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EIGENVECTORS OF THREE TERM RECURRENCE TOEPLITZ MATRICES AND RIORDAN GROUP

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Abstract: Eigenvalues of tridiagonal (including main) Toeplitz matrices are analytically known under some regular distance to the main diagonal. Any eigenvector may be easily computed then, through a backward process; instead, we give an analytical form for each component through the reciprocation of the underlied trinomial. More generally, the connection to the Riordan group follows some bilinear iterative process.

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1. Introduction

Toeplitz matrices are intimately related to many problems involving difference equations. Finding explicit formulas for eigenvalues and eigenvectors drastically speed up underlied applications. Recently, we gave an analytical formula for eigenvalues of $n \times n$ Toeplitz real matrices with 2 diagonals apart main diagonal and possibly circulant perturbations, see [1]. Eigenvector associated with an eigenvalue is retrieved through a forward substitution scheme. At that time, despite we noticed the connection with the Pascal triangle, we were not aware of [2, 3] articles devoted to the reciprocation of quadratic polynomials. The purpose of this note is to emphasize the connection to the Pascal triangle that turns the computation of the eigenvectors to an analytical scheme too and

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rephrases their article as the reciprocation of a special trinomial.

2. An Iterative Process to Calculate the Reciprocal of Some Trinomials

We borrow notations from [2, 3], in particular $\mathbb{K}, \mathbb{K}[[x]]$ denotes a field with characteristic 0 and the ring of formal power series over $\mathbb{K}$ respectively; $\omega(f)$ stands for the order of the series $f = \sum_{n\geq 0} f_n x^n$, i.e. $f_{\omega(f)} \neq 0$ while $f_i = 0$ for all $i < \omega(f)$. It is worth recalling the contractive property of a linear approximation.

Proposition 1. (see [2]) *The map $d : \mathbb{K}[[x]] \times \mathbb{K}[[x]] \mapsto \mathbb{R}_+$ defined by $d(f,g) = 1/2^{\omega(f-g)}$ is a complete ultrametric on $\mathbb{K}[[x]]$. Moreover $d(f,g) \leq 1/2^{k+1}$ iff $T_k(f) = T_k(g)$, namely expansions coincide upto k-th order.*

Proposition 2. (see [2]) *Let $f, h \in \mathbb{K}[[x]]$ with $f(0) = 0$, then the linear map $P : \mathbb{K}[[x]] \mapsto \mathbb{K}[[x]]$ defined by $P(S) = fS + h$ is 1/2-contractive independently on f and h. More precisely, $d(P(S_1), P(S_2)) = d(S_1, S_2)/2^{\omega(f)}$ and the unique fixed point of P is $h/(1-f)$*

Corollary 1. (see [2]) *Let $f, g \in \mathbb{K}[[x]]$ with $g(0) \neq 0$, if $f/g = \sum_{n\geq 0} d_n x^n$ then the linear map $P(S) = (g(0)-g)/g(0)S + f/g(0)$ has a unique fixed point f/g and*

$$d_0 = f_0/g_0, d_n = f_n/g_0 - \sum_{i=1}^{n} g_i d_{n-i}/g_0 \text{ for all } n \geq 1$$

Hence the iterative process starts at $P(0), P^2(0) = P(P(0)), P^3(0)\ldots$ In [1], we resolve the kernel problem for the eigenvalues as

$$R(x) = (bx^{l+r} + (d-\lambda)x^r + a)V(x).$$

Once eigenvalues are computed, we solve the Toeplitz recurrences backwards (forwards equivalently) to retrieve an associated eigenvector; however, it could be better to look conversely for

$$V(x) = R(x)/(bx^{l+r} + (d-\lambda)x^r + a)$$

whence the reciprocation of the trinomial $1 - bx^p + cx^{p+1}$ after some notational simplifications (the rewritings $(\lambda - d)/a \mapsto b, b/a \mapsto c$ make notations agreeing with [3]) and sign restrictions ($-1 - bx^p + cx^{p+1}$ is analogous).

Theorem 1. *Let $Q(x) = 1 - bx^p + cx^{p+1}$ be a trinomial associated with higher order three term recurrence and the linear map $P_Q(S) = (bx^p - cx^{p+1})S + 1$, if $1/Q = \sum_{k\geq 0} d_k x^k$ then*

$$\begin{aligned} P_Q^{n+1}(0) =& T_{np}(1/Q) + x^{np}\big(\sum_{k\geq 1} d_{n,k}x^k\big) \\ \text{with } & d_{n,0} = b^n, d_{n,1} = -nb^{n-1}c, \\ & d_{n,k} = bd_{n-1,k} - cd_{n-1,k-1} \text{ for all } k > 1, \\ & d_{n,k} = 0 \text{ for all } k > n. \end{aligned}$$

Proof.

$$\begin{aligned} P_Q^1(0) =& 1 \Rightarrow d_{0,0} = 1, \\ P_Q^2(0) =& (1 + bx^p) - cx^{p+1} \Rightarrow d_{1,0} = b, d_{1,1} = -c, \\ P_Q^3(0) =& (1 + bx^p - cx^{p+1} + b^2x^{2p}) + x^{2p+1}(-2bc + c^2x) \\ & \Rightarrow d_{2,0} = b^2, d_{2,1} = -2bc, d_{2,2} = c^2 \end{aligned}$$

establish the first step. Suppose the result is true for $n-1$

$$P_Q^n(0) = \sum_{k=0}^{(n-1)p} d_k x^k + x^{(n-1)p}\left(\sum_{k=1}^{n-1} d_{n-1,k}x^k\right),$$

then

$$\begin{aligned} P_Q^{n+1}(0) &= 1 + (bx^p - cx^{p+1})\sum_{k=0}^{(n-1)p} d_k x^k + x^{np}\left(\sum_{k=1}^{n-1} bd_{n-1,k}x^k - cd_{n-1,k}x^{k+1}\right) \\ &= \sum_{k=0}^{np} d_k x^k + x^{np+1}\left(-cd_{(n-1)p} - cd_{n-1,n-1}x^n\right) \\ & x^{np}\left(\sum_{k=1}^{n-1}(bd_{n-1,k} - cd_{n-1,k-1})x^k\right) \\ &= \sum_{k=0}^{np} d_k x^k + x^{np}\left(\sum_{k=1}^{n} d_{n,k}x^k\right), \end{aligned}$$

where $d_{n,1} = -cd_{(n-1)p} + bd_{(n-1),1} = -cb^{n-1} - b(n-1)b^{n-2}c = -nb^{n-1}c$ and $d_{n,n} = -cd_{n-1,n-1} = (-1)^n c^n$ actually fulfills the induction on entries since $d_{n,0} = d_{np} = b^n$ and $bd_{n-1,n} = 0$, respectively. Actually, $d_{k+p} = bd_k - cd_{k-1}$ follows first equality and since $d_{np-1} = 0$ under $np + n < (n+1)p$, it gives the required equality $d_{np} = b^n$. □

Let $C_Q = (d_{n,k})_{\mathbb{N}\times\mathbb{N}}$ the lower triangular matrix above then it is well known [4] that it is Riordan; moreover, $d_{np+k} = d_{n,k}$ as stated in the proof for all $1 \le k \le n$. Let $p_n(x) = \sum_{k=0}^{n} d_{n,k}x^k$ the polynomials associated with the rows in C_Q; we have proven $d_{n,0} = b^n$, $d_{n,n} = (-1)^n c^n$ therefore, we get a counterpart to Corollary 1

Corollary 2.

$$\begin{aligned} d_{n,k} =&(-1)^k \binom{n-1}{k} b^{n-1-k}c^k, \text{ for all } k > 0,\\ P_Q^{n+1}(0) =&T_{np}(1/Q) + x^{np}\big((b-cx)^n - b^n\big). \end{aligned}$$

Proof. By induction, for all $k > 0$

$$\begin{aligned} d_{n,k} =&bd_{n-1,k} - cd_{n-1,k-1}\\ =&b(-1)^k \binom{n-2}{k} b^{n-2-k}c^k - c(-1)^{k-1}\binom{n-2}{k-1} b^{n-2-k+1}c^{k-1}\\ =&(-1)^k \binom{n-1}{k} b^{n-1-k}c^k,\\ p_n(x) =&(b-cx)^n, \end{aligned}$$

whence the result using the first column and diagonal entries for the last equality. □

This proves the connection with the Pascal triangle we had quoted earlier. Notice the power discrepancy $d_{n,0} = d_{np} = b^n$ compared with $(-1)^0\binom{n-1}{0}b^{n-1}c^0$ from the corollary to prove the result and secondly that $T_{np}(1/Q)$ is actually the Taylor expansion upto np as notation intended to highlight.

3. A Bilinear Map and Doubly Iterative Process

Let us consider the ubiquitous circular shift matrix $S_n = (s_{i(i+1)} = 1)$ for all i modulo the size n and its Toeplitz symmetrization $T_{S_n} = (I - S_n)(I - S_n)^t$ whose diagonal equals 2 and wrapped lower and upper diagonals equal -1. A direct application of the kernel method (see [1]), with no circulant perturbation, leads to the spectrum $\Lambda(T_{S_n}) = \big\{2(1-\cos\frac{2l\pi}{n}), l = 0 \cdots n-1\big\}$. Using the perturbation corners $0, 0$ it leads to the standard shift S_∞ and Toeplitz symmetrization T_{S_∞} whose spectrum is $\Lambda(T_{S_\infty}) = \big\{2(1-\cos\frac{l\pi}{n}), l = 0 \cdots n-1\big\}$. Instead of applying forward substitution to compute the eigenvector associated with an eigenvalue in the spectrum, previous section suggests to consider the bilinear map $P^{k+1}(S,T) = fP^k(S,T) - hP^{k-1}(S,T)$ with initial conditions $P^0(S,T) = S, P^{-1}(S,T) = T$.

Proposition 3. *Let $f(x) = x$, $h(x) = 1$ then the bilinear map $P^{k+1}(S,T) = fP^k(S,T) - hP^{k-1}(S,T)$ generates a lower triangular Riordan array $R = (r_{m,k})$ related to the Pascal triangle*

$$\begin{aligned} r_{m+1,m+1} =&1, \quad r_{m+1,1} = (-1)^{(m+1)/2}, \\ r_{m+1,m-2p} =& - r_{m,m-2p}, \quad p \geq 0, \\ r_{m+1,m-2p-1} =& r_{m,m-2p-1} + r_{m,m-2p-2}, \quad p \geq 1, \end{aligned}$$

and expands as

$$\begin{aligned} P^m(S,T) =& Sp_m(x) - Tp_{m-1}(x), \\ p_m(x) =& \sum_{j=0}^{m/2} (-1)^j \binom{m-j}{j} x^{m-2j}. \end{aligned}$$

Proof.

$$\begin{aligned} P^0(S,T) =& S \Rightarrow r_{1,1} = 1 = (-1)^0, \\ P^1(S,T) =& xS - T \Rightarrow r_{2,2} = 1, r_{2,1} = (-1)^1, \\ P^2(S,T) =& x(xS - T) - S = x^2S - xT - S \\ & \Rightarrow r_{3,3} = 1, r_{3,1} = -1, r_{3,2} = -1, \\ P^3(S,T) =& x(x^2S - xT - S) - xS + T = x^3S - x^2T - 2xS + T \\ & \Rightarrow r_{4,4} = 1, r_{4,3} = -1, r_{4,2} = -2 = r_{3,1} + r_{3,2}, r_{4,1} = 1, \end{aligned}$$

initialize induction with even-odd partitioning in x

$$\begin{aligned} P^{2k}(S,T) =& Sp_{2k}(x) - Tq_{2k-1}(x), \\ P^{2k+1}(S,T) =& Sp_{2k+1}(x) - Tq_{2k}(x), \end{aligned}$$

for even and odd polynomials $p(x), q(x)$ whose degree occurs in subscript. Then, the induction is verified since

$$\begin{aligned} P^{2k+2}(S,T) =& xP^{2k+1}(S,T) - P^{2k}(S,T) \\ =& S(xp_{2k+1}(x) - p_{2k}(x)) - T(xq_{2k}(x) - q_{2k-1}(x)), \\ p_{2k+2}(x) =& xp_{2k+1}(x) - p_{2k}(x), \end{aligned}$$

$$\begin{aligned}
q_{2k+1}(x) =& xq_{2k}(x) - q_{2k-1}(x),\\
P^{2k+3}(S,T) =& xP^{2k+2}(S,T) - P^{2k+1}(S,T)\\
=& S(xp_{2k+2}(x) - p_{2k+1}(x)) - T(xq_{2k+1}(x) - q_{2k}(x)),\\
p_{2k+3}(x) =& xp_{2k+2}(x) - p_{2k+1}(x),\\
q_{2k+2}(x) =& xq_{2k+1}(x) - q_{2k}(x).
\end{aligned}$$

Let $r_{mk}x^{k-1}$ be the monomial in $P^m(S,T)$. The result follows by noticing $q_k(x) = p_k(x) =$ for all indices $k \geq 0$, and assigning S, T to monomials according to either parity. Indeed,

$$\begin{aligned}
p_{m+1}(x) =& \sum_{j=0}^{m/2}(-1)^j \binom{m-j}{j} x^{m+1-2j} - \sum_{j=0}^{(m-1)/2} (-1)^j \binom{m-1-j}{j} x^{m-1-2j}\\
=& x^{m+1} + \sum_{j=1}^{m/2+1} (-1)^j \binom{m-j}{j} x^{m+1-2j} +\\
& \sum_{j=1}^{(m-1)/2+1} (-1)^j \binom{m-1-(j-1)}{j-1} x^{m-1-2(j-1)}\\
=& \sum_{j=0}^{(m+1)/2} (-1)^j \binom{m+1-j}{j} x^{m+1-2j}
\end{aligned}$$

agrees with the induction too. □

Equivalently, the Riordan array is made of pairs of diagonals with alternating signs and values $\binom{m/2+k-1}{k-1}$ for $k \geq 1$; however, in spite of forward substitution, we had better in row major ordering (upto S and T scaling factors)

Corollary 3.

$$\begin{aligned}
r_{m,m} =& 1, \quad r_{m,m-1} = -1,\\
r_{m,m-2k} =& (-1)^k \binom{m-2k}{m-2k-1}, \quad k > 0,\\
r_{m,m-2k-1} =& (-1)^{k-1} \binom{m-2k-1}{m-2k-2}, \quad k > 0,\\
r_{m,1} =& (-1)^{m/2}.
\end{aligned}$$

Setting, $S = T = v_1 = 1$ yields for unnormalized eigenvectors, in standard shift case, an integral combination of binomials in cosine powers: $v_m = \sum_{k\geq 1} r_{m,k} 2^{k-1} cos^{k-1}(l\pi/n)$ since $v_{m+1} = (2-\lambda)v_m - v_{m-1}$ with $v_2 = (1-\lambda)v_1 = (2-\lambda)v_1 - v_1$.

In circular shift case, $S = v_1$ and $T = v_n$ leads to $v_{m+1} = v_1 p_m(x) - v_n p_{m-1}(x)$. Similarly, backward substitution leads to $v_{n-m} = v_n p_m(x) - v_1 p_{m-1}(x)$. If $n = 2s$ is even, then v_n is solved w.r.t. v_1 first and applied to the remaining components for $m = 1, n-1$

$$v_{s+1} = v_1 p_s(x) - v_n p_{s-1}(x) = v_{n-(s-1)} = v_n p_{s-1}(x) - v_1 p_{s-2}(x),$$

$$v_n = v_1 \frac{p_s(x) + p_{s-2}(x)}{2p_{s-1}(x)} = xv_1/2,$$

$$v_{m+1} = (2p_m(x) - xp_{m-1}(x))v_1/2.$$

Starting with $v_1 = 2$, we retrieve coefficients of Lucas/Cardan polynomials with unsigned triangle $t_{mk} = t_{(m-1)k} + t_{(m-2)(k-1)}$, since 2 factor scales up cosine coefficient while x factor shifts 1 column left and 2 rows above:

$$v_{m+1} = \sum_{k=0}^{(m+1)/2} (-1)^k t_{m,k} 2^{m-2k} cos^{m-2k}(2l\pi/n),$$

$$t_{m,k} = \frac{m}{m-k}\binom{m-k}{k}.$$

If $n = 2s+1$ is odd then $v_{s+1} = v_1 p_s(x) - v_n p_{s-1}(x) = v_{n-s} = v_n p_s(x) - v_1 p_{s-1}$ simply leads to $v_1 = v_n = 1$ and the unnormalized eigenvector components, for all $m = 1, n-1$

$$v_{m+1} = \sum_{k=0}^{m/2} (-1)^k \binom{m-k}{k} 2^{m-2k} cos^{m-2k}(2l\pi/n) - \sum_{k=0}^{(m-1)/2} (-1)^k \binom{m-1-k}{k} 2^{m-1-2k} cos^{m-1-2k}(2l\pi/n).$$

To complete the properties of $p_m(x)$ polynomials, notice the simple relationship:

Proposition 4. *Convolution* $c_{(m+1)n}(x) = p_m(x)p_{n-2}(x) - p_{m-1}(x)p_{n-1}(x)$ *simplifies as* $c_{(m+1)n}(x) = \sum_{j=0}^{(n-2-m)/2} (-1)^j \binom{n-2-m-j}{j} x^{n-2-m-2j}$ *for all* $m = n-2 \ldots 1$

Proof. By a backward induction to meet increasing degrees, we have:

$$c_{nn}(x) = 0, \quad c_{(n-1)n}(x) = 1,$$

$$\begin{aligned} c_{(m+2)n}(x) = (xp_m(x) - p_{m-1}(x))p_{n-2}(x) - (xp_{m-1}(x) - p_{m-2}(x))p_{n-1}(x) \\ = xc_{(m+1)n}(x) - c_{mn}(x), \end{aligned}$$

$$\begin{aligned} c_{mn}(x) &= xc_{(m+1)n}(x) - c_{(m+2)n}(x) \\ &= x^{n-m} + \sum_{j=1}^{(n-1-m)/2} (-1)^j x^{n-1-m-2j} \binom{n-2-m-j}{j} \\ &+ \sum_{j=1}^{(n-1-m)/2} (-1)^j x^{n-1-(m+2)-2(j-1)} \binom{n-1-(m+2)-(j-1)}{j-1} \\ &= x^{n-m} + \sum_{j=1}^{(n-1-m)/2} (-1)^j x^{n-1-m-2j} \left(\binom{n-2-m-j}{j} + \binom{n-2-m-j}{j-1} \right) \\ &= \sum_{j=0}^{(n-1-m)/2} (-1)^j x^{n-1-m-2j} \binom{n-1-m-j}{j}, \end{aligned}$$

where the summand limit takes care of either lowest degree. □

We leave to the reader, extension to multilinear case and greater distance apart main diagonal as in 2 hops shift case S_n^2, S_∞^2 whose eigenvalues are given by the kernel method, respectively $2(1 - \cos \frac{4l\pi}{n})$, for $l = 0 \cdots n-1$ and $2(1 - \cos \frac{l\pi}{n}), 2(1 - \cos \frac{(l+1)\pi}{n+1})$ for $l = 0, 2, \cdots n-2$ and even n, or $2(1 - \cos \frac{(l-1)\pi}{n}), 2(1 - \cos \frac{l\pi}{n+1})$ for $l = 2, 4, \cdots n-1$ and odd n.

Finally, notice that for complex shift $s_{i(i+1)} = \sqrt{-1}$ and Hermitian Toeplitz matrices $(I - S)(I - S)^*$, we arrive at the same real eigenvalues and the easily solvable eigenvector recurrence $v_{m+1} = \sqrt{-1}^m p_m(x)v_1 - \sqrt{-1}^{m-1} p_{m-1}(x)v_n$.

4. Concluding Remarks

In this note, we extend the reciprocation of a quadratic polynomial to more general trinomials. It gives a direct connection to the Pascal triangle for higher order three term recurrence Toeplitz matrices. Another direct consequence is an analytical form for all eigenvector components of such matrices. It improves over former forward substitution scheme for both accuracy, in big sized matrix case, and time complexity, since all components may be computed in parallel on the contrary to forward substitution.

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