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A REMARK ON THE CHROMATIC POLYNOMIALS OF INCOMPARABILITY GRAPHS OF POSETS

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Abstract: The incomparability graph $I(P)$, associated with a poset P, has vertex set P and edge ij whenever i and j are incomparability in P. In this remark we present an identity involving chromatic polynomials of incomparability graphs of a sequence of posets. The result can be regarded as a generalization of a symmetrical Eulerian identity.

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1. Introduction

Let $(P, \prec)$ be a strict partially ordered set (or poset), i.e., P is a set partially ordered by a transitive irreflexive relation $\prec$ (see, e.g. [8]). A permutation $\pi : P \to P$ has a drop at an element $x \in P$ if $\pi(x) \prec x$. Denote by $\delta_P(k)$ the number of permutations on P which have k drops.

There are several ways in which one can associate a graph to a given poset P. Here we consider the incomparability graph, denoted by $I(P)$, defined as follows. The vertex set of $I(P)$ consists of the elements of P. Elements x and y are adjacent if they are incomparable in P with respect to $\prec$, i.e., neither $x \prec y$ nor $y \prec x$ hold. For more information on the interrelation between posets and graphs see the survey paper [9] as well as [1].

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In this remark, we deal with the chromatic polynomials of incomparability graphs of a sequence of posets. Given a graph G and an integer k, the chromatic polynomial $\chi_G(k)$ counts the number of k-vertex-colorings of G. The chromatic polynomial is related with more general Tutte polynomial, which has various applications in graph theory as well as statistical physics, see [5]. The main result in this remark is the following.

Theorem 1. *Let $1 \le a \le b$ be two integers. Suppose that $\{(P_i, \prec_i)\}_{i=1}^{b+1}$ are $b+1$ posets with cardinalities $|P_i| = a+i-1$ for $i = 1, 2, \cdots, b+1$. We have*

$$\sum_{i=1}^{b+1}(a+i)\sum_{j=0}^{i}(-1)^j\binom{a+b+1}{a+i}\binom{a+i}{j}\chi_{I(P_i)}(i-j)$$
$$= \sum_{i=b-a+1}^{b+1}(a+i)\sum_{j=0}^{i}(-1)^j\binom{a+b+1}{a+i}\binom{b+i}{j}\chi_{I(P_i)}(i-j). \quad (1)$$

Observe that the both sides of equation (1) are almost the same except for the ranges of summation. The only restriction imposed on the sequence of posets in Theorem 1 is their cardinalities. If we take $P_1 = [a], P_2 = [a+1], \cdots, P_{b+1} = [a+b]$ with standard linear order $<$, then $I(P_i)$ is the empty graph on $a+i-1$ vertices for $i = 1, \cdots, b+1$. Hence, the corresponding chromatic polynomials $\chi_{I(P_i)}(k) = k^{a+i-1}$ for $i = 1, \cdots, b+1$, so that Theorem 1 reduces to the following symmetrical Eulerian identity proposed in [4].

Corollary 2. (see [4]) *For positive integers a and b,*

$$\sum_{k=a}^{a+b}\binom{a+b}{k}\left\langle {k \atop a-1} \right\rangle = \sum_{k=b}^{a+b}\binom{a+b}{k}\left\langle {k \atop b-1} \right\rangle. \quad (2)$$

This identity arises in a recent study of descents in permutations which have a restriction on their largest drop, see [3].

2. Proof of Theorem 1

In this section, we present a proof of Theorem 1 based on the generating function method.

Recall that $\delta_Q(k)$ is the number of permutations on poset Q which have k drops. By an equivalent definition in [2], the Eulerian number $\left\langle {n \atop k} \right\rangle$ counts the number of permutations on $[n]$ which have k drops. Therefore, we may have a correspondence between $\delta_Q(k)$ and $\left\langle {n \atop k} \right\rangle$ when $|Q| = n$.

Let $\{(Q_k, \prec_k)\}_{k=0}^{\infty}$ be a sequence of posets with cardinalities $|Q_k| = k$. Throughout this paper, we use the convention $\delta_{Q_0}(0) = 0$ with $Q_0 = \emptyset$, which corresponds to the convention $\left\langle {0 \atop 0} \right\rangle = 0$ (the more common convention is $\left\langle {0 \atop 0} \right\rangle = 1$, see [7]).

The generating function for $\delta_{Q_n}(i)$ is shown to be given by (c.f. equation (7.56) in [7])

$$E(w, z) = \frac{e^z - e^{wz}}{e^{wz} - we^z} = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \delta_{Q_n}(i) w^i \frac{z^n}{n!}. \tag{3}$$

Therefore, we obtain

$$(e^{wz} - we^z) E(w, z) = e^z - e^{wz}. \tag{4}$$

We first calculate the left-hand side of (4). Employing (3), we have

$$\begin{aligned}
e^{wz} E(w, z) &= \sum_{k=0}^{\infty} \frac{(wz)^k}{k!} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \delta_{Q_n}(i) w^i \frac{z^n}{n!} \\
&= \sum_{k=0}^{\infty} \frac{(wz)^k}{k!} \sum_{n'=k}^{\infty} \sum_{i'=k}^{\infty} \delta_{Q_{n'-k}}(i' - k) w^{i'-k} \frac{z^{n'-k}}{(n' - k)!} \\
&= \sum_{k=0}^{\infty} \sum_{n'=k}^{\infty} \sum_{i'=k}^{\infty} \frac{1}{k!} \delta_{Q_{n'-k}}(i' - k) w^{i'} \frac{z^{n'}}{(n' - k)!} \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \sum_{i=k}^{\infty} \binom{n}{k} \delta_{Q_{n-k}}(i - k) w^i \frac{z^n}{n!},
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
we^z E(w, z) &= w \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \delta_{Q_n}(i) w^i \frac{z^n}{n!} \\
&= w \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{n'=k}^{\infty} \sum_{i'=1}^{\infty} \delta_{Q_{n'-k}}(i' - 1) w^{i'-1} \frac{z^{n'-k}}{(n' - k)!} \\
&= \sum_{k=0}^{\infty} \sum_{n'=k}^{\infty} \sum_{i'=1}^{\infty} \frac{1}{k!} \delta_{Q_{n'-k}}(i' - 1) w^{i'} \frac{z^{n'}}{(n' - k)!} \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \sum_{i=1}^{\infty} \binom{n}{k} \delta_{Q_{n-k}}(i - 1) w^i \frac{z^n}{n!}.
\end{aligned} \tag{6}$$

Next, we expand the right-hand side of (4) and get

$$e^z - e^{wz} = \sum_{k=0}^{\infty} \frac{(1-w^k)z^k}{k!}. \tag{7}$$

Expanding the corresponding sums in (5)–(7) and identifying coefficients of $w^i z^n$ in (4), we obtain for $n > i \geq 1$,

$$\sum_{k=0}^{i} \binom{n}{k} \delta_{Q_{n-k}}(i-k) - \sum_{k=0}^{n-i} \binom{n}{k} \delta_{Q_{n-k}}(i-1) = 0. \tag{8}$$

Recall that we have

$$\delta_{Q_n}(m) = \delta_{Q_n}(n-m-1) \tag{9}$$

for integers $n \geq 0$ and $m \geq 0$. Setting $n = a+b$ and $i = a$ in (8), we obtain

$$\sum_{k=a}^{a+b} \binom{a+b}{k} \delta_{Q_k}(a-1) = \sum_{k=b}^{a+b} \binom{a+b}{k} \delta_{Q_k}(b-1). \tag{10}$$

We will need the following lemma regarding chromatic polynomials of incomparability graphs.

Lemma 3. (see [2], [6]) *Suppose $(P, \prec)$ is a poset of size n and $k \geq 0$ is an integer. Then*

$$\delta_P(n-k) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} \chi_{I(P)}(k-j). \tag{11}$$

By our assumptions in Theorem 1 and Lemma 3, we may rewrite the left-hand side of (10) as

$$\begin{aligned}
\sum_{i=1}^{b+1} \binom{a+b}{a+i-1} \delta_{P_i}(a-1) &= \sum_{i=1}^{b+1} \sum_{j=0}^{i} (-1)^j \binom{a+i}{j} \\
&\quad \times \chi_{I(P_i)}(i-j) \binom{a+b}{a+i-1} \\
&= \sum_{i=1}^{b+1} \sum_{j=0}^{i} (-1)^j \frac{a+i}{a+b+1} \binom{a+b+1}{a+i} \\
&\quad \times \binom{a+i}{j} \chi_{I(P_i)}(i-j).
\end{aligned} \tag{12}$$

Similarly, the right-hand side of (10) is

$$\begin{aligned}\sum_{i=b-a+1}^{b+1}\binom{a+b}{a+i-1}\delta_{P_i}(b-1) &= \sum_{i=b-a+1}^{b+1}\sum_{j=0}^{i}(-1)^j\binom{b+i}{j} \\ &\quad\times\chi_{I(P_i)}(i-j)\binom{a+b}{a+i-1} \\ &= \sum_{i=b-a+1}^{b+1}\sum_{j=0}^{i}(-1)^j \\ &\quad\times\frac{a+i}{a+b+1}\binom{a+b+1}{a+i} \\ &\quad\times\binom{b+i}{j}\chi_{I(P_i)}(i-j). \qquad (13)\end{aligned}$$

Equating (12) to (13), we get (1). The proof is then complete.

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