\omega^{\frac{1-\alpha}{\alpha}}i}{\alpha}\mathfrak{F}_\alpha\left[t\ u(t)\right](\omega)$$

□

2.3. Convolution Products

We known that the Fourier transform applies the convolution product to a point wise product, then we have

$$\mathfrak{F}[f * g] = \mathfrak{F}[f].\mathfrak{F}[g] \tag{2.15}$$

Now, we obtain an analogue relation for our FFT that are given by the following

Theorem 1. *Let f and g be functions belonging to $\phi(\mathbb{R})$. Then*

$$\mathfrak{F}_\alpha[f * g] = \mathfrak{F}_\alpha[f].\mathfrak{F}_\alpha[g]. \tag{2.16}$$

Proof. By the definition we have

$$\mathfrak{F}_\alpha[f * g] = \int_{\mathbb{R}} e^{i\omega^{1/\alpha}t}\int_{\mathbb{R}} f(x-t)g(x)dxdt \tag{2.17}$$

Applying the Fubini's theorem and making the change of variable $u = t - x$, we obtain

$$\mathfrak{F}_\alpha[f * g] = \int_{\mathbb{R}} g(x)\int_{\mathbb{R}} e^{i\omega^{1/\alpha}(u+x)}f(u)dudx$$

$$= \int_{\mathbb{R}} g(x) e^{i\omega^{1/\alpha} x} \int_{\mathbb{R}} e^{i\omega^{1/\alpha} u} f(u) du dx$$

$$= \mathfrak{F}_\alpha[f].\mathfrak{F}_\alpha[g],$$

which is the thesis. □

If we consider the convolution $\circ$ we have

Theorem 2. *Let f and g be functions belonging to $\phi(\mathbb{R})$, $0 < \alpha \leq 1$. Then*

$$\mathfrak{F}_\alpha[f \circ g](\omega) = \left(\int_0^{+\infty} e^{i(-\omega^{1/\alpha})u} f(u) du \right) (\mathfrak{F}_\alpha g)(\omega) \tag{2.18}$$

Proof. From Definition 4 we have

$$\mathfrak{F}_\alpha[f \circ g](\omega) = \int_{\mathbb{R}} e^{i\omega^{1/\alpha} t} (f \circ g)(t) dt$$

$$= \int_{\mathbb{R}} e^{i\omega^{1/\alpha} t} \int_0^{+\infty} f(s-t) g(s) ds dt$$

$$= \int_{\mathbb{R}} \int_0^{+\infty} e^{i\omega^{1/\alpha} t} f(s-t) g(s) ds dt. \tag{2.19}$$

Applying Fubini's Theorem and making the change of variable $u = s - t$, we have

$$\mathfrak{F}_\alpha[f \circ g](\omega) = \int_{\mathbb{R}} g(s) \int_0^{+\infty} e^{i\omega^{1/\alpha}(s-u)} f(u) du ds$$

$$= \int_{\mathbb{R}} e^{i\omega^{1/\alpha} s} g(s) ds \int_0^{+\infty} e^{i(-\omega^{1/\alpha})u} f(u) du$$

$$= \left(\int_0^{+\infty} e^{i(-\omega^{1/\alpha})u} f(u) du \right) (\mathfrak{F}_\alpha g)(\omega),$$

that is the thesis of Theorem 2. □

Particular Case. if $Dom f = \mathbb{R}^+$ it results

$$\mathfrak{F}_\alpha[f \circ g](\omega) = \mathfrak{F}[f](-\omega^{1/\alpha}).\mathfrak{F}_\alpha[g](\omega) \tag{2.20}$$

3. Fractional Fourier Transform of Fractional Riemann-Liouville Operators

In this last section we consider the Riemann-Liouville fractional operators and we show the results of applies our FFT to them.

Previously we need some elementary definitions and results.

Definition. Let u be a locally integrable function on (a,∞). The Riemann-Liouville integral of order α, $0<\alpha\leq 1$ of the function u is given by

$$ {}_aI_x^\alpha u(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}u(t)dt. \tag{3.1}$$

Analogously for f a locally integrable function on $(-\infty,b)$ we have

$$ {}_xI_b^\alpha u(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}u(t)dt \tag{3.2}$$

When $a=-\infty$ in (3.1), and $b=+\infty$ in (3.2) and if u is a function of the space $S(\mathbb{R})$, we have

$$ {}_{-\infty}I_x^\alpha u(x) = {}_{-\infty}W_x^\alpha u(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-t)^{\alpha-1}u(t)dt \tag{3.3}$$

and

$$ {}_xI_{+\infty}^\alpha u(x) = {}_xW_{+\infty}^\alpha u(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}u(t)dt \tag{3.4}$$

that usually are known as Weyl fractional integrals of order α. cf [2], [6] and [7]. Here, $\Gamma(\alpha)$ denotes the Euler Gamma function defined by

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt, \qquad Re(z)>0$$

For $\alpha>1$, and $t>0$, let $j_\alpha(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, be the singular kernel of Riemann-Liouville.

Taking into account (3.1), (3.2) we may writte

$$ {}_{-\infty}I_x^\alpha u(x) = \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} * u\right)(x), \tag{3.5}$$

where $*$ denotes the classical convolution, and

$$ {}_xI_{+\infty}^\alpha u(x) = \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} \circ u\right)(x), \tag{3.6}$$

where $\circ$ denotes the convolution defined by (2.7).

Frequently the integral ${}_{-\infty}I_x^{\alpha}u(x)$ is denoted $I_+^{\alpha}u(x)$, and ${}_xI_{+\infty}^{\alpha}u(x)$ is denoted $I_-^{\alpha}u(x)$.

From this expression of the fractional integrals we are able to evaluate its fractional Fourier transform. In fact we have the following

Lemma 1. *Let u be a function belonging to $\phi(\mathbb{R})$, the Lizorkin space. For $0 < \alpha \leq 1$, $0 < \beta \leq 1$ and $\omega \neq 0$. Then*

$$\mathfrak{F}_{\alpha}[I_+^{\beta}u](\omega) = |\omega|^{-\beta/\alpha} c_{\beta} \mathfrak{F}_{\alpha}[u(t)](\omega), \tag{3.7}$$

where

$$c_{\beta} = \cos\left(\frac{\beta\pi}{2}\right) + i \ \mathrm{sgn}\,(\omega^{1/\alpha}) sen\left(\frac{\beta\pi}{2}\right) \tag{3.8}$$

Proof. From the definition (2.8) we have

$$\mathfrak{F}_{\alpha}[I_+^{\beta}u](\omega) = \mathfrak{F}_{\alpha}\left[\frac{t^{\beta-1}}{\Gamma(\beta)} * u\right](\omega) = \mathfrak{F}_{\alpha}\left[\frac{t^{\beta-1}}{\Gamma(\beta)}\right](\omega).\mathfrak{F}_{\alpha}\left[u(t)\right](\omega) \tag{3.9}$$

Then, we need evaluate the FFT of the singular Riemann-Liouville kernel $j_{\beta}(t)$. Then, we have

$$\mathfrak{F}_{\alpha}\left[\frac{t^{\beta-1}}{\Gamma(\beta)}\right](\omega) = \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}} e^{i\omega^{1/\alpha}t} t^{\beta-1} dt =$$

$$= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} e^{i\omega^{1/\alpha}t} t^{\beta-1} dt =$$

$$= \frac{1}{\Gamma(\beta)} \left[\int_0^{+\infty} \cos\left(\omega^{1/\alpha}t\right) t^{\beta-1} dt + i \int_0^{+\infty} sen\left(\omega^{1/\alpha}t\right) t^{\beta-1} dt\right] \tag{3.10}$$

Applying formulae (3.761.9) from [1] we have

$$\int_0^{\infty} \cos(\omega^{1/\alpha}t) t^{\beta-1} dt = \Gamma(\beta) \left|\omega^{1/\alpha}\right|^{-\beta} \cos\left(\frac{\beta\pi}{2}\right) \tag{3.11}$$

and by formulae (3.761.4) from [1] we have

$$\int_0^{\infty} sen(\omega^{1/\alpha}t) t^{\beta-1} dt = \Gamma(\beta) \left|\omega^{1/\alpha}\right|^{-\beta} sen\left(\frac{\beta\pi}{2}\right) \mathrm{sgn}\,(\omega^{1/\alpha}) \tag{3.12}$$

Replacing (3.11) and (3.12) in (3.10) it result:

$$\mathfrak{F}_\alpha\left[\frac{t^{\beta-1}}{\Gamma(\beta)}\right](\omega) = \frac{1}{\Gamma(\beta)}\left[|\omega^{1/\alpha}|^{-\beta}\cos\left(\frac{\beta\pi}{2}\right)\Gamma(\beta)\right.$$
$$\left. +i\ \mathrm{sgn}\,(\omega^{1/\alpha})|\omega^{1/\alpha}|^{-\beta} sen\left(\frac{\beta\pi}{2}\right)\Gamma(\beta)\right]$$
$$= |\omega|^{-\beta/\alpha}\left[\cos\left(\frac{\beta\pi}{2}\right) + i\ \mathrm{sgn}\,(\omega^{1/\alpha}) sen\left(\frac{\beta\pi}{2}\right)\right]. \quad (3.13)$$

Then, replacing (III.10) in (III.9), we obtain

$$\mathfrak{F}_\alpha[I_+^\beta u](\omega) = |\omega|^{-\beta/\alpha}\left[\cos\left(\frac{\beta\pi}{2}\right)\right.$$
$$\left. +i\ \mathrm{sgn}\,(\omega^{1/\alpha}) sen\left(\frac{\beta\pi}{2}\right)\right]\mathfrak{F}_\alpha[u(t)](\omega). \quad (3.14)$$

□

Particular Case. If $\beta = \alpha$ we obtain

$$\mathfrak{F}_\alpha[I_+^\alpha u](\omega) = \frac{1}{\Gamma(\alpha)}\left[|\omega^{1/\alpha}|^{-\alpha}\right.$$
$$\left.\times\cos\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha) + i\mathrm{sgn}\,(\omega^{1/\alpha})|\omega^{1/\alpha}|^{-\alpha} sen\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha)\right]\mathfrak{F}_\alpha[u(t)](\omega)$$
$$= |\omega|^{-1}\left[\cos\left(\frac{\alpha\pi}{2}\right) + i\mathrm{sgn}\,(\omega^{1/\alpha}) sen\left(\frac{\alpha\pi}{2}\right)\right]\mathfrak{F}_\alpha[u(t)](\omega). \quad (3.15)$$

Remark. If $\omega = 0$, taking into account that $I_+^\alpha u\ \epsilon\ \phi(\mathbb{R})$, it results

$$\mathfrak{F}_\alpha[I_+^\beta u](0) = \mathfrak{F}[I_+^\beta u](0) = 0. \quad (3.16)$$

Lemma 2. *Let u be a function belonging to $\phi(\mathbb{R})$, the Lizorkin space. For $0 < \alpha \leq 1$, $0 < \beta \leq 1$ and $\omega \neq 0$ its result*

$$\mathfrak{F}_\alpha[I_-^\beta u](\omega) = |\omega|^{-\beta/a} c_\beta \mathfrak{F}_\alpha[u(t)](\omega), \quad (3.17)$$

where

$$c_\beta = \cos\left(\frac{\beta\pi}{2}\right) - i\ \mathrm{sgn}\,\omega^{1/\alpha} sen\left(\frac{\beta\pi}{2}\right) \quad (3.18)$$

Proof. From the definition formula (II.8) and from (II.18), by Theorem 2 we have

$$\mathfrak{F}_\alpha[I_-^\beta u](\omega) = \mathfrak{F}_\alpha\left[\frac{t^{\beta-1}}{\Gamma(\beta)}\circ u\right](\omega) = \mathfrak{F}\left[\frac{t^{\beta-1}}{\Gamma(\beta)}\right](-\omega^{1/\alpha}).\mathfrak{F}_\alpha\left[u(t)\right](\omega) \quad (3.19)$$

Now, we need the Fourier transform of the kernel $j_\beta = \frac{t^{\beta-1}}{\Gamma(\beta)}$, evaluated at the point $(-\omega^{1/\alpha})$. Then

$$\mathfrak{F}\left[\frac{t^{\beta-1}}{\Gamma(\beta)}\right](-\omega^{1/\alpha}) = \frac{1}{\Gamma(\beta)}\int_{\mathbb{R}} e^{i(-\omega^{1/\alpha})t}t^{\beta-1}dt$$

$$= \frac{1}{\Gamma(\beta)}\int_0^{+\infty} e^{i(-\omega^{1/\alpha})t}t^{\beta-1}dt$$

$$= \frac{1}{\Gamma(\beta)}\left[\int_0^{+\infty}\cos\left(\omega^{1/\alpha}t\right)t^{\beta-1}dt - i\int_0^{+\infty} sen\left(\omega^{1/\alpha}t\right)t^{\beta-1}dt\right]$$

$$= \frac{1}{\Gamma(\beta)}\left[|\omega^{1/\alpha}|^{-\beta}\cos\left(\frac{\beta\pi}{2}\right)\Gamma(\beta) - i\ \mathrm{sgn}\,(\omega^{1/\alpha})|\omega^{1/\alpha}|^{-\beta} sen\left(\frac{\beta\pi}{2}\right)\Gamma(\beta)\right]$$

$$= |\omega|^{-\beta/\alpha}\left[\cos\left(\frac{\beta\pi}{2}\right) - i\mathrm{sgn}\,(\omega^{1/\alpha}) sen\left(\frac{\beta\pi}{2}\right)\right]. \tag{3.20}$$

Then, replacing (3.17) in (3.16), we have

$$\mathfrak{F}_\alpha[I_-^\beta u](\omega) = |\omega|^{-\beta/\alpha}\left[\cos\left(\frac{\beta\pi}{2}\right) - i\mathrm{sgn}\,(\omega^{1/\alpha}) sen\left(\frac{\beta\pi}{2}\right)\right]\mathfrak{F}_\alpha[u(t)](\omega). \tag{3.21}$$

□

Particular Case. If $\beta = \alpha$ we obtain

$$\mathfrak{F}_\alpha[I_-^\alpha u](\omega) =$$

$$\frac{1}{\Gamma(\alpha)}\left[|\omega^{1/\alpha}|^{-\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha) - i\mathrm{sgn}\,(\omega^{1/\alpha})|\omega^{1/\alpha}|^{-\alpha} sen\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha)\right]\mathfrak{F}_\alpha[u(t)](\omega)$$

$$= |\omega|^{-1}\left[\cos\left(\frac{\alpha\pi}{2}\right) - i\mathrm{sgn}\,(\omega^{1/\alpha}) sen\left(\frac{\alpha\pi}{2}\right)\right]\mathfrak{F}_\alpha[u(t)](\omega). \tag{3.22}$$

Remark. If $\omega = 0$, taking into account that $I_-^\alpha u \in \phi(\mathbb{R})$, it results

$$\mathfrak{F}_\alpha[I_-^\beta u](0) = \mathfrak{F}[I_-^\beta u](0) = 0. \tag{3.23}$$

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