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TECHNOLOGICAL DIFFUSION AND BENTHAMITE UTILITY IN THE RAMSEY MODEL: AN ANALYTICAL EXPLORATION

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Abstract: We investigate the dynamic effects of assuming a Benthamite utility function into the modified Ramsey model with technological diffusion introduced by Ferrara and Guerrini [7]. In addition, we derive a closed form solution for the model when capital's share is equal to the reciprocal of the intertemporal elasticity of substitution.

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1. Introduction

Ramsey's article [24] was the first in the long history of economics to introduce a dynamic method, i.e. a calculus of variation, to examine the question of how much a country would need to save and invest in order to maximize welfare. Whereas the Ramsey model is a good approximation of the real world, it describes the economic reality incompletely since it abstracts from technological diffusion. In addition, the Ramsey model assumes a constant population growth rate, an hypothesis which is not completely realistic, because population growing exponentially can be arbitrarily large. Recently, Accinelli and Brida [1], Bucci and Guerrini [4], Ferrara and Guerrini [6]-[10], Germanà and

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Guerrini [11], and Guerrini [13]-[21], have explored the implications of studying the neoclassical growth models within a framework where the change over time of the labor force is governed by the logistic law or by a bounded population growth rate. Duczynski [5] have analyzed the role of technological diffusion in the Ramsey model.

Ferrara and Guerrini [7] have combined within the same framework these two different research lines. In this paper, motivated by the work of Ferrara and Guerrini [8], who introduced a Benthamite formulation for the utility function into the Ramsey model with logistic population growth rate, we wish to investigate the dynamic effects of assuming a Benthamite utility function into Ferrara and Guerrini's model [7]. This set-up leads the economy to be described by a four dimensional dynamical system, whose unique non-trivial steady state equilibrium is a saddle point with a three dimensional stable manifold. Three stable roots, rather than only one as in basic neoclassical models, determine the speed of convergence. The crucial determinant of the asymptotic speed of convergence is the larger of the three negative eigenvalues. Finally, following Smith [25], we show that our model becomes analytically tractable when the intertemporal elasticity of substitution is equal to the reciprocal of capital's share.

2. The Model

We consider a closed economy populated by a fixed number of identical infinitely lived households that, for simplicity, is normalized to one. The household size L evolves according to the following law

$$\frac{\dot{L}}{L} = a - bL \equiv n(L), \tag{1}$$

with $a > b > 0$, L_0 normalized to one, and a dot over a variable which denotes its time derivative. The time argument is suppressed to ease the burden of notations. equation (1) is known as the Verhulst equation (see [26]), and the underlying population model is known as the logistic model. Let C be aggregate consumption. In contrast to Ferrara and Guerrini [7], we are supposing that the society is weighted by numbers so that its welfare responds to total population as well as capital per capita consumption. This means that the the felicity function is now multiplied by the size of the family (Benthamite welfare function). Each household maximizes its dynastic utility

$$\int_0^\infty u(C/L)Le^{-\rho t}dt, \tag{2}$$

where $\rho > 0$ is the rate of time preference. The instantaneous utility function takes the constant intertemporal elasticity of substitution (CIES) form

$$u(C/L) = \frac{(C/L)^{1-\theta}}{1-\theta}, \tag{3}$$

where $1/\theta > 0$ is the elasticity of intertemporal substitution. If $\theta = 1$, then (3) becomes $u(C/L) = \ln(C/L)$. As in Duczynski [5], output Y is produced with a Cobb-Douglas technology $Y = K^\alpha(AL)^{1-\alpha}$, where K denotes the capital stock, A is the labor-augmenting technological progress, and $\alpha \in (0, 1)$ is the capital share in production. Labor productivity is governed by the law of motion

$$\frac{\dot{A}}{A} = g + \lambda\frac{\tau A^L - A}{A}, \tag{4}$$

where g, λ, τ are positive parameters, and A^L is the level of technology in the world's technological leader. We assume $A/(\tau A^L) \leq 1$, and $\tau \geq 1$, so that the economy converges to a lower steady-state than the level of the leading country. In (4), the terms g reflects domestic forces of the technological innovations, the term $\lambda(\tau A^L - A)/A$ corresponds to technological diffusion from the leading country. The equation of motion for A^L is $\dot{A}^L/A^L = g$, so that $A^L = A_0^L e^{gt}$. An assumption similar to (4) was proposed by Nelson and Phelps [22], but in a context with no parameter τ, and λ meaning a positive function of the domestic human-capital intensity. The household's budget constraint is $Y = I+C$, where I is the gross investment. Regarding the capital stock, it accumulates according to

$$\dot{K} = I - \delta K, \tag{5}$$

where $\delta > 0$ is the depreciation rate.

3. The Optimization Problem

The household's optimization problem is to maximize its dynastic utility (2) subject to constraints (1), (4), and (5). It is well-known that this problem can be dealt with the Pontryagin maximum principle (see Pontryagin et al [23]). Let H be the current-value Hamiltonian of our problem

$$H = \frac{C^{1-\theta}L^{\theta}}{1-\theta} + \mu\left[K^{\alpha}(AL)^{1-\alpha} - C - \delta K\right], \tag{6}$$

where μ is the co-state variable associated to (5). By applying the maximum principle, we obtain that the dynamics of K, C, L, μ must satisfy the following conditions

$$C^{-\theta}L^{\theta} = \mu, \tag{7}$$
$$\dot{\mu} = -\mu[\alpha K^{\alpha-1}(AL)^{1-\alpha} - \rho - \delta], \tag{8}$$
$$\dot{K} = K^{\alpha}(AL)^{1-\alpha} - C - \delta K. \tag{9}$$

Furthermore, we assume the usual transversality condition $\lim_{t\to\infty} e^{-\rho t}\mu K = 0$. Log-differentiating (7) with respect to time, we get the equality $\dot{C}/C - \dot{L}/L = -(1/\theta)\dot{\mu}/\mu$. Thus, using (8), we see that the co-state variable μ can be eliminated from the above equations. Next, let $y = Y/(AL)$, $k = K/(AL)$, and $c = C/(AL)$ denote output, capital stock, and consumption per effective labor unit, respectively. Log-differentiation of k (resp. c) gives $\dot{k}/k = \dot{K}/K - \dot{A}/A - \dot{L}/L$ (resp. $\dot{c}/c = \dot{C}/C - \dot{A}/A - \dot{L}/L$). In addition, set $x = \tau A^{L}/A$. By log-differentiating x, and using (4), we get $\dot{A}/A = g - \dot{x}/x$, and

$$\dot{x} = \lambda x(1-x). \tag{10}$$

Therefore, using (9), after rearrangement, we obtain that the economy is described by

$$\dot{k} = k^{\alpha} - \left(\delta + g - \frac{\dot{x}}{x} + \frac{\dot{L}}{L}\right)k - c, \tag{11}$$
$$\dot{c} = \frac{c}{\theta}\left[\alpha k^{\alpha-1} - \rho - \delta - \theta\left(g - \frac{\dot{x}}{x}\right)\right], \tag{12}$$

together with equations (1), (10), and the transversality condition

$$\lim_{t\to\infty} e^{-[\rho-(1-\theta)g]t}c^{-\theta}kx^{-(1-\theta)}L = 0. \tag{13}$$

Contrary to Ferrara and Guerrini [7], equation (12) shows that population has now no effect on the growth rate of consumption. Finally, given $k_0 > 0$, $c_0 > 0$, and $x_0 > 0$, this Cauchy problem has a unique solution (k, c, L, x) defined on $[0, +\infty)$ (see Birkhoff and Rota [2]).

4. The Equilibrium

In this section, we focus on characterizing a possible state of the economy where the growth rates of c, k, L and x are equal to zero. Such a situation is defined as steady state.

Lemma 1. *The unique non-trivial steady state of the economy is*

$$k_* = \left(\frac{\alpha}{\rho+\delta+\theta g}\right)^{\frac{1}{1-\alpha}}, \quad L_* = \frac{a}{b}, \tag{14}$$

$$c_* = \frac{\rho+(1-\alpha)\delta+(\theta-\alpha)g}{\alpha}, \quad x_* = 1. \tag{15}$$

Proof. Equations (14)-(15) are obtained from equating (1), (10)-(12) to zero, and solving the resulting system. □

Remark 1. The transversality condition (13) holds at the steady state if $\rho > (1-\theta)g$ as $\lim_{t\to\infty} e^{-[\rho-(1-\theta)g]t} c_*^{-\theta} k_* x_*^{-(1-\theta)} L_* = 0$. Thus, the positiveness of c_* follows from $\rho+\theta g+(1-\alpha)\delta-\alpha g > (1-\alpha)(g+\delta) > 0$.

Proposition 1. *The steady state described by equations (14)-(15) is a saddle point.*

Proof. Linearizing equations (1), (10)-(12) around the steady state yields

$$\begin{bmatrix} \dot{k} \\ \dot{c} \\ \dot{L} \\ \dot{x} \end{bmatrix} = J^* \begin{bmatrix} k-k_* \\ c-c_* \\ L-L_* \\ x-x_* \end{bmatrix}.$$

Computing the elements of the Jacobian matrix $J^* = (J^*_{ij})$, we get

$$J^*_{11} = \rho-(1-\theta)g, \;\; J^*_{12} = -1, \;\; J^*_{13} = bk_*, \;\; J^*_{14} = \lambda k_*, \;\; J^*_{21} = -\frac{(1-\alpha)\alpha c_* k_*^{\alpha}}{\theta},$$

$$J^*_{22} = J^*_{23} = 0, \;\; J^*_{24} = \lambda c_*, \;\; J^*_{31} = J^*_{32} = J^*_{34} = 0,$$

$$J^*_{33} = -a, \;\; J^*_{41} = J^*_{42} = J^*_{43} = 0, \;\; J^*_{44} = -\lambda.$$

There exists a general result in the theory of differential equations, known as the Hartman-Grobman Theorem (see Guckenheimer and Holmes [12]), which guarantees that, if the Jacobian matrix calculated at the equilibrium point has no zero or purely imaginary eigenvalues, then, in a neighbourhood of the equilibrium point, the qualitative properties of a nonlinear system are preserved by the linearization. In our case, two roots of the Jacobian matrix J^* are $\xi_1 = -a$ and $\xi_2 = -\lambda$, and the other two roots can be derived solving $\xi^2 - [\rho - (1-\theta)g]\xi - (1-\alpha)\alpha c_* k_*^\alpha/\theta = 0$. We find the following two real roots

$$\xi_{3,4} = \frac{\rho - (1-\theta)g \pm \sqrt{[\rho-(1-\theta)g]^2 + \frac{4(1-\alpha)\alpha c_* k_*^\alpha}{\theta}}}{2}.$$

Their signs can be determined looking at the trace and determinant of J^*, as the trace (resp. determinant) of a matrix is also equal to the sum (reps. product) of its eigenvalues. After simplification, we get

$$\det(J^*) = (-a)(-\lambda)\left[-\frac{(1-\alpha)\alpha c_* k_*^\alpha}{\theta}\right], \quad trace\,(J^*) = \rho - (1-\theta)g - a - \lambda.$$

From these we can derive that

$$\xi_3\xi_4 = -\frac{(1-\alpha)\alpha c_* k_*^\alpha}{\theta} < 0, \quad \xi_3 + \xi_4 = \rho - (1-\theta)g > 0.$$

Thus, one root is negative and one root is positive. Hence, J^* has one (real) stable root and three (real) unstable roots. This proves that the steady state is (locally) a saddle point (Blume and Simon [3]). □

Since there are three negative eigenvalues, the stable manifold is an hyperplane going through the steady state. This hyperplane is generated by the associated eigenvectors to the three negative eigenvalues of J^*. Let us assume that ξ_4 is the positive root of J^*. With three stable roots, say, for example, $0 > \xi_1 > \xi_2 > \xi_3$, the speeds of adjustment change over time, although asymptotically all variables converge to their respective equilibria at the rate of the slower growing eigenvalue, $-\xi_1$ (asymptotic speed).

5. Explicit Solutions

Let us consider the dynamical system (1), (10)-(12) and perform the following transformations of variables

$$u = k^{1-\alpha}, \quad v = c/k. \tag{16}$$

If we substitute these definitions of u and v into equations (11), (12), we obtain

$$\frac{\dot{k}}{k} = u^{-1} - \left(\delta + g - \frac{\dot{x}}{x} + \frac{\dot{L}}{L}\right) - v, \tag{17}$$

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left[\alpha u^{-1} - \rho - \delta - \theta\left(g - \frac{\dot{x}}{x}\right)\right]. \tag{18}$$

From (16), upon log-differentiation, we get $\dot{u}/u = (1-\alpha)\dot{k}/k$, $\dot{v}/v = \dot{c}/c - \dot{k}/k$. Therefore, combining these with equations (17), (18), we obtain

$$\dot{u} = (1-\alpha)\left[-\left(\delta + g - \frac{\dot{x}}{x} + \frac{\dot{L}}{L}\right) - v\right]u + (1-\alpha),$$

$$\dot{v} = \left(\frac{\alpha}{\theta} - 1\right)u^{-1}v + \left[-\frac{\rho}{\theta} + \left(1 - \frac{1}{\theta}\right)\delta + \frac{\dot{L}}{L}\right]v + v^2. \tag{19}$$

Following Smith [25], we impose the restriction $\theta = \alpha$, so that the term $u^{-1}v$ disappears from equation (19). Hence, the dynamical system (1), (10)-(12) becomes

$$\dot{u} = (1-\alpha)\left[-\left(\delta + g - \frac{\dot{x}}{x} + \frac{\dot{L}}{L}\right) - v\right]u + (1-\alpha), \tag{20}$$

$$\dot{v} = \left[-\frac{\rho}{\alpha} + \left(1 - \frac{1}{\alpha}\right)\delta + \frac{\dot{L}}{L}\right]v + v^2, \tag{21}$$

$$\dot{L} = L(a - bL), \tag{22}$$

$$\dot{x} = \lambda x(1-x). \tag{23}$$

We are now going to show that such a system can be solved analytically.

Lemma 2.

$$L = \frac{ae^{at}}{a - b + be^{at}}, \qquad L_\infty = \lim_{t\to\infty} L = \frac{a}{b}, \tag{24}$$

$$x = \frac{x_0 e^{\lambda t}}{1 - x_0 + x_0 e^{\lambda t}}, \quad x_\infty = \lim_{t\to\infty} x = 1. \tag{25}$$

Proof. equation (22) is a Bernoulli's differential equations. The change of variables $z = L^{-1}$ is made to transform it into a linear first-order differential equation in z, whose solution is easily found. Similarly for (23). □

Proposition 2. *For all t, the time path of capital and consumption per effective labor unit are*

$$k = x_0^{-1} e^{-(\delta+g)t} x L^{-1} \varphi \left\{ k_0^{1-\alpha} + (1-\alpha) \int_0^t \left[x_0^{-1} e^{-(\delta+g)t} x L^{-1} \varphi \right]^{-(1-\alpha)} dt \right\}^{\frac{1}{1-\alpha}},$$

$$c = c_0 k_0^{-1} L \varphi^{-1} e^{(\delta - \frac{\rho+\delta}{\alpha})t} k,$$

where

$$\varphi = 1 - \frac{c_0}{k_0} \int_0^t e^{[-\frac{\rho}{\alpha} + (1-\frac{1}{\alpha})\delta]t} L dt. \qquad (26)$$

Proof. First, we start solving equation (21) since this is a Bernoulli's differential equation in v with the following general solution

$$v = e^{\int_0^t [-\frac{\rho}{\alpha} + (1-\frac{1}{\alpha})\delta + \frac{\dot{L}}{L}] dt} \left(v_0^{-1} - \int_0^t e^{\int_0^t [-\frac{\rho}{\alpha} + (1-\frac{1}{\alpha})\delta + \frac{\dot{L}}{L}] dt} dt \right)^{-1}. \qquad (27)$$

A direct calculation shows that

$$\int_0^t \left[-\frac{\rho}{\alpha} + \left(1 - \frac{1}{\alpha}\right)\delta + \frac{\dot{L}}{L} \right] dt = \left[-\frac{\rho}{\alpha} + \left(1 - \frac{1}{\alpha}\right)\delta \right] t + \ln L.$$

Consequently, we get

$$v = v_0 e^{[-\frac{\rho}{\alpha} + (1-\frac{1}{\alpha})\delta]t} L \varphi^{-1} = \frac{d}{dt}\left(-\ln \varphi\right). \qquad (28)$$

Next, we consider equation (20). It is a linear differential equation in v, whose general solution is given by

$$u = e^{-(1-\alpha)\int_0^t \left(\delta + g - \frac{\dot{x}}{x} + \frac{\dot{L}}{L} + v\right) dt} \left[u_0 + (1-\alpha) \int_0^t e^{(1-\alpha)\int_0^t \left(\delta + g - \frac{\dot{x}}{x} + \frac{\dot{L}}{L} + v\right) dt} dt \right]. \qquad (29)$$

equation (28) yields

$$\int_0^t \left(\delta + g - \frac{\dot{x}}{x} + \frac{\dot{L}}{L} + v \right) dt = (\delta + g)t - \ln \frac{x}{x_0} + \ln L - \ln \varphi. \qquad (30)$$

Hence, substituting (30) into equation (29), we get

$$u = \left[x_0^{-1} e^{-(\delta+g)t} x L^{-1} \varphi\right]^{1-\alpha} \left\{ u_0 + (1-\alpha) \int_0^t \left[x_0^{-1} e^{-(\delta+g)t} x L^{-1} \varphi\right]^{-(1-\alpha)} dt \right\}.$$

The statement now follows from being $k = u^{\frac{1}{1-\alpha}}$ and $c = u^{\frac{1}{1-\alpha}} v$. □

Lemma 3. *The transversality condition* (13) *implies* $\lim_{t\to\infty} \varphi = 0$.

Proof. The assumption $\theta = \alpha$, and the limits of (24), (25), imply that

$$\lim_{t\to\infty} e^{-[\rho-(1-\alpha)g]t} c^{-\alpha} k x^{-(1-\alpha)} L = 0 \Longleftrightarrow \lim_{t\to\infty} e^{-[\rho-(1-\alpha)g]t} c^{-\alpha} k = 0. \qquad (31)$$

Substituting in (31) the formulas for k, c of Proposition 2, we have

$$\lim_{t\to\infty} \varphi \left\{ k_0^{1-\alpha} + (1-\alpha) x_0^{1-\alpha} \int_0^t \left[e^{-(\delta+g)t} x L^{-1} \varphi\right]^{-(1-\alpha)} dt \right\} = 0. \qquad (32)$$

Differentiating equations (24), (25) and (26) with respect to time, we obtain $\dot{L} > 0$, $\dot{x} < 0$ and $\dot{\varphi} < 0$, respectively. Therefore, the function $L \geq 1$ increases monotonically, while the functions x and $\varphi \leq 1$ decrease monotonically. Consequently, the integrals $\int_0^t \left[e^{-(\delta+g)t} x L^{-1} \varphi\right]^{-(1-\alpha)} dt$ and $\int_0^t e^{(1-\alpha)(\delta+g)t} dt$ must have the same behaviour in the long run. From being

$$\int_0^t e^{(1-\alpha)(\delta+g)t} dt = \left[e^{(1-\alpha)(\delta+g)t} - 1\right] / (1-\alpha)(\delta+g),$$

we can conclude that the expression inside the parenthesis of formula (32) diverges. The statement is now immediate. □

Remark 2. $k_0/c_0 = \int_0^\infty e^{[-\frac{\rho}{\alpha} + (1-\frac{1}{\alpha})\delta]t} L dt.$

6. Conclusion

In this paper we consider a modified version of the Ramsey model, obtained by introducing technological diffusion, a logistic law formulation for the evolution of population growth, and a Benthamite formulation for the utility function. Within this framework, the model is shown to have a unique non-trivial steady state equilibrium, which is proved to be a saddle point with a three dimensional stable manifold. With three stable roots now determining the speed of convergence, we have that the speeds of adjustment change over time, although asymptotically all variables converge to their respective equilibria at the rate of the slower growing eigenvalue. Finally, we demonstrate that the model has an exact solution when capital's share is equal to the reciprocal of the intertemporal elasticity of substitution.

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