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HOMOGENEOUS POLYNOMIALS WITH TWO SETS COMPUTING THEIR SYMMETRIC TENSOR RANK

E. Ballico

Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let $X_{m,d} \subset \mathbb{P}^{\binom{m+d}{m}-1}$ be the order d Veronese embedding of $\mathbb{P}^m$. Here we classify points $P \in \mathbb{P}^{\binom{m+d}{m}-1}$ with symmetric rank $d+1$ and whose symmetric rank is computed by at least 2 sets $A, B \subset \mathbb{P}^m$ such that no 3 of the points of $A \cup B$ are collinear. In these cases the symmetric rank is computed by an infinite (and one-dimensional) family of subsets of $X_{m,d}$.

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1. Introduction

Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. For any $P \in \mathbb{P}^n$ let $r_X(P)$ be the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$. The integer $r_X(P)$ is called the X-rank of P (see [5]). Let $\mathcal{S}(X, P)$ the set of all $S \subset X$ computing $r_X(P)$, i.e. the set of all subsets $S \subset X$ such that $\sharp(S) = r_X(P)$ and $P \in \langle S \rangle$. Notice that any $S \in \mathcal{S}(X, P)$ is linearly independent and $P \notin \langle S' \rangle$ for any $S' \subsetneq S$. For every integer $t \geq 1$ let $\sigma_k(X)$ denote the closure in $\mathbb{P}^n$ of all $(k-1)$-dimensional linear spaces spanned by t points of Y. Set $\sigma_0(X) := \emptyset$. For any $P \in \mathbb{P}^n$ the border X-rank $b_X(P)$ is the minimal integer $t \geq 1$ such that $P \in \sigma_t(X)$, i.e. the only integer $t \geq 1$ such that $P \in \sigma_t(X) \setminus \sigma_{t-1}(X)$. If $\sigma_{k-1}(X) \neq \mathbb{P}^n$, then a general $P \in \sigma_k(X)$ has X-rank k. An important problem is to find conditions on X and P such that $\sharp(\mathcal{S}(X, P)) = 1$. In [1] we

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consider the construction of examples of (X,P) in which X is a smooth curve and $\sharp(\mathcal{S}(X,P))$ is a prescribed integers. Here we consider the case in which X is a Veronese embedding of $\mathbb{P}^m$, $m \ge 2$. In this case the X-rank is called the symmetric rank. Fix positive integers m, d. Let $\nu_d : \mathbb{P}^m \to \mathbb{P}^{\binom{m+d}{m}-1}$ be the order d Veronese embedding of $\mathbb{P}^m$. Set $X_{m,d} := \nu_d(\mathbb{P}^m)$. We always assume $m \ge 2$, because the problem is essentially empty in the case $m = 1$. Our starting point was an example from [2] which we recall here (see [2], Example 1, for the case $m = 2$).

Example 1. Assume $m = 2$ and $d \ge 4$. Let $D \subset \mathbb{P}^2$ be a smooth conic. Fix sets $S, S' \subset D$ such that $\sharp(S) = \sharp(S') = d + 1$ and $S \cap S' = \emptyset$. Since no 3 points of D are collinear, the sets S, S' and $S \cup S'$ are in linearly general position. Since $h^0(D, \mathcal{O}_D(d)) = 2d + 1$ and D is projectively normal, we have $h^1(\mathcal{I}_S(d)) = h^1(\mathcal{I}_{S'}(d)) = 0$ and $h^1(\mathcal{I}_{S\cup S'}(d)) = 1$. Thus $\nu_d(S)$ and $\nu_d(S')$ are linearly independent and $\langle \nu_d(S)\rangle \cap \langle \nu_d(S')\rangle$ is a unique point. Call P this point. Obviously $r_X(P) \le d + 1$. We first check that $b_{X_{2,d}}(P) \ge d + 1$. Assume $b_X(P) \le d$ and take Z computing $b_{X_{2,d}}(P)$. We may apply a small part of the proof of [2], Theorem 1, to P, S, Z (even if a priori S may not compute $b_X(P)$). We get the existence of a line L such that $\deg(Z \cap L) < \sharp(S \cap L)$ and $\deg(Z \cap L) + \sharp(S \cap L) \ge d + 2$. Since $d \ge 4$, we get $\sharp(S \cap L) \ge 3$, contradiction. By construction $\sharp(\mathcal{S}(X_{2,d}, P)) \ge 2$. If $m \ge 3$ we get a similar example taking a 2-dimensional linear subspace $N \subset \mathbb{P}^m$ and taking D, S, S', P as above. Indeed, it is well-known that every $A \in \mathcal{S}(X_{m,d}, P)$ is contained in $\nu_d(N)$. By construction $\sharp(\mathcal{S}(X_{2,d}, P)) \ge 2$. Lemma 2 below will give that $\mathcal{S}(X_{2,d}, P)$ is infinite, one-dimensional and with a unique positive-dimensional irreducible component.

Theorem 1. *Fix $P \in \mathbb{P}^{\binom{m+d}{m}-1}$ such that there are finite subsets $A, B \subset X_{m,d}$ such that $\sharp(A) = \sharp(B) = d + 1$, $P \in \langle A\rangle \cap \langle B\rangle$, $P \notin \langle A'\rangle$ for any $A' \subsetneqq A$, $P \notin \langle B'\rangle$ for any $B' \subsetneqq B$ and $A \ne B$. Then $A \cap B = \emptyset$. Write $A = \nu_d(S)$ and $B = \nu_d(S')$. Assume that no 3 points of $S \cup S'$ are collinear. Then $m \ge 2$ and there are a plane $N \subset \mathbb{P}^m$ and D, S, S' as in Example 1. If $d \ge 4$, then $\mathcal{S}(X, P)$ is one-dimensional and $E \in \mathcal{S}(X_{m,d}, P)$ if and only if $E = \nu_d(F)$ with $F \subset D$ and $\nu_d(F)$ computing the $\nu_d(D)$-rank of P.*

If one of the sets A or B is very special, then often the other must be special, too.

Proposition 1. *Fix $P \in \mathbb{P}^{\binom{m+d}{m}-1}$ such that there are finite subsets $A, B \subset X_{m,d}$ such that $\sharp(A) = \sharp(B) = d + 1$, $P \in \langle A\rangle \cap \langle B\rangle$, $P \notin \langle A'\rangle$ for any $A' \subsetneqq A$, $P \notin \langle B'\rangle$ for any $B' \subsetneqq B$ and $A \ne B$. Write $A = \nu_d(S)$ and $B = \nu_d(S')$.*

Assume the existence of a plane $N \subseteq \mathbb{P}^2$ and a smooth conic $D \subset N$ such that $S \subset D$. Then $S' \subset D$ and A, B are as in Example 1.

Lemma 1. *Fix positive integers $m \geq 2$, $d \geq 2$ and $s \leq 2d+2$. Let $E \subset \mathbb{P}^m$ be a subset such that $\sharp(S) = s$ and no 3 of its points are collinear. Then $h^1(\mathcal{I}_E(d)) > 0$ if and only if $s = 2d+2$ and there is a plane $N \subseteq \mathbb{P}^m$ and a smooth conic $D \subset N$ such that $E \subset D$.*

Proof. The "if" part is obvious. To check the other implication we assume $h^1(\mathbb{P}^m, \mathcal{I}_E(d)) > 0$. We mention that the case $s \leq 2d+1$ is classical. First assume $m = 2$. Let $D \subset \mathbb{P}^2$ be a smooth conic containing the maximal number of points of E. Now assume $m > 2$ and that the result is true in $\mathbb{P}^m$. In $\mathbb{P}^m$ we use induction on d (leaving to the reader the case $d = 1$ in which no such E exists). Let $M \subset \mathbb{P}^m$ be a hyperplane such that $\sharp(M \cap E)$ is maximal. If $E \subset M$, then it is sufficient to use the inductive assumption. Hence we may assume $E \cap M \neq E$. Thus $\sharp(E \cap M) \leq 2d+1$. The inductive assumption gives $h^1(M, \mathcal{I}_{M \cap E}(d)) = 0$. Since $M \cap E$ is maximal, we have $\sharp(M \cap E) \geq 3$. Thus $\sharp(E \setminus (E \cap M)) \leq 2(d-1)+1$. The inductive assumption on d gives $h^1(\mathbb{P}^m, \mathcal{I}_{E \setminus (E \cap M)}(d-1)) = 0$. Hence Castelnuovo's inequality gives $h^1(\mathbb{P}^m, \mathcal{I}_E(d)) = 0$, contradiction. □

Lemma 2. *Fix an integer $d \geq 2$, a rational normal curve $C \subset \mathbb{P}^{2d}$ and $P \in \mathbb{P}^{2d} \setminus \sigma_d(C)$. Then $r_C(P) = d+1$ and $\mathcal{S}(X, P)$ is one-dimensional. Moreover, $\mathcal{S}(X, P)$ has a unique positive-dimensional irreducible component.*

Proof. Let $J(C, \dots, C) \subset C^{d+1} \times \mathbb{P}^{2d}$ be the abstract join of $d+1$ copies of C and $\pi : J(C, \dots, C) \to \mathbb{P}^{2d}$ the proper morphism induced by the projection $C^{d+1} \times \mathbb{P}^{2d} \to \mathbb{P}^{2d}$. Since the secant varieties and the joins of any curve have the expected dimension, we have $\dim(J(C, \dots, C)) = 2d+1$, $b_C(P) = d+1$, and π is surjective. A classical theorem of Sylvester gives $r_C(P) = d+1$ (see [3], Theorem 1). Hence $\pi^{-1}(P)$ contains infinitely many reduced fibers, i.e. $\mathcal{S}(C, P)$ is positive-dimensional. Fix $S, S' \in \mathcal{S}(C, P)$ such that $S \neq S'$. Since any subset of C with cardinality at most $2d+1$ is linearly independent (see [2], Lemma 1), gives $S \cap S' = \emptyset$. Since $\dim(C) = 1$ and C is irreducible, we get that $\mathcal{S}(C, P)$ has at most one positive-dimensional irreducible component. □

Proof of Theorem 1. Since $d+1 \geq 3$ and no 3 points of S are collinear, $m \geq 2$. By [2], Lemma 1, we have $h^1(\mathcal{I}_{S \cup S'}(d)) = 0$. Lemma 1 gives the existence of a plane $N \subseteq \mathbb{P}^2$ and a smooth conic $D \subset N$ such that $S \cup S' \subset D$. Take P, N, A, B, S, S' as in Example 1. Fix another element $\nu_d(S'')$ of $\mathcal{S}(X, P)$. The " only if " part of the theorem just proved gives $S'' \cap S' = S'' \cap S = \emptyset$ and the existence of smooth conics D', D'' such that $S \cup S'' \subset D'$ and $S' \cup S'' \subset D''$. Since two different smooth conics have at most 4 common points and $\sharp(S) = d+1 \geq 5$, we het $D' = D'' = D$. Notice that $P \in \langle \nu_d(D) \rangle$ and that $\nu_d(D)$ is a rational normal curve of its linear span $\langle \nu_d(D) \rangle \cong \mathbb{P}^{2d}$. Since $b_{\nu_d(D)}(P) \leq b_{X_{m,d}}(P)) = d+1$ and the secant varieties of any curve have the expected dimension, we get $P \in \langle \nu_d(D) \rangle \setminus \sigma_d(\nu_d(D))$. Hence Lemma 2 concludes the proof. □

Proof of Proposition 1. By [2], Lemma 1, we have $h^1(\mathbb{P}^m, \mathcal{I}_{S \cup S'}(d)) > 0$. As in the proof of Lemma 1 we get $S \cap S' = \emptyset$. First assume $m = 2$. Set $E := S' \setminus (S' \cap D)$. In order to obtain a contradiction we assume $E \neq \emptyset$. Thus $\deg((S \cup S') \cap D)) \leq 2d+1$. Thus $h^1(\mathbb{P}^m, \mathcal{I}_{S \cup S'}(d)) > 0$. Castelnuovo's inequality gives $h^1(\mathbb{P}^2, \mathcal{I}_E(d-2)) > 0$. By [3], Lemma 1, there is a line $L \subset \mathbb{P}^2$ such that $\sharp(L \cap E) \geq d$. Set $F := (S \cup S') \cap L$. First assume $\sharp(F) \leq d+1$. Thus $h^1(L, \mathcal{I}_F(d)) = 0$. Since $\sharp(F) \geq d$, we have $\sharp((S \cup S') \setminus F) \leq d+2$. Since $h^1(L, \mathcal{I}_F(d)) = 0$, Castelnuovo's inequality gives $h^1(\mathbb{P}^2, \mathcal{I}_{(S \cup S') \setminus F}(d-1)) = 0$. Hence there is a line $R \subset \mathbb{P}^2$ such that $\sharp(R \cap ((S \cup S') \setminus F)) \geq d+1$. Since at least $d-1$ of the points of $(S \cup S') \setminus F$ are contained in D and no 3 of the points of D are collinear, we got a contradiction. Now assume $\sharp(F) \geq d+2$. Thus $\sharp(S \cup S') \setminus F) \leq d$. Thus $h^1(\mathbb{P}^2, \mathcal{I}_{(S \cup S') \setminus F}(d-1)) = 0$. Since $S \cap S' = \emptyset$, we get $\langle \nu_d(S) \rangle \cap \langle \nu_d(S') \rangle = \langle \nu_d(S \cap L) \rangle \cap \langle \nu_d(S' \cap L) \rangle$. Since $S \cap L \subsetneq S$, we got a contradiction.

Now assume $m > 2$. Fix a hyperplane $M \subset \mathbb{P}^m$ containing N and such that $\sharp(M \cap S')$ is maximal among the set of all hyperplanes containing M. Use the inductive step of the proof of Lemma 1. □

Decreasing the assumptions we loose the uniqueness of Example 1.

Under suitable (very strong) assumptions we will find out the the next example satisfies another uniqueness statement.

Example 2. Fix an integer $d \geq 2$. Assume $m \geq 2$ and fix a plane $N \subseteq \mathbb{P}^n$ and lines $L, R \subset N$ such that $L \neq R$. Fix a subset $W \subset (L \cup R) \setminus L \cap R$ such that $\sharp(W \cap L) = \sharp(W \cap R) = d+1$. Take any decomposition $W = S \sqcup S'$ with $\sharp(S) = \sharp(S') = d+1$. Since $h^0(L \cup L, \mathcal{O}_{L \cup R}(d)) = 2d+1$, $L \cup R$ is arithmetically Cohen-Macaulay and each line of $L \cup R$ contains at most $d+1$ points of W, the set $\langle \nu_d(S) \rangle \cap \langle \nu_d(S') \rangle$ is a single point. We have $\{S, S'\} \subseteq \mathcal{S}(X_{m,d}, P)$. Notice that either $\sharp(S \cap L) \geq (d+1)/2$ or $\sharp(S' \cap R) \geq (d+1)/2$.

Proposition 2. *Fix $P \in \mathbb{P}^{\binom{m+d}{m}-1}$ such that there are finite subsets $A, B \subset X_{m,d}$ such that $\sharp(A) = \sharp(B) = d+1$, $P \in \langle A \rangle \cap \langle B \rangle$, $P \notin \langle A' \rangle$ for any $A' \subsetneq A$, $P \notin \langle B' \rangle$ for any $B' \subsetneq B$ and $A \neq B$. Write $A = \nu_d(S)$ and $B = \nu_d(S')$. Assume $\sharp(T \cap (S \cup S')) \leq d+1$ for every line $T \subset \mathbb{P}^m$ and the existence of a* *line $L \subset \mathbb{P}^m$ such that $\sharp(S \cap L) \geq (d+1)/2$. Then there are a line $R \neq L$ such that $W := S \cup S'$ is as in Example 2.*

Proof. By [2] we have $S \cap S' = \emptyset$ and $r_{X_{m,d}}(P)) = d+1$. First assume $m = 2$. Set $F := S \cup S' \setminus (S \cup S') \cap L$. Since $\sharp(S \cup S') \cap L) \leq d+1$ we have $h^1(L, \mathcal{I}_{(S \cup S') \cap L}(d)) = 0$. Since a line is arithmetically Cohen-Macaulay, we get $h^1(\mathbb{P}^2, \mathcal{I}_{(S \cup S') \cap L}(d)) = 0$. Hence [2], Lemma 1, and Castelnuovo's inequality gives $h^1(\mathbb{P}^2, \mathcal{I}_F(d-1)) > 0$. Since $\sharp(F) \leq 2(d-1)+1$, [3], Lemma 1, gives the existence of a line R such that $\sharp(F \cap R) \geq d+1$. Since $S \cup S'$ is reduced, $F \cap L = \emptyset$. Thus $R \neq L$. By assumption we have $\sharp(F \cap R) = d+1$ and the point $R \cap L$ is not contained in $S \cup S'$. Thus we are in the situation described in Example 2. The case $m > 2$ easily follows by induction on m as in the proof of Lemma 1. □

Remark 1. Fix $P \in \mathbb{P}^{\binom{m+d}{m}-1}$ such that $b_{X_{m,d}}(P) \leq d+1$. By [4], Lemma 2.1.5, or [3], Proposition 1, $b_{X_{m,d}}(P)$ is the minimal integer t such that there is a zero-dimensional smoothable subscheme $Z \subset X_{m,d}$ such that $P \in \langle Z \rangle$. Hence [2], Theorem 1 gives that all points P appearing in the statements of Theorem 1 and Propositions 1 and 2 have border rank $d+1$.

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