International Journal of Pure and Applied Mathematics
Volume 66 No. 4 2011, 415-425

VOLUME IN EUCLIDEAN SPACES

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Abstract: Structuralism in mathematics is now a maturity method, describing itself as a powerful tool for ordering and classifying the basic knowledge of mathematical objects. After it mostly exhausted the study of structures as such, current mathematics witnesses a return to a basic model, with the rebirth of interest for the very practical. In addition, in cases of particular areas, many results that cannot be extended to the general structural can take place. On the other hand, results in the general structure can be obtained in specific areas through more natural methods, taking into account the particularities of these spaces.

AMS Subject Classification: 31D05, 60J45
Key Words: compact sets, volume, basis of compact neighborhoods, μ-division

1. Introduction

In this introduction we clarify some notation and define some basic concepts. Here $|\cdot|$ denotes the Euclidean norm throughout. $d_C(x)$ or $d(x, C)$ denotes the Euclidean distance of the point $x \in \mathbb{R}^n$ from the set $C \subset \mathbb{R}^n$, defined as

$$d_C(x) := \inf_{y \in C} |x - y|.$$

We use $d(A, B)$ to denote the distance between the sets A and B in $\mathbb{R}^n$, that is

$$d(A, B) := \inf\{|a - b| \mid a \in A \text{ and } b \in B\}.$$

It is known [7, Volume II, p. 133] that A is closed set and B is compact set,

Received: December 10, 2010

and there exist $a \in A$ and $b \in B$ such that $d(A,B) = |a-b|$ (otherwise, we will prove this result, combining more results from [3, p. 119], [6, p. 62] and [7, Volume I, p. 180]).

If $A \subset \mathbb{R}^n$ is a non-empty set, the diameter of A is defined as

$$\operatorname{diam}(A) := \sup\{|x-y| \mid x \in A \text{ and } y \in A\}.$$

If $\operatorname{diam}(A) < \infty$ we say that A is a bounded set.

We use $\operatorname{int}A$ to denote the set of interior points of the set A, and $\operatorname{cl}A$ to denote the closure of A. $\operatorname{Fr}A$ is the boundary of A, that is $\operatorname{Fr}A = \operatorname{cl}A \setminus \operatorname{int}A$.

Let $X \subset \mathbb{R}^n$. By $K(X)$ we denote the family of all compact subsets of X. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact subsets in X.

If $(K_n)_{n\in\mathbb{N}}$ is decreasing (in respect to the inclusions of sets) and $\bigcap_{n\in\mathbb{N}} K_n =: K$, then we denote this by $K_n \downarrow K$. Obviously K is a compact set.

If $K \in K(X)$ and $(K_n)_{n\in\mathbb{N}}$ is increasing (in respect to the inclusions of sets) and $K = \bigcup_{n\in\mathbb{N}} K_n$, then we denote this by $K_n \uparrow K$.

Counter Example. Let $B_n := \{x \in \mathbb{R}^n \mid |x| \le n\}$. Then $B_n \in K(\mathbb{R}^n)$ and $\bigcup_{n\in\mathbb{N}} B_n = \mathbb{R}^n$ is not compact.

If the $(t_n)_{n\in\mathbb{N}}$ is a monotone decreasing sequence of strictly positive numbers such that $t_n \to 0$ as $n \to \infty$, then we denote this by $t_n \downarrow 0$.

Throughout the present paper, for $t > 0$, Q_t and U_t are definite as:

$$Q_t := \{x \in \mathbb{R}^n \mid d(x,K) \le t\}$$

and

$$U_t := \{x \in \mathbb{R}^n \mid d(x,K) < t\}.$$

In the first instance we observe that $K \subset U_t \subset Q_t$.

Under the above notations our attention focuses on the following family:

$$(Q_{t_n})_{n\in\mathbb{N}}, \text{ where } t_n \downarrow 0. \tag{1.1}$$

We recall that $V \subset \mathbb{R}^n$ is said to be a neighborhood of A, if $A \subset \operatorname{int} V$.

Also, a family $\mathcal{B}$ of neighborhoods of $A \subset \mathbb{R}^n$ is called basis of neighborhoods of A if for every neighborhood V of A, then there exists $V_1 \in \mathcal{B}$ such that $V_1 \subset V$.

Also, it is known ([7, Volume I, p. 104], [8, p. 58], [3, p. 114]) that $K \subset \mathbb{R}^n$ is compact set if and only if K is closed and bounded set of $\mathbb{R}^n$.

2. Results

Definition 1. Let X be a subset of $\mathbb{R}^n$. We say that a function $\mu : K(X) \to \mathbb{R}$ is a volume on X if the following properties are fulfilled:

a) monotony: If $K, L \in K(X)$ and $K \subset L$, then $\mu(K) \leq \mu(L)$;

b) additivity: If $K, L \in K(X)$ then $\mu(K \cup L) \leq \mu(K) + \mu(L)$
and
if $K \cap L = \phi$ then $\mu(K \cup L) = \mu(K) + \mu(L)$;

c) regularity: If $K_n \downarrow K$ then $\mu(K_n) \downarrow \mu(K)$, as $n \to \infty$.

Example. 1) For $a \in \mathbb{R}^n$ we define $\mu_a : K(\mathbb{R}^n) \to \mathbb{R}$ by

$$\mu_a(K) = \begin{cases} 1 & \text{if } a \in K, \\ 0 & \text{otherwise.} \end{cases}$$

μ is called the volume Dirac concentred in a.

2) Let $X \subset \mathbb{R}^n$ be a discrete closed set. Then $K(X) = \{A \subset X \mid A \text{ is finite set}\}$. We define the volume $\mu : K(X) \to \mathbb{R}$ by $\mu(K) = \text{card} K$.

3) The trivial extension of a volume.

If X is closed set and μ is a volume on $X \subset \mathbb{R}^n$, we define the volume $\mu' : K(\mathbb{R}^n) \to \mathbb{R}$ by

$$\mu'(K) = \mu(K \cap X).$$

4) The restriction of a volume.

If $X \subset \mathbb{R}^n$ and μ is a volume on $\mathbb{R}^n$, then we define the volume μ on X by: $\mu'(K) = \mu(K)$ (we see that $K(X) \subset K(\mathbb{R}^n)$).

Consequently it is sufficient to study the volume on $\mathbb{R}^n$.

Lemma 1. *The following properties are fulfilled:*

(i) Q_t is a compact neighborhood of K, U_t is a open neighborhood of K, for every $t > 0$ and

$$\text{diam}(Q_t) \leq \text{diam}(K) + 2t;$$

(ii) $(Q_t)_{t>0}$ is a basis of compact neighborhoods of K;

(iii) $(Q_{t_n})_{n \in \mathbb{N}}$ is a countable basis of compact neighborhoods of K and $Q_{t_n} \downarrow K$.

Proof. (i) We define a real valued function φ on $\mathbb{R}^n$ by $\varphi(x) = d(x, K)$. Since [7, Volume I, p. 180] $|\varphi(x) - \varphi(y)| = |d(x, K) - d(y, K)| \leq d(x, y)$ for all $x, y \in \mathbb{R}^n$, we infer that φ is (Lipschitz and therefore) continuous. Thus

$$Q_t = \varphi^{-1}([0, t]) \text{ and } U_t = \varphi^{-1}((-1, t)),$$

that is Q_t is closed and U_t is open set.

Let us prove that Q_t is bounded set. Indeed, if $x, y \in Q_t$ there exist $a, b \in K$ such that $d(x, K) = d(x, a)$ and $d(y, K) = d(y, b)$ (as K is compact and $\{x\}, \{y\}$ are closed).

Thus $d(x, y) \leq d(x, a) + d(a, b) + d(b, y) \leq 2t + d(a, b)$, hence $\text{diam}(Q_t) \leq 2t + \text{diam}(k) < \infty$, that is Q_t is bounded set.

(ii) Follows immediately from (iii)

(iii) Denote $\mathcal{B} := \{Q_{t_n} \mid n \in \mathbb{N}\}$. Let be U an open neighborhood of K. Since K is compact and $\mathbb{R}^n \setminus U$ is closed, there [7, Volume II, p. 33] exists $a \in K$ and $b \in \mathbb{R}^n \setminus U$ such that

$$d(K, \mathbb{R}^n \setminus U) = d(a, b) > 0 \text{ [7, Volume II, p. 107]}.$$

Since $d(a, b) > 0$ and $t_n \downarrow 0$, there exists $n_0 \in \mathbb{N}$ such that $0 < t_{n_0} < d(a, b)$.

We check that $Q_{t_{n_0}} \subset U$ and $\bigcap_{n \in \mathbb{N}} Q_{t_n} = K$.

Indeed, if $x \in Q_{t_{n_0}}$, then $d(x, K) \leq t_{n_0} < d(a, b) = d(K, \mathbb{R}^n \setminus U)$ whence $x \notin \mathbb{R}^n \setminus U$, that is $x \in U$.

Now, let us prove that $\bigcap_{n \in \mathbb{N}} Q_{t_n} = K$.

One of the inclusions, namely $K \subset \bigcap_{n \in \mathbb{N}} Q_{t_n}$ holds immediately, because $K \subset Q_{t_n}$ for every $n \in \mathbb{N}$.

Reciprocally, from the assumption $x \in \bigcap_{n \in \mathbb{N}} Q_{t_n}$, it follows that for all $n \in \mathbb{N}$, $d(x, K) < t_n$, that is $d(x, K) = 0$ (as $t_n \downarrow 0$), whence $x \in \text{cl}K = K$, and this ends the proof. □

Corollary 1. *If μ is a volume on $\mathbb{R}^n$, K is a compact subset of $\mathbb{R}^n$ and $\varepsilon > 0$ is given, then there exists Q a compact neighborhood of K such that $0 \leq \mu(Q) - \mu(K) < \varepsilon$.*

Indeed, we consider the sequence $(Q_{t_n})_n$ from the Lemma 1 above, so $\mu(Q_{t_n}) \to \mu(K)$ as $n \to \infty$.

Theorem 1. *Let μ be a volume on $\mathbb{R}^n$ and K a compact subset of $\mathbb{R}^n$. Then there exists $(K_n)_{n \in \mathbb{N}}$ a countable basis of compact neighborhoods of K such that $\mu(\text{Fr}K_n) = 0$, for all $n \in \mathbb{N}$.*

Proof. We construct the $(t_n)_{n \in \mathbb{N}}$ monotone decreasing sequence of strictly positive numbers such that $t_n \to 0$ as $n \to \infty$ and such that $\mu(\text{Fr}Q_{t_n}) = 0$, for each $n \in \mathbb{N}$.

First, being given $\varepsilon > 0$ we consider $I = (0, \varepsilon)$ and we define:

$$A_n := \left\{ t \in I \mid \mu(\text{Fr}Q_t) > \frac{\mu(Q_\varepsilon)}{n} \right\}.$$

We will prove that card $A_n \leq n$. To this ends, we suppose that there exist $t_1, t_2, \ldots, t_p \in A_n$ with $p > n$ and $0 < t_1 < t_2 < \ldots < t_p < \varepsilon$.

It is easy to see that:

$$\begin{aligned}&\bigcup_{i=1}^{p} \mathrm{Fr}Q_{t_i} \subset Q_\varepsilon,\\ &\mathrm{Fr}Q_{t_j} \cap \mathrm{Fr}Q_{t_j} = \phi, \text{ for } i \neq j,\\ &(\mathrm{Fr}Q_t = Q_t \setminus \mathrm{int}\, Q_t \subset Q_t \setminus U_t = \{x \in \mathbb{R}^n \mid d(x, K) = t\}).\end{aligned}$$

Therefore

$$\mu(Q_\varepsilon) \geq \mu\left(\bigcup_{i=1}^{p} \mathrm{Fr}Q_{t_i}\right) = \sum_{i=1}^{p} \mu(\mathrm{Fr}Q_{t_i}) > p \cdot \tfrac{\mu(Q_\varepsilon)}{n}$$

and $p > n$, that is $\mu(Q_\varepsilon) > \mu(Q_\varepsilon)$ and this contradiction proves that card$A_n \leq n$.

Set $A := \bigcup_n A_n$. It follows that A is countable set. But I is uncountable set, hence $I \setminus A \neq \phi$. For $t \in I \setminus A$ it follows that $\mu(\mathrm{Fr}Q_t) \leq \frac{\mu(Q_\varepsilon)}{n}$ for all n, hence $\mu(\mathrm{Fr}Q_t) = 0$. Taking into account the above result we define inductively $\varepsilon_1 = 1$ and $t_1 \in (0, \varepsilon_1)$ with $\mu(\mathrm{Fr}Q_{t_1}) = 0, \ldots, \varepsilon_n = \min\left\{\frac{1}{n}, t_{n-1}\right\}$ and $t_n \in (0, \varepsilon_n)$ with $\mu(\mathrm{Fr}Q_{t_n}) = 0, \ldots$.

Now, considering $K_n := Q_{t_n}$, the proof is complete. □

Theorem 2. *Let μ be a real valued function on $K(\mathbb{R}^n)$ with the monotony and regularity properties. Then the following statements are equivalent:*

(b) $\mu(K \cup L) \leq \mu(K) + \mu(L)$, *for every* $K, L \in K(\mathbb{R}^n)$ *and if* $K \cap L = \phi$, *then* $\mu(K \cup L) = \mu(K) + \mu(L)$;

(b′) $\mu(\phi) = 0$ *and* $\mu(K \cup L) = \mu(K) + \mu(L)$ *if* $\mu(K \cap L) = 0$;

(b″) $\mu(\phi) = 0$ *and* $\mu(K \cup L) = \mu(K) + \mu(L) - \mu(K \cap L)$ *for all* $K, L \in K(\mathbb{R}^n)$.

Proof. (b) ⇒ (b′) follows from (b) for $K = L = \phi$.

Let K and L be two compact subsets of $\mathbb{R}^n$ such that $\mu(K \cap L) = 0$.

In keeping with Lemma 1, then there exists $(Q_n)_{n \in \mathbb{N}}$ a countable basis a compact neighborhoods of $Q := K \cap L$ and $Q_n \downarrow Q$.

We further consider the following compact subsets:

$$\begin{cases} Q'_n := K \cap Q_n \text{ and } Q'' := L \cap Q_n, & n \in \mathbb{N}, \\ K_n := K \setminus \mathrm{int}Q_n \text{ and } L_n := L \setminus \mathrm{int}Q_n, & n \in \mathbb{N}. \end{cases} \tag{2.1}$$

Obviously, $Q'_n \downarrow Q$ and $Q''_n \downarrow Q$, because $K \subset Q'_n \subset Q_n$, $L \subset Q''_n \subset Q_n$ and $Q_n \downarrow K$. We check that

$$\begin{cases} K = K_n \cup Q'_n \text{ and } L = L_n \cup Q''_n, \ n \in \mathbb{N}, \\ K_n \cap L_n = \phi, \ n \in \mathbb{N}. \end{cases} \tag{2.2}$$

Indeed,

$$\begin{cases} K = K \cap [Q_n \cup (\mathbb{R}^n \setminus \mathrm{int} Q_n)] = (K \cap Q_n) \cup (K \setminus \mathrm{int} Q_n) = Q'_n \cup K_n, \\ K_n \cap L_n = (K \cap L) \setminus Q_n = \phi, \text{ because } K \cap L \subset \mathrm{int} Q_n. \end{cases}$$

Analogously we prove that $L = Q'' \cup L_n$.

Combining (b) with the monotone properties of μ and (2.2) we conclude:

$$\begin{cases} \mu(K) \leq \mu(K_n) + \mu(Q'_n) \leq \mu(K_n) + \mu(Q_n), & n \in \mathbb{N}, \\ \mu(L) \leq \mu(L_n) + \mu(Q'_n) \leq \mu(L_n) + \mu(Q_n), & n \in \mathbb{N}. \end{cases}$$

By adding the preceding inequalities, we obtain

$$\mu(K) + \mu(L) \leq \mu(K_n) + \mu(L_n) + 2\mu(Q_n), \quad n \in \mathbb{N}. \tag{2.3}$$

Since $K_n \cap L_n = \phi$, then by (b) we have:

$$\mu(K_n \cup L_n) = \mu(K_n) + \mu(L_n) + \mu(Q_n), \quad n \in \mathbb{N}. \tag{2.4}$$

But $K_n \cap L_n \subset K \cup L$ (see (2.1), hence

$$\mu(K_n \cup L_n) \leq \mu(K \cup L), \quad n \in \mathbb{N}. \tag{2.5}$$

Using (2.4) and (2.5) in (2.3) we conclude:

$$\mu(K \cup L) + 2\mu(Q_n) \geq \mu(K) + \mu(L), \quad n \in \mathbb{N}. \tag{2.6}$$

Passing to the limit in (2.6) we will write:

$$\mu(K \cup L) \geq \mu(K) + \mu(L)$$

(as $Q_n \downarrow Q, \mu(Q_n) \to \mu(Q) = 0$ as $n \to \infty$).

Therefore $\mu(K \cup L) = \mu(K) + \mu(L)$, if $\mu(K \cap L) = 0$.

(b$'$) $\Rightarrow$ (b$''$) Let K and L be two compact subsets of $\mathbb{R}^n$, and $Q := K \cap L$.

According to Theorem 1 there exists $(Q_n)_{n \in \mathbb{N}}$ a countable basis of neighborhoods of Q such that $\mu(\mathrm{Fr} Q_n) = 0$, for each $n \in \mathbb{N}$ and $Q_n \downarrow Q$.

Together the notations from the preceding implication we consider

$$Q'''_n := (K \cup L) \cap Q_n = Q'_n \cup Q''_n.$$

We see that $Q'''_n \downarrow Q$, because $Q \subset Q'''_n$ and $Q_n \downarrow Q$.

We check that:

$$K_n \cap Q'_n \subset \mathrm{Fr}Q_n \text{ and } L_n \cap Q''_n \subset \mathrm{Fr}Q_n, \quad n \in \mathbb{N}, \tag{2.7}$$

and

$$K \cup L = K_n \cup L_n \cup Q''_n, \quad n \in \mathbb{N}. \tag{2.8}$$

Indeed,

$$\begin{aligned} &K_n \cap Q'_n \subset K_n \cap Q_n \subset Q_n \setminus \mathrm{int}Q_n = \mathrm{Fr}Q_n,\\ &L_n \cap Q''_n \subset K_n \cap Q_n \subset Q_n \setminus \mathrm{int}Q_n = \mathrm{Fr}Q_n,\\ &K_n \cup L_n \cup Q'''_n = (K_n \cup Q'_n) \cup (L_n \cup Q''_n) = K \cup L. \end{aligned}$$

From the (2.7) we obtain: $\mu(K_n \cap Q'_n) = \mu(L_n \cap Q''_n) = 0$, $n \in \mathbb{N}$, hence (applying (b$'$) and $\mu(\mathrm{Fr}Q_n) = 0$) we have:

$$\begin{cases} \mu(K) = \mu(K_n \cup Q'_n) = \mu(K_n) + \mu(Q'_n), & n \in \mathbb{N},\\ \mu(L) = \mu(L_n \cup Q''_n) = \mu(L_n) + \mu(Q''_n), & n \in \mathbb{N}. \end{cases}$$

From here we derive

$$\begin{cases} \mu(K_n) = \mu(K) - \mu(Q'_n) \to \mu(K) - \mu(Q), & \text{as } n \to \infty\\ \mu(L_n) = \mu(L) - \mu(Q''_n) \to \mu(L) - \mu(Q), & \text{as } n \to \infty. \end{cases} \tag{2.9}$$

Using (2.7) (i.e. $(K_n \cup L_n) \cap Q'''_n \subset \mathrm{Fr}Q_n$) and (2.8) and (b$'$) it follows that

$$\begin{aligned} \mu(K \cup L) &= \mu(K_n \cup L_n) + \mu(Q'''_n)\\ &= (\text{see } (2.4))\ \mu(K_n) + \mu(L_n) + \mu(Q'''_n),\ n \in \mathbb{N}. \end{aligned} \tag{2.10}$$

Passing to the limit in (2.10) and taking in account of (2.9) it follows:

$$\begin{aligned} &\mu(K \cup L) = \mu(K) - \mu(Q) + \mu(L) - \mu(Q) + \mu(Q),\\ &\text{that is} \quad \mu(K \cup L) = \mu(K) + \mu(L) - \mu(K \cap L). \end{aligned}$$

(b$''$) $\Rightarrow$ (b). Let K and L be two compact subsets of $\mathbb{R}^n$.

According to (b$''$)

$$\mu(K \cup L) = \mu(K) + \mu(L) - \mu(K \cap L) \text{ and } \mu(\phi) = 0.$$

Therefore $\mu(K \cup L) \le \mu(K) + \mu(L)$.

If $K \cap L = \phi$ it follows that

$$\mu(K \cup L) = \mu(K) + \mu(L) - \mu(\phi) = \mu(K) + \mu(L).$$

The proof is complete. □

Corollary 2. *Let $(K_i)_{i\in I}$ be a finite family of compacts of $\mathbb{R}^n$ and μ be volume on $\mathbb{R}^n$. Then*

$$\mu\left(\bigcup_{i\in I} K_i\right) \leq \sum_{i\in I} \mu(K_i)$$

and $\mu(\bigcup_{i\in I} K_i) = \sum_{i\in I} \mu(K_i)$ if and only if $\mu(K_i \cap K_j) = 0$, for $i \neq j$.

Proof. By induction according to $n = \text{card } I$.

First, for $n = 1$, the assertion is evident, and for $n = 2$ it is mentioned in Definition 1. If $\mu(\bigcup_{i=1}^{n-1} K_i) \leq \sum_{i=1}^{n-1} \mu(K_i)$ then

$$\mu\left(\bigcup_{i=1}^{n-1} K_i \cup K_n\right) \leq \mu\left(\bigcup_{n=1}^{n-1} K_i\right) + \mu(K_n) \leq \sum_{i=1}^{n} \mu(K_i).$$

Now we assume that $\mu(\bigcup_{i=1}^{n} K_i) = \sum_{i=1}^{n} \mu(K_i)$ and that by contrary exist two distinct numbers i_0 and j_0 such that $\mu(K_{i_0} \cup K_{j_0}) > 0$. Then we infer that $\mu(K_{i_0} \cup (\bigcup_{i\neq i_0} K_i)) > 0$. Denote $K' := \bigcup_{i\neq i_0} K_i$ and we deduce that

$$\begin{aligned} \mu\left(\bigcup_{i=1}^{n} K_i\right) &= \mu(K_{i_0} \cup K') = \mu(K_{i_0}) + \mu(K') - \mu(K_{i_0} \cap K') \\ &< \mu(K) + \mu(K') \leq \mu(K_{i_0}) + \sum_{i\neq i_0} \mu(K_i) = \mu\left(\bigcup_{i=1}^{n} K_i\right), \\ &\text{hence}\mu\left(\bigcup_{i=1}^{n} K_i\right) < \mu\left(\bigcup_{i=1}^{n} K_i\right), \text{ contradiction.} \end{aligned}$$

Conversely, we assume that the assertion is true for n and let $K_1, \ldots, K_{n+1}$ compact sets with $\mu(K_i \cap K_j) = 0$ for any pair of distinct elements i and j. Then

$$\begin{aligned} \mu\left(\bigcup_{i=1}^{n+1} K_i\right) &= \mu\left(\bigcup_{i=1}^{n} K_i \cup K_{n+1}\right) \\ &= \mu\left(\bigcup_{i=1}^{n} K_i\right) + \mu(K_{n+1}) - \mu\left(K_{n+1} \cap \left(\bigcup_{i=1}^{n} K_i\right)\right). \end{aligned}$$

But

$$\begin{aligned} &\mu\left(K_{n+1} \cap \left(\bigcup_{i=1}^{n} K_i\right)\right) \\ &= \mu\left(\bigcup_{i=1}^{n} (K_{n+1} \cap K_i)\right) \leq \sum_{i=1}^{n} \mu(K_{n+1} \cap K_i) = 0. \end{aligned}$$

From these we derive

$$\mu\left(\bigcup_{i=1}^{n+1} K_i\right) = \mu\left(\bigcup_{i=1}^{n} K_i\right) + \mu(K_{n+1}) = \sum_{i=1}^{n+1} \mu(K_i).$$

□

Definition 2. Let K be a compact subset of $\mathbb{R}^n$ and μ volume on $\mathbb{R}^n$. We say that a finite family $(K_i)_{i\in I}$ of compact subsets of $\mathbb{R}^n$ is a μ-division of K if:

(i) $K = \bigcup_{i\in I} K_i$;

(ii) $\mu(K_i \cap K_j) = 0$, for $i \neq j$.

Remark. In this case (by Corollary 2), $\mu(K) = \sum_{i\in I} \mu(K_i)$.

Theorem 3. *Let K be a compact subset of $\mathbb{R}^n$ and μ be a volume on $\mathbb{R}^n$. Then, for every $\varepsilon > 0$, there exists $(K_i)_{i\in I}$ a μ-division of K such that* $\mathrm{diam} K_i < \varepsilon$, *for all $i \in I$.*

Proof. We choose $x \in K$. According to Theorem 1, there exists Q_x a compact neighborhood of $\{x\}$ such that $\mu(\mathrm{Fr}Q_x) = 0$ and $\mathrm{diam}(Q_x) < \varepsilon$; this is possible because $0 < t < \frac{\varepsilon}{2}$ implies $\mathrm{diam}(Q_t) \leq \mathrm{diam}\{x\} + 2t < \varepsilon$.

But K is compact and $K \subset \bigcup_{x\in K} Q_x$, hence there exists $I \subset K$ a finite set such that: $K \subset \bigcup_{x\in I} Q_x$.

Moreover:

(i) $K \subset \cup_{x\in I} Q_x$;

(ii) $\mathrm{diam}(Q_x) < \varepsilon$ for all $x \in I$;

(iii) $\mu(\mathrm{Fr}Q_x) = 0$ for all $x \in I$.

Consider J the family of all non-empty subsets of I. For each $F \in J$ we define (as in [4, p. 138]:

$$Q_F = \cap_{x\in F} Q_x \setminus \cup_{x\notin F} \mathrm{int} Q_x.$$

The set Q_F is compact and $Q_F \subset Q_x$ if $x \in F$, thus according to (ii), $\mathrm{diam}(Q_F) < \varepsilon$ for all $F \in J$.

We denote $K_F := K \cap Q_F$, hence $\mathrm{diam}(K_F) < \varepsilon$ for all $F \in J$.

Further, if $F \neq F'$, then there exists $x_0 \in F \setminus F'$, hence $Q_F \cap Q'_F \subset Q_{x_0} \setminus \mathrm{int} Q_{x_0} = \mathrm{Fr} Q_{x_0}$, hence $\mu(Q_F \cap Q_{F'}) = 0$.

Since $K_F \cap K_{F'} \subset Q_F \cap Q_{F'}$, it follows that $\mu(K_F \cap K_{F'}) = 0$, for $F \neq F'$.

Now, we prove that $K = \bigcup_{F\in J} K_F$.

One of the inclusions, namely $\bigcup_{F\in J} K_F$ is clearly, because $K_F \subset K$, for all $F \in J$.

Reciprocally, from the assumption $x \in K$ it follows that there exists $y \in I$ such that $x \in Q_y$.

Thus, the set $I(x)$ defined as $I(x) := \{y \in I \mid x \in Q_y\}$ is nonempty set.

We check that $x \in K_{I(x)} = K \cap Q_{I(x)}$.

Indeed,

$$\begin{cases} x \in K \Rightarrow x \in Q_y, & \text{for all } y \in I(x) \Rightarrow x \in \bigcap_{y\in I(x)} Q_y, \\ x \notin Q_y, & \text{for all } y \notin I(x) \Rightarrow x \notin \bigcup_{y\notin I(x)} \mathrm{int} Q_y. \end{cases}$$

Therefore $x \in K \Rightarrow x \in \bigcup_{y\in I(x)} Q_y \setminus \bigcup_{y\notin I(x)} \mathrm{int}Q_y = Q_{I(x)}$, that is $K \subset \bigcup_{x\in I} K_{I(x)}$, so the proof of Theorem 3 is complete. □

Theorem 4. *Let μ be a volume on $\mathbb{R}^n$ and $(K_n)_{n\in\mathbb{N}}$ a sequence monotone increasing of compact sets such that $\bigcup_{n\in\mathbb{N}} K_n =: K$ is compact (i.e. $K_n \uparrow K$).*

Then $\mu(K_n) \uparrow \mu(K)$ as $n \to \infty$.

Proof. First, $K_n \subset K_{n+1}$ implies $\mu(K_n) \le \mu(K_{n+1})$, i.e. $(\mu(K_n))_{n\in\mathbb{N}}$ is a monotone increasing sequence.

But $K_n \subset \bigcup_{n\in\mathbb{N}} K_n = K$ implies $\mu(K_n) \le \mu(K)$, for all $n \in \mathbb{N}$, hence $(\mu(K_n))_{n\in\mathbb{N}}$ is bounded sequence.

Therefore $(\mu(K_n))_{n\in\mathbb{N}}$ is convergent sequence and

$$\lim_n \mu(K_n) = \sup\{\mu(K_n) \mid n \in \mathbb{N}\} \le \mu(K).$$

Further, we construct inductively the sequence monotone increasing $(Q_n)_{n\in\mathbb{N}}$ of compacts sets such that:

(i) Q_n is a compact neighborhood of K_n, $n \in \mathbb{N}$;

(ii) $\mu(Q_n) - \mu(K_n) < \varepsilon$, $n \in \mathbb{N}$.

For this end, we invoke Corollary 1. Thus, there exists Q_1 a compact neighborhood of K, such that $\mu(Q_1) - \mu(K_1) < \varepsilon$.

Suppose that we have chosen:

$$Q_1 \subset Q_2 \subset \ldots \subset Q_n \text{ such that } K_i \subset \mathrm{int}Q_i \text{ and } \mu(Q_i) - \mu(K_i) < \varepsilon,$$

for $i \in \{1, 2, \ldots, n\}$.

Further, $\mu(K_{n+1} \cup Q_n) = \mu(K_{n+1}) + \mu(Q_n) - \mu(K_{n+1} \cap Q_n)$.

$$K_n \subset Q_n \cap K_{n+1} \text{ implies } \mu(K_n) \le \mu(Q_n \cap K_{n+1}),$$

hence

$$\mu(K_{n+1} \cup Q_n) \le \mu(K_{n+1}) + \mu(Q_n) - \mu(K_n) < \mu(K_{n+1}) + \varepsilon.$$

Putting $\varepsilon' = \mu(K_{n+1}) + \varepsilon - \mu(K_{n+1} \cup Q_n) > 0$ and invoking Corollary 1, there exists Q_{n+1} a compact neighborhood of $K_{n+1} \cup Q_{n+1}$ such that $\mu(Q_{n+1}) - \mu(K_{n+1} \cup Q_{n+1}) < \varepsilon'$, that is $\mu(Q_{n+1}) - \mu(K_{n+1}) < \varepsilon$.

Moreover, $Q_n \subset Q_n \cup K_{n+1} \subset Q_{n+1}$. Thus, the sequence $(Q_n)_{n\in\mathbb{N}}$ is constructed.

As $K = \bigcup_{n\in\mathbb{N}} K_n \subset \bigcup_{n\in\mathbb{N}} Q_n$, K is compact set and $(\mathrm{int}Q_n)_{n\in\mathbb{N}}$ is increasing, we conclude that there exists $n_0 \in \mathbb{N}$, such that $K \subset \bigcup_{n=0}^{n_0} Q_n \subset Q_{n_0}$.

Therefore $\sup\{\mu(K_n) : n \in \mathbb{N}\} \le \mu(K) \le \mu(Q_{n_0}) \le \mu(K_{n_0}) + \varepsilon \le \sup\{\mu(K_n) : n \in \mathbb{N}\} + \varepsilon$.

But $\varepsilon > 0$ is chosen arbitrarily, hence $\mu(K) = \sup\{\mu(K_n) \mid n \in \mathbb{N}\}$.

The proof of Theorem 4 is complete. □

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