International Journal of Pure and Applied Mathematics
Volume 67 No. 2 2011, 141-147

NOTE ON A CHARACTERIZATION OF α-MODIFIED AND TRUNCATED POISSON DISTRIBUTION

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Abstract: Poisson, binomial and negative binomial distributions as elements of a family power-series distributions were characterized by relating their first and second-order moments (see [5]). We give similar characterizations of α-modified and truncated Poisson distributions.

AMS Subject Classification: 60E05, 62E10
Key Words: power-series distribution, moments, characterization, α-modified and truncated Poisson distribution

1. Introduction

Recall that a discrete random variables X has power-series distribution (PSD) if its probability function (pf) is given by

$$P[X = x] = \frac{a(x)\theta^x}{f(\theta)}, \quad x \in T \subseteq \mathbb{N}_\circ, \tag{1}$$

where $\mathbb{N}_\circ = \mathbb{N} \cup \{0\}$, $f(\theta) = \sum_{x \in T} a(x)\theta^x$, θ is an unknown parameter taking

Received: December 3, 2010

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values in some open interval Θ, $a(x)\theta^x > 0$, coefficients $a(x)$ are nonnegative and independent of θ (see [6], [7]).

The characterizations of Poisson, binomial and negative binomial distributions are as follows (cf. [5]).

Theorem 1. *Suppose that a random variable X is distributed according to (1) and has finite expectation. Then X follows a Poisson, binomial or negative binomial distribution if and only if*

$$\frac{E_\theta(X(X-1))}{E_\theta^2 X} = c \quad \textit{for all} \quad \theta \in \Theta$$

for $c = 1$, $c = (n-1)/n$ for some positive integer $n \neq 1$, or $c = (n-1)/n$ for some negative integer n, respectively, where $\Theta = (0, k)$ for some constant k.

The aim of this note is to give counterparts of these characterizations for a α-modified Poisson distribution and for a truncated at 0 Poisson distribution.

2. Characterizations

We start with a α-modified Poisson distribution. Note that α-modified distributions and their applications were studied in [1], [2], and [3]. We are interested in a α-modified Poisson distribution with *pf*:

$$P[X = x] = \frac{\lambda^x}{x!} D_x(1-\lambda)e^\lambda, \quad x = 0, 1, 2, \ldots;\ 0 < \lambda < 1, \tag{2}$$

and $D_x = (-1+\alpha)^x = \Delta^x 0! = x! \sum_{i=0}^{k} \frac{(-1)^i}{i!}$, where α is Riordan's combinatorial symbol defined by $\alpha_k \equiv \alpha^k = k!$ (see [8]). One can see that this distribution is a member of power-series distribution with

$$a(x) = \frac{D_x}{x!},\ \theta = \lambda \text{ and } f(\theta) = f(\lambda) = (1-\lambda)^{-1}e^{-\lambda}.$$

A characterization of a α-modified Poisson distribution is contained in the following theorem.

Theorem 2. *Suppose that the random variable X is distributed according to (1) with $0 < \theta = \lambda < 1$, and has finite expectation. Then X follows a α-modified Poisson distribution if and only if*

$$\frac{\lambda^2 E_\lambda(X(X-1)) - E_\lambda^2 X}{\lambda^2 E_\lambda^2 X} = 1, \quad \textit{for all} \quad \lambda \in (0,1). \tag{3}$$

Proof. The "only if" part. Note that the probability generating function of the distribution in (2) is

$$G_X(s) = \frac{1-\lambda}{1-s\lambda}e^{\lambda(1-s)}.$$

Hence

$$E_\lambda X = G^{'}(1) = \frac{\lambda^2}{1-\lambda}$$

and

$$E_\lambda(X(X-1)) = G^{''}(1) = \frac{\lambda^2(1+\lambda^2)}{(1-\lambda)^2}.$$

Therefore

$$\frac{E_\lambda(X(X-1))}{(E_\lambda X)^2} = 1 + \frac{1}{\lambda^2}$$

which proves (3).

The same result one can get taking into account that for X having PSD we have

$$E_\lambda X = \lambda\frac{f^{'}(\lambda)}{f(\lambda)} \quad \text{and} \quad E_\lambda X^2 = \lambda\frac{f^{'}(\lambda)}{f(\lambda)} + \lambda^2\frac{f^{''}(\lambda)}{f(\lambda)} \tag{4}$$

(see [4]), and

$$\frac{E_\lambda(X(X-1))}{(E_\lambda X)^2} = \frac{f^{''}(\lambda)f(\lambda)}{(f^{'}(\lambda))^2}.$$

Hence for a α-modified Poisson distribution where $f(\lambda) = (1-\lambda)^{-1}e^{-\lambda}$ we get (3).

The "if" part. From the assumption (3) we have

$$\frac{E_\lambda(X(X-1))}{(E_\lambda X)^2} = 1 + \frac{1}{\lambda^2}, \quad \text{for all } \lambda \in (0,1),$$

and from the assumption that X belongs to a family PSD, by (4), we obtain

$$\frac{E_\lambda(X(X-1))}{(E_\lambda X)^2} = \frac{f^{''}(\lambda)}{f(\lambda)} \Big/ \left[\frac{f^{'}(\lambda)}{f(\lambda)}\right]^2.$$

Therefore we have

$$\frac{f^{''}(\lambda)}{f(\lambda)} \Big/ \left[\frac{f^{'}(\lambda)}{f(\lambda)}\right]^2 = 1 + \frac{1}{\lambda^2},$$

or

$$\frac{f''(\lambda)}{f(\lambda)} = \left(1 + \frac{1}{\lambda^2}\right)\left[\frac{f'(\lambda)}{f(\lambda)}\right]^2. \tag{5}$$

But

$$\left[\frac{f'(\lambda)}{f(\lambda)}\right]' = \frac{f''(\lambda)}{f(\lambda)} - \left[\frac{f'(\lambda)}{f(\lambda)}\right]^2$$

or

$$\frac{f''(\lambda)}{f(\lambda)} = \left[\frac{f'(\lambda)}{f(\lambda)}\right]' + \left[\frac{f'(\lambda)}{f(\lambda)}\right]^2 \tag{6}$$

and by (5) we get

$$\left[\frac{f'(\lambda)}{f(\lambda)}\right]' + \left[\frac{f'(\lambda)}{f(\lambda)}\right]^2 = \left(1 + \frac{1}{\lambda^2}\right)\left[\frac{f'(\lambda)}{f(\lambda)}\right]^2$$

which leads to

$$\left[\frac{f'(\lambda)}{f(\lambda)}\right]' = \frac{1}{\lambda^2}\left[\frac{f'(\lambda)}{f(\lambda)}\right]^2.$$

Hence

$$\frac{f'(\lambda)}{f(\lambda)} = \frac{\lambda}{1 - c_1\lambda}$$

which gives

$$f(\lambda) = \frac{c_2 e^{-\lambda/c_1}}{(1 - c_1\lambda)^{1/c_1^2}}, \tag{7}$$

where c_1 and c_2 are arbitrary allowing to use the theory of power series in (7). With $c_1 = c_2 = 1$ we have

$$f(\lambda) = (1 - \lambda)^{-1} e^{-\lambda},$$

which ends the proof. □

As a complement of the moments discussed in Theorem 2 we give a list of moments ($m_r := EX^r$, $\mu_r := E(X - EX)^r$) of a α-modified Poisson distribution which can be useful in statistical inferences:

$$m_1 = \frac{\lambda^2}{1-\lambda}, \quad m_2 = \frac{\lambda^4 - \lambda^3 + 2\lambda^2}{(1-\lambda)^2},$$
$$m_3 = \frac{\lambda^6 - 3\lambda^5 + 7\lambda^4 - 3\lambda^3 + 4\lambda^2}{(1-\lambda)^3},$$
$$m_4 = \frac{\lambda^8 - 6\lambda^7 + 19\lambda^6 - 25\lambda^5 + 32\lambda^4 - 5\lambda^3 + 8\lambda^2}{(1-\lambda)^4},$$
$$\mu_2 = \frac{\lambda^2(2-\lambda)}{(1-\lambda)^2}, \quad \mu_3 = \frac{\lambda^4 - 3\lambda^3 + 4\lambda^2}{(1-\lambda)^3},$$
$$\mu_4 = \frac{3\lambda^6 - 13\lambda^5 + 16\lambda^4 - 5\lambda^3 + 8\lambda^2}{(1-\lambda)^4}.$$

Skewness coefficient and kurtosis are as follows:

$$\gamma = \frac{\lambda^2 - 3\lambda + 4}{\lambda(2-\lambda)^{3/2}}, \quad \kappa = \frac{3\lambda^4 - 13\lambda^3 + 16\lambda^2 - 5\lambda + 8}{\lambda^2(2-\lambda)^2}.$$

Now we present a similar characterization of a Poisson distribution truncated at 0, i.e. the distribution with the probability function

$$P[X = x] = \frac{\lambda^x}{x!}\frac{e^{-\lambda}}{1-e^{-\lambda}}, \quad x \in T = \mathbb{N}.$$

We see that this distribution is a member of GPSD (generalized power series distribution, cf. [7]) with

$$a(x) = \frac{1}{x!}, \quad \theta = \lambda, \quad f(\theta) = f(\lambda) = e^{\lambda} - 1.$$

Theorem 3. *Suppose that a random variable X is distributed according to* (1) *with $T = \mathbb{N}$, and has finite expectation. Then X follows a Poisson distribution truncated at* 0 *if and only if*

$$\frac{e^{\lambda}E_{\lambda}(X(X-1)) + E^2_{\lambda}X}{e^{\lambda}E^2_{\lambda}X} = 1, \quad \lambda \in (0, +\infty). \tag{8}$$

Proof. The "only if" part. From the assumption that X belongs to a family PSD we have

$$E_{\lambda}X = \lambda\frac{f^{'}(\lambda)}{f(\lambda)} \quad \text{and} \quad E_{\lambda}X^2 = \lambda\frac{f^{'}(\lambda)}{f(\lambda)} + \lambda^2\frac{f^{''}(\lambda)}{f(\lambda)}.$$

Hence

$$\frac{E_\lambda(X(X-1))}{(E_\lambda X)^2} = \frac{f''(\lambda)f(\lambda)}{(f'(\lambda))^2}.$$

Therefore for a Poisson distribution truncated at 0 with $f(\lambda) = e^\lambda - 1$ we have

$$\frac{E_\lambda(X(X-1))}{(E_\lambda X)^2} = 1 - e^{-\lambda},$$

which proves (8).

The "if" part. From the assumption (8) and from the assumption that X belongs to a family PSD we have

$$\frac{E_\lambda(X(X-1))}{(E_\lambda X)^2} = \frac{f''(\lambda)}{f(\lambda)} \Big/ \left[\frac{f'(\lambda)}{f(\lambda)}\right]^2.$$

Hence

$$\frac{f''(\lambda)}{f(\lambda)} = (1 - e^{-\lambda}) \left[\frac{f'(\lambda)}{f(\lambda)}\right]^2$$

and by (6)

$$\left[\frac{f'(\lambda)}{f(\lambda)}\right]' = -e^{-\lambda} \left[\frac{f'(\lambda)}{f(\lambda)}\right]^2$$

which leads to

$$f(\lambda) = c_2(c_1 e^\lambda - 1)^{1/c_1} \tag{9}$$

for arbitrary c_1 and c_2 allowing to use the theory of power series in (9). With $c_1 = c_2 = 1$ we have

$$f(\lambda) = e^\lambda - 1$$

which ends the proof. □

References

[1] S. Berg, J. Jaworski, Modified binomial and Poisson distributions with application in random mapping theory, *J. Statist. Plann. Infer.*, **18** (1988), 313-322.

[2] S. Berg, K. Nowicki, Statistical inference for a class of modified power series distributions with applications to random mapping theory, *J. Statist. Plann. Infer.*, **28** (1991), 247-261.

[3] S. Chakraborty, On some new α-modified binomial and Poisson distributions and their applications, *Comm. Statist. Theory Methods*, **37** (2008), 1755-1769.

[4] R. Gupta, Modified power series distribution and some of its applications, *Sankhyā, Ser. B*, **36** (1974), 288-298.

[5] A. Kyriakoussis, Gang Li, A. Papadopoulos, On characterization and goodness-of-fit test of some discrete distribution families, *J. Statist. Plann. Infer.*, **74** (1998), 215-228.

[6] A. Noack, A class of random variables with discrete distributions, *Ann. Math. Stat.*, **21** (1950), 127-132.

[7] G.P. Patil, Certain properties of the generalized power series distribution, *Ann. Inst. Stat. Math.*, **14** (1962), 179-182.

[8] J. Riordan *Combinatorial Identities*, John Wiley and Sons, New York (1968).

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