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ON THE X-RANKS OF POINTS ON $\mathbb{P}^n$ ($X \subset \mathbb{P}^n$ A CURVE), THE TANGENT DEVELOPABLE OF X, AND THE UNIQUENESS OF SETS $S \subset X$ COMPUTING X-RANKS

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Abstract: Let $X \subset \mathbb{P}^n$, $n \geq 4$, be an integral projective curve. Let $\tau(X) \subset \mathbb{P}^n$ be the tangent developable of X. For each $P \in \mathbb{P}^n$ let $r_X(P)$ be the minimal cardinality of a set $S \subset X$ spanning X and $\mathcal{S}(X,P)$ the set of all $S \subset X$ computing $r_X(P)$. Here we give many cases with $\sharp(\mathcal{S}(X,P)) = 1$ and show that if X is smooth, then $r_X(P) \geq 3$ for a general $P \in \tau(X)$.

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1. Introduction

Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. We start with a question for which we do not know the answer.

Question 1. Let $Y \subset \mathbb{P}^n$, $n \geq 3$, be an integral and non-degenerate curve. Is it true that $(T_QY \cap Y)_{red} = \{Q\}$ for a general $Q \in X$?

H. Kaji proved the result for smooth curves (see [6]). Easy examples show that Question 1 fails in positive characteristic, even for smooth curves. It would be nice to have easy geometric or cohomological conditions on Y which impliy an affermative answer to Question 1 for that curve (in positive characteristic); even for smooth curves it would be important. The higher dimensional version

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of Question 1 is not true, even for smooth varieties (see [7]).

Now we explain why we raised this classical question (here again $X \subset \mathbb{P}^n$ is an integral and non-degenerate subvariety). For any $P \in \mathbb{P}^n$ let $r_X(P)$ be the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$. The integer $r_X(P)$ is called the X-rank. Let $\mathcal{S}(X, P)$ be the set of all $S \subset X$ computing $r_X(P)$, i.e. the set of all subsets $S \subset X$ such that $\sharp(S) = r_X(P)$ and $P \in \langle S \rangle$. Notice that any $S \in \mathcal{S}(X, P)$ is linearly independent and $P \notin \langle S' \rangle$ for any $S' \subsetneq S$. An important problem is to find conditions on X and P such that $\sharp(\mathcal{S}(X, P)) = 1$ (see [3], Theorem 1.2.6 and Remark 1.2.7). Let $\tau(X) \subset \mathbb{P}^n$ denote the tangent developable of X, i.e. the closure in $\mathbb{P}^n$ of the union of all tangent spaces $T_Q X$, $Q \in X_{reg}$. For every integer $t \geq 1$ let $\sigma_k(X)$ denote the closure in $\mathbb{P}^n$ of all $(k-1)$-dimensional linear spaces spanned by t points of Y. Set $\sigma_0(X) := \emptyset$. If $\sigma_{k-1}(X) \neq \mathbb{P}^n$, then a general $P \in \sigma_k(X)$ has X-rank k. Set $\sigma_k^0(X) := \{P \in \sigma_k(X) \setminus \sigma_{k-1}(X) := r_X(P)\}$. Very seldom $\sharp(\mathcal{S}(X, P)) = 1$ for a general $P \in \mathbb{P}^n$ (it happens for the rational normal curve); necessary conditions are that X is not defective and that $t := (n+1)/(\dim(X)+1) \in \mathbb{N}$; under this assumption the uniqueness of $\mathcal{S}(X, P)$ is equivalent to having degree 1 the morphism from the abstract join of t copies of X to the embedded join, i.e. to $\sigma_t(X) = \mathbb{P}^n$; these varieties are studied but no general classification is in sight.

Remark 1. Now we fix an integer $k \geq 2$ such that $k(\dim(X)+1) \leq n$. There is an easy to check condition which implies that $\sharp(S(X, P)) = 1$ for a general $P \in \sigma_k(X)$: it is sufficient to assume that X is not weakly $(k-1)$-defective (see [5], Proposition 1.5). No curve is weakly defective (see [4] or [5], Remark 1.2). Hence if $\dim(X) = 1$, then the generic uniqueness holds (i.e. $\sharp(\mathcal{S}(X, P)) = 1$) for a general $P \in \sigma_k(X)$ if $2k \leq n$.

For a refinement of this uniqueness remark, see Section 2. Theorem 1 below is related to another uniqueness result, if we allow degree 2 connected subschemes of X_{reg}. Fix $Q \in X_{reg}$ and any $P \in T_Q X \setminus \{Q\}$. The line $\langle \{P, Q\} \rangle$ intersects X in a scheme containing a degree 2 connected component with Q as its reduction (a tangent vector). Thus a general $P \in \tau(X)$ is contained in a line spanned by an unreduced degree 2 subscheme of X.

Using Kaji's Theorem for smooth curves we are able to prove the following result.

Theorem 1. *Let $C \subset \mathbb{P}^n$, $n \geq 4$, be a smooth and non-degenerate curve. Then $r_C(P) \geq 3$ for a general $P \in \tau(X)$.*

See Corollary 2 for the corresponding result for singular curves (in which we cannot use an affirmative answer to Question 1).

Then we introduce a related topic concerning the uniqueness of the sets computing $r_X(P)$, i.e. when $\sharp(\mathcal{S}(X,P)) = 1$ (see Section 2)

2. Refinements of the Uniqueness Part

Notation 1. Take $X \subset \mathbb{P}^n$ with arbitrary dimension. Let $\alpha(X)$ be the maximal integer such that any subset of X with cardinality at most $\alpha(X)$ is linearly independent.

The proof of the following lemma is just the proof of [1], Lemma 1.

Lemma 1. *Fix $P \in \mathbb{P}^n$ such that there are two zero-dimensional subschemes A, B of X with the following properties: $A \neq B$, $P \in \langle A\rangle \cap \langle B\rangle$, $P \notin \langle A'\rangle$ for any subscheme $A' \subsetneqq A$ and $P \notin \langle B'\rangle$ for any subscheme $B' \subsetneqq B$. Then $h^1(\mathbb{P}^n, \mathcal{I}_{A\cup B}(1)) > 0$.*

Corollary 1. *X of arbitrary dimension. For every integer $t \le \lfloor \alpha(X)/2 \rfloor$ and every $P \in \sigma_t^0(X)$ we have $\sharp(\mathcal{S}(X,P)) = 1$.*

Proof. Assume the existence of $A, B \in \mathcal{S}(X,P)$ such that $A \neq B$. Since $\deg(A \cup B) \le 2t \le \alpha(X)$, Lemma 1 gives a contradiction. □

If X is a Veronese embedding, then Corollary 2 is just [3], Theorem 1.2.6.

From now on we assume that X is a smooth curve, C.

Notation 2. Let $C \subset \mathbb{P}^n$ be a smooth, connected and non-degenerate curve. Let $\beta(C)$ be the maximal integer such that every zero-dimensional subscheme of C with degree at most $\beta(C)$ is linearly independent.

Proposition 1. *Fix an integer $k \le \lfloor \beta(C)/2 \rfloor$ and any $P \in \sigma_k(C) \setminus \sigma_{k-1}(C)$. Then there exists a unique zero-dimensional scheme $Z \subset C$ such that $\deg(Z) \le k$ and $P \in \langle Z\rangle$. Moreover $\deg(Z) = k$ and $P \notin \langle Z'\rangle$ for all $Z' \subsetneqq Z$.*

Proof. The existence part is stated in [2], Lemma 1, which in turn is just an adaptation of some parts of the beautiful paper [3]. The uniqueness part is true by Lemma 1 and the definition of the integer $\beta(C)$. □

Fix an integer $k \ge 2$. Let $\mathrm{Hilb}^k(C)$ denote the set of all zero-dimensional subschemes of degree k of the smooth curve C. It is a smooth and integral projective variety of dimension k. Let $A(k)$ be the set of all non-decreasing sequences $w_1 \ge w_2 \ge \cdots \ge w_s > 0$ such that $\sum_{i=1}^{s} w_i = k$. For each such sequence $\underline{w} = (w_1, \dots, w_s, 0, \dots)$ set $\ell(\underline{w}) = s$ (the length of $\underline{w}$). For any $Z \in \mathrm{Hilb}^k(C)$ such that $\deg(Z) = k$ there is a unique $\underline{w} = (w_1, \dots, w_s) \in A(k)$ such that the connected components of Z have degree $w_1, \dots, w_s$. We will say that $\ell(\underline{w})$ is the type of Z. Let $\mathrm{Hilb}^k(C)(\underline{w})$ be the subset of $\mathrm{Hilb}^k(C)$ formed by all subschemes with type $\underline{w}$. The set $\mathrm{Hilb}^k(C)(\underline{w})$ is a locally closed and irreducible subvariety of $\mathrm{Hilb}^k(C)$ and $\dim(\mathrm{Hilb}^k(C)(\underline{w})) = \ell(\underline{w})$. We get a partition

$$\mathrm{Hilb}^k(C) = \sqcup_{\underline{w}\in A(k)}\mathrm{Hilb}^k(C)(\underline{w}).$$

Set $\sigma_k(C)(\underline{w}) := \{P \in \sigma_k(C) \setminus \sigma_{k-1}(C) : Z \in \mathrm{Hilb}^k(C)(\underline{w}) \text{ such that } P \in \langle Z\rangle\}$. By [2], Lemma 1, we have

$$\sigma_k(C) \setminus \sigma_{k-1}(C) = \cup_{\underline{w}\in A(k)}\sigma_k(C)(\underline{w}).$$

Theorem 2. *Fix an integer $k \le \lfloor \beta(C)/2\rfloor$. We have an algebraic partition*

$$\sigma_k(C) \setminus \sigma_{k-1}(C) = \sqcup_{\underline{w}\in A(k)}\sigma_k(C)(\underline{w})$$

and each $\sigma_k(C)(\underline{w})$ is irreducible and of dimension $k-1+\ell(\underline{w})$.

Proof. Use Proposition 1 to get that this union is a disjoint union. For the same reason two different elements $Z_1, Z_2 \in \sigma_k(C)(\underline{w})$ give disjoint and non-empty subsets of $\sigma_k(C)\setminus\sigma_{k-1}(C)$, i.e. $\langle Z_1\rangle \cap \langle Z_2\rangle \cap (\sigma_k(C)\setminus\sigma_{k-1}(C)) = \emptyset$. Each $Z \in \mathrm{Hilb}^k(C)$ is linearly independent. Since $\mathrm{Hilb}^k(C)(\underline{w})$ is a locally closed and irreducible subvariety of $\mathrm{Hilb}^k(C)$ and $\dim(\mathrm{Hilb}^k(C)(\underline{w})) = \ell(\underline{w})$, we get $\dim(\sigma_k(C)(\underline{w})) = k-1+\ell(\underline{w})$. $\square$

3. Proof of Theorem 1

Proposition 2. *Let $X \subset \mathbb{P}^n$, $n \ge 4$, be an integral projective curve.*

(a) For a general $Q \in X_{reg}$ the tangential projection of X from the line T_QX is birational onto its image.

(b) Take a general $(P_1, P_2) \in X_{reg}\times X_{reg}$. Then there is no $P_3 \in X_{reg}$ such that the 3-dimensional linear subspace $\langle T_{P_1} \cup T_{P_2}X\rangle$ contains a tangent line $T_{P_3}X$ with $P_3 \in X_{reg}\setminus\{P_1, P_2\}$.

Proof. In the set-up of part (b) the linear space $\langle T_{P_1}\cup T_{P_2}X\rangle$ has dimension 3, because X is not strange, by our characteristic zero assumption. Since $n \ge 4$ and no curve is weakly defective, [4], Proposition 2.5, gives part (b). We will use part (b) to prove part (a). Fix a general $P_1 \in X_{reg}$ and assume that $\ell_{T_{P_1}X,X}$ is not birational onto it image. Thus for a general $P_2 \in X_{reg}$ there is $P_3 \in X_{reg}\setminus(\{P_2\}\cup(T_{P_1}X \cap X))$ such that $\ell_{T_{P_1}X,X}(P_2) = \ell_{T_{P_1}X,X}(P_3)$. Moreover, for general P_2 we also have $\ell_{T_{P_1}X,X}(T_{P_2}X) = \ell_{T_{P_1}X,X}(T_{P_3}X)$. Hence $T_{P_3}X \subset \langle T_{P_1}X\cup T_{P_2}X\rangle$. Part (b) gives a contradiction. $\square$

Proposition 3. *Fix $Q \in X_{reg}$. Let $\ell_{T_QX,X} : X \setminus X \cap T_QX \to \mathbb{P}^{n-2}$ denote the rational map induced by the linear projection from T_QX.*

(a) If $\ell_{T_QX,X}$ is birational onto its image and $(T_QX \cap X)_{red} = \{Q\}$, then $r_X(P) \geq 3$ for a general $P \in T_QX$.

(b) If $r_X(P) = 2$ for a general $P \in T_QX$ and $(T_QX \cap X)_{red} = \{Q\}$, then $\ell_{T_QX,X}$ is not birational onto its image.

(c) If $\sharp((T_XQ \cap X))_{red}) \geq 2$, then $r_X(P) \leq 2$ for all $P \in T_QX$.

Proof. Part (c) is obvious. Hence from now on we assume $(T_QX \cap X)_{red} = \{Q\}$. Fix $P \in T_QX \subset \{Q\}$. We have $r_X(P) = 2$ if and only if there are $A_P, B_P \in X$ such that $A_P \neq B_P$ and $P \in \langle\{A_P, B_P\}\rangle$. Since $(T_QX \cap X)_{red} = \{Q\}$ and $P \in T_QX$, we have $Q \notin \{A_P, B_P\}$. Hence if $r_X(P) = 2$ for infinitely many $P \in T_QX$, then $\ell_{T_QX,X}$ is not birational onto its image. Similarly part (a) follows by counting the dimensions of the lines (different from T_QX) intersecting X at at least 2 points and intersecting $T_QX \setminus \{Q\}$. □

Corollary 2. *A general $P \in \tau(X)$ has $r_X(P) = 2$ if and only if a general tangent line of X meets X at another point.*

Proof. Apply Propositions 3 and 2. □

Proof of Theorem 1. Apply Corollary 2 and the quoted theorem of Kaji (see [6]). □

Example 1. Let $F \subset \mathbb{P}^n$, $n \geq 4$, be an integral and non-degenerate 3-fold, which is ruled by planes. Let $X \subset F$ be an integral projective curve. Assume that for a general plane $\Pi \subset F$ the set $(\Pi \cap X)_{red}$ spans Π. X has a one-dimensional family of mutually intersecting pairs of secant lines. Hence X is non-degenerate and there is a one-dimension set $T \subset \sigma_2^0(X)$ such that $\sharp(\mathcal{S}(X,P)) \geq 2$ for all $P \in T$. Given any F as above we may construct several $X \subset F$ as above just taking the intersection of F with 2 sufficiently general hypersurfaces, none of them being a hyperplane.

Example 2. Assume that X is contained in a ruled surface F and that the general fiber of F intersects X in at least $x \geq 3$ points. Then there is a one-dimension subset Γ of $\sigma_2^0(X)$ such that $\sharp(\mathcal{S}(X,P)) \geq \binom{x}{2} \geq 3$.

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