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ON THE SECANT VARIETIES OF WEAKLY 0-DEFECTIVE PROJECTIVE VARIETIES

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Abstract: Let $X \subset \mathbb{P}^n$ be an integral non-degenerate subvariety, which is weakly 0-defective, i.e. such that its dual variety is not a hypersurface. Here we prove that in many cases the secant varieties of X have the expected dimension.

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1. Introduction

Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. For any $P \in \mathbb{P}^n$ let $r_X(P)$ be the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$. The integer $r_X(P)$ is called the X-rank. Let $\mathcal{S}(X, P)$ the set of all $S \subset X$ computing $r_X(P)$, i.e. the set of all subsets $S \subset X$ such that $\sharp(S) = r_X(P)$ and $P \in \langle S \rangle$. Notice that any $S \in \mathcal{S}(X, P)$ is linearly independent and $P \notin \langle S' \rangle$ for any $S' \subsetneq S$. An important problem is to find conditions on X and P such that $\sharp(\mathcal{S}(X, P)) = 1$. For every integer $t \geq 1$ let $\sigma_k(X)$ denote the closure in $\mathbb{P}^n$ of all $(k-1)$-dimensional linear spaces spanned by t points of Y. Set $\sigma_0(X) := \emptyset$. If $\sigma_{k-1}(X) \neq \mathbb{P}^n$, then a general $P \in \sigma_k(X)$ has X-rank k. Very seldom $\sharp(\mathcal{S}(X, P)) = 1$ for a general $P \in \mathbb{P}^n$ (see [2] for a full classification when X is smooth and $\dim(X) \leq 3$). Now we fix an integer $k \geq 2$ such that $k(\dim(X) + 1) \leq n$. There is an easy to check condition which implies that $\sharp(S(X, P)) = 1$ for a general $P \in \sigma_k(X)$: it

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is sufficient to assume that X is not weakly $(k-1)$-defective (see [3], Proposition 1.5), i.e. it is sufficient to assume that for a general $S \subset X$ such that $\sharp(S) = k$ the points of S are isolated component of the contact locus of a general hyperplane of $\mathbb{P}^n$ tangent to X at each point of X. No curve is weakly defective (see [1] or [3], Remark 1.2). Here we are interested in the opposite case, i.e. want to state some criterion for $\sharp(S)(X, P)$ when X is weakly defective in the strongest possible sense, i.e. it is weakly 0-defective. From the very definition we get that X is weakly 0-defective if and only if its dual variety $X^* \subset \mathbb{P}^{n*}$ is not a hypersurface. In this case the integer $\nu_0(X) := n-1-\dim(X^*)$ will be called the *weak defect* of X or *dual defect* of X. Since $\mathrm{char}(\mathbb{K}) = 0$ the contact locus of a general $H \in X^*$ is a linear space. By definition this linear space has dimension $\nu_0(X)$. These varieties (even smooth ones) are quite frequent: $2x > m > x$, X is a smooth $\mathbb{P}^x$-bundle over a variety of dimension $m - x$, the fiber of the ruling are embedded as linear subspaces, then $\nu_0(X) \geq 2x - m$ (see [5], §0) (the converse holds if X is smooth and $\nu_0(X) \geq m/2$ (see [4], [5]). Thus for each fixed $m \geq 3$ there are many such $X \subset \mathbb{P}^n$ for all $n \gg 0$. For any $P \in X_{reg}$ let $V(X, P)$ (resp. $W(X, P)$) be the union of all lines (resp. all $\nu_0(X)$-dimensional linear subspaces)$L \subset X$ such that $P \in X$. Obviously $W(X, P) \subseteq V(X, P) \subseteq (X \cap T_P X)_{red}$ for every $P \in X_{reg}$. Moreover, P is contained in every irreducible component of $V(X, P)$ and in every irreducible component of $W(X, P)$. Thus both $V(X, P)$ and $W(X, P)$ are connected. There is a non-empty open subset U of X_{reg} such that all $V(X, P)$ and all $W(X, P)$, $P \in U$, are " dimensionally and combinatorially the same ". Set $m := \dim(X)$. Let $v(X)$ be the minimal dimension of an irreducible component of $V(X, P)$ for any $P \in U$.

Proposition 1. *Fix an integer $s \geq 2$ such that $s(m + 1) - 1 < n$. Fix a general $S \subset X_{reg}$ such that $\sharp(S) = s$. Set $\Psi_S := X \cap \langle \cup_{P \in S} \in T_P X\rangle) \setminus (\cup_{P \in S} V(X, P))$. Assume $\dim(\langle \cup_{P \in S} T_P X\rangle) = s(m + 1) - 1$ and $\dim(\Psi_S) \leq v(X) - 1$. Then $\sharp(\mathcal{S}(X, Q)) = 1$ for a general $Q \in \sigma_s(X)$.*

Proof. Since $\dim(\langle \cup_{P \in S} \in T_P X\rangle) = s(m+1)-1$, Terracini's Lemma gives $\dim(\sigma_s(X)) = s(m + 1) - 1$, i.e. that X is not $(s - 1)$-defective (with the terminology of [1]).

Claim. *We claim that if the Gauss mapping of X is not generically finite, then X is 1-defective.*

Proof of Claim. Take a general $A \in \sigma_2(X)$. By Terracini's Lemma there are generall $(P_1, P_2) \in X_{reg} \times X_{reg}$ such that $T_A \sigma_2(X) = \langle T_{P_1} X \cup T_{P_2} X\rangle$. Since the Gauss mapping is not generically finite and each P_i is general in X_{reg}, there is a line $L_i \subset X$, $i = 1, 2$, such that $P_i \in L_i$ and $T_{P_i} X = T_{Q_i} X$ for all $Q_i \in (L_i \cap X_{reg})$. Thus $\langle T_{Q_1}X \cup T_{Q_2}X\rangle = T_A\sigma_2(X)$ for each $(Q_1, Q_2) \in (L_1 \cap X_{reg}) \times (L_2 \cap X_{reg})$.

Since 1-defectivity implies $(s-1)$-defectivity, the claim gives that the Gauss mapping of X is generically finite. Hence a general tangent space to X is tangent at a unique point of X. Hence P is an isolated pointed of $\mathrm{Sing}(X \cap T_PX)$ for every $P \in S$. Assume $\sharp(S)(X,Q) \geq 2$. Take $S_1 \in \mathcal{S}(X,Q) \setminus \{S\}$. Since Q is general, for dimensional reasons S_1 is general in the symmetric product of s copies of X. Since both $\langle \cup_{P\in S} \in T_PX\rangle$ and $\langle \cup_{P_1 \in S_1} \in T_{S_1}X\rangle$ are the tangent space of $\sigma_s(X)$ at Q (Terracini's Lemma), our assumption on Ψ_S gives $\cup_{P\in S}V(S,P) = \cup_{P_1\in S_1}V(X,P_1)$. Thus for each $P \in S$ there is $P_1 \in S$ such that $V(X,P) = V(X,P_1)$. Thus T_PX is tangent to X at P_1, contradiction. □

The statement of Proposition 1 may seemed to be tautological. However, notice that quite often the sets $V(X,P) = W(X,P)$, P general in X, are the maximal linear subspaces contained in X and containing P. This is the essential observation which allows us to carry over some inductive proofs (see Proposition 3). For each integer $s \geq 1$ fix a general $S \subset X$ such that $\sharp(S) = s$. Let $\mathcal{H}(-2S)$ be the set of all hyperplanes of $\mathbb{P}^n$ containing $\langle \cup_{P\in S}T_PX\rangle$. For any $H \in \mathcal{H}(-2S$ the contact locus $\Delta_{H,S}$ of H is the union of the closures of the irreducible components of $\{P \in X_{reg} : Y_PX \subset H\}$ which intersects s. Let $\nu_s(X)$ be the maximal dimension of one of the irreducible components of $\Delta_{H,S}$, H general in $\mathcal{H}(-2S)$. □

Question 1. What are the possible sequences $\nu_s(X)$, $s \geq 1$, for weakly 0-defective m-dimensional subvarieties of $\mathbb{P}^n$? What are the possible strings of integers $\nu_i(X)$, $0 \leq i \leq s$, for weakly 0-defective m-dimensional subvarieties of $\mathbb{P}^n$ such that $\dim(\sigma_{s+1}(X)) = (s+1)(m+1) - 1$?

Proposition 2. *Fix positive integers n, m, s, x such that $m \geq 3$, $x < m < 3x$ and $n \geq (s+1)(m+1) - 1$. Set $k := 2x - m$. Fix any smooth and connected $(m-x)$-dimensional projective variety Y and a $\mathbb{P}^x$-bundle X over Y. Then there is an embedding $X \subset \mathbb{P}^n$ with $\nu_i(X) = k$ for all $0 \leq i \leq s$, $\dim(\sigma_{s+1}(X)) = (s+1)(m+1) - 1$ and such that every fiber of the ruling $u : X \to Y$ is embedded as a linear subspace.*

Proof. Let $A \subset \mathbb{P}^y$ ben integral m-dimensional projective variety such that for a general $P \in A$ there is an x-dimensional linear subspace L contained in A_{reg} and passing through P. As remarked by A. Landman and M. Reid, a dimensional count shows that A is weakly 0-defective and $\nu_0(A) \geq 2x - m = k$ (see (see [4], p. 895). Write $X = \mathbb{P}(E)$ for some vector bundle E on Y such that $\mathrm{rank}(E) = x + 1$. We fix the tautological line bundle $\mathcal{O}_X(1)$ on $X = \mathbb{P}(E)$

and we twist it by $u^*(R)$ with $R \in \mathrm{Pic}(Y)$ and R sufficiently ample. We get a very ample $L := \mathcal{O}_X(1) \otimes u^*(R)$ with $h^0(X, L) \gg n$. Let $v : Y \to X$ be any section of the ruling. Let $j := X \hookrightarrow \mathbb{P}^r$ be the associated embedding. Fix $S \subset j(v(Y_{reg}))$ such that $\sharp(S) = s+1$. For each $P \in S$ the linear space $T_{j(P)}X$ is the linear span of $j(u^{-1}(v(P))$ and $T_P(j(Y))$. Thus by Terracini's Lemma we may assume $\dim(\sigma_{s+1}(j(X))) = (s+1)(m+1) - 1$ (for a fixed s) if we take R sufficiently ample. Take as embedding of X into $\mathbb{P}^n$ a general linear projection of $j(X)$. Since $n \geq (s+1)(m+1)-1$ and the linear projection is general, we still get $\dim(\sigma_{s+1}(X)) = (s+1)(m+1)-1$. It is sufficient to check the assertions on the weak defectivities $\nu_i(X)$, $0 \leq i \leq s$, of X. For very positive R and general linear projections we may also assume that $v(Y) \subset X \subset \mathbb{P}^n$ is non-degenerate and not defective. As claimed at the beginning of the proof dimensional count gives $\nu_0(X) \geq k$. For R sufficiently ample we may assume that for a general $P \in j(X)$ the x-dimensional linear space image of the fiber of the ruling of X through P is the union of all lines contained in $j(X)$ and contained in P. Thus we also get $\nu_0(j(X)) = k$. Taking a general projection the same holds true for $X \subset \mathbb{P}^n$. To get $\mu_i(X) \leq k$ for all $1 \leq i \leq s$, we (perhaps) need to take a more ample R. For sufficiently ample R we may assume that for each integer i such that $1 \leq i \leq s-1$ and a general $S_i \subset X$ such that $\sharp(S_i) = i$ the tangential projection of X for $\langle \cup_{P \in S_i} T_P X \rangle$ is birational onto its image and for a gerneral point Q of the closure of its image $V(X', Q)$ is the x-dimensional linear subspace obtained projecting the corresponding fiber of the ruling $X \to Y$. At each step we need to change R to carry over the existence of S_{i+1} from the existence of S_i. It is the control of all general $V(X', Q)$ which allows us to do the induction. Then we take a general projection into $\mathbb{P}^n$. □

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References

[1] L. Chiantini, C. Ciliberto, Weakly defective varieties, *Trans. Amer. Math. Soc.*, **454**, No. 1 (2002), 151-178.

[2] C. Ciliberto, M. Mella, F. Russo, Varieties with one apparent double point, *J. Algebraic Geom.*, **13**, No. 3 (2004), 475-512.

[3] C. Ciliberto, F. Russo, Varieties with minimal secant degree and linear systems of maximal dimension on surfaces, *Adv. Math.*, **206**, No. 1 (2006), 1-50.

[4] L. Ein, Varieties with small dual varieties. I, *Invent. Math.*, **86**, No. 1 (1986), 63-74.

[5] L. Ein, Lawrence varieties with small dual varieties. II, *Duke Math. J.*, **52**, No. 4 (1985), 895-907.

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