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THE BLACK-SCHOLES EQUATION AND CERTAIN QUANTUM HAMILTONIANS

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Abstract: In this paper a quantum mechanics is built by means of a non-Hermitian momentum operator. We have shown that it is possible to construct two Hermitian and two non-Hermitian type of Hamiltonians using this momentum operator. We can construct a generalized supersymmetric quantum mechanics that has a dual based on these Hamiltonians. In addition, it is shown that the non-Hermitian Hamiltonians of this theory can be related to Hamiltonians that naturally arise in the so-called quantum finance.

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1. Introduction

In the origins of quantum mechanics P.A.M. Dirac [6] observed that commuta-

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tion relations

$$[x_i, x_j] = [P_i, P_j] = 0, \qquad [x_i, P_j] = i\delta_{ij}, \tag{1.1}$$

are satisfied by the operators $x_i, P_j = -i\partial_j$ and also by the set

$$x_i, \qquad P_{(f)j} = -i\partial_j + i\partial_j f, \tag{1.2}$$

with f been an arbitrary function, see [6]. For different reasons, operators (1.2) were discarded, for example, the Hamiltonian $H_f = \frac{P_f^2}{2}$ is non-Hermitian and it could have a non-real spectrum. However, it has recently been shown that there are non-Hermitian operators with real spectrum [3]. Studies of non-Hermitian Hamiltonians and their applications in physics can be found in the papers [7], [4] and [10].

In this paper, we will show that, using the operator $P_{(f)j}$, four different types of Hamiltonians can be built, two of these Hermitians and the other two, non-Hermitians. We will show that, from two Hermitian Hamiltonians in one dimension, it is possible to construct a supersymmetric mechanics, and that using one of the two non-Hermitian Hamiltonians a generalized supersymmetric mechanic can be constructed. Moreover, we will show that this new supersymmetric quantum mechanics has a dual and the ground state of the corresponding Hamiltonians will be found.

As a second point of this work, it is shown how the operator $P_{(f)j}$ can also be used to build some of non-Hermitian Hamiltonians that naturally arise in the so-called quantum finance.

2. Non-Hermitian Hamiltonians

In this section, we will study the quantum mechanics that emerges when the operator $P_{(f)j}$ is considered. As an starting point, we have to notice that the operator $P_{(f)i}$ is given by the transformation

$$P_{(f)i} = e^f P_i e^{-f}, \tag{2.1}$$

and also that it is not Hermitian.

With $\vec{P}_{(f)}$ we can construct four Hamiltonians, two of them Hermitians

$$H_1 = \alpha^2 \vec{P}^{\dagger}_{(f)} \cdot \vec{P}_{(f)} = \alpha \left(\vec{P}^{\,2} + \nabla^2 f + \left(\vec{\nabla} f\right)^2 \right), \tag{2.2}$$

$$H_2 = \alpha^2 \vec{P}_{(f)} \cdot \vec{P}^{\dagger}_{(f)} = \alpha \left(\vec{P}^{\,2} - \nabla^2 f + \left(\vec{\nabla} f \right)^2 \right) \tag{2.3}$$

and two non-Hermitians

$$\begin{aligned} H_3 &= \beta^2 \vec{P}^{\dagger}_{(f)} \cdot \vec{P}^{\dagger}_{(f)} \\ &= \beta^2 \left(\vec{P}^{\,2} - 2i \vec{\nabla} f \cdot \vec{P} - \nabla^2 f - \left(\vec{\nabla} f \right)^2 \right), \qquad (2.4) \\ H_4 &= \beta^2 \vec{P}_{(f)} \cdot \vec{P}_{(f)} \\ &= \beta^2 \left(\vec{P}^{\,2} + 2i \vec{\nabla} f \cdot \vec{P} + \nabla^2 f - \left(\vec{\nabla} f \right)^2 \right). \qquad (2.5) \end{aligned}$$

These Hamiltonians are obtained naturally in different contexts. In the following section, we will see that, they can be used to obtain a generalized version of the supersymmetric quantum mechanics.

3. Supersymmetric Quantum Mechanics

In the one dimensional case, the Hamiltonians H_1 and H_2 are given by

$$\begin{aligned} H_1 &= \alpha^2 \left(P^{\,2} + \frac{d^2 f}{dx^2} + \left(\frac{df}{dx} \right)^2 \right), \qquad (3.1) \\ H_2 &= \alpha^2 \left(P^{\,2} - \frac{d^2 f}{dx^2} + \left(\frac{df}{dx} \right)^2 \right). \qquad (3.2) \end{aligned}$$

Moreover, if

$$f(x) = \int_0^x W(u) du, \tag{3.3}$$

then

$$\begin{aligned} H_1 &= \alpha^2 \left(P^{\,2} + \frac{dW}{dx} + W^2 \right), \qquad (3.4) \\ H_2 &= \alpha^2 \left(P^{\,2} - \frac{dW}{dx} + W^2 \right). \qquad (3.5) \\ & \qquad (3.6) \end{aligned}$$

This Hamiltonians can be used to form the matrix

$$h = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}. \tag{3.7}$$

Now, defining

$$Q = \begin{pmatrix} 0 & \alpha P_{(f)} \\ 0 & 0 \end{pmatrix}, \tag{3.8}$$

we have

$$h = \{Q, Q^\dagger\}, \qquad Q^2 = 0, \quad \{Q, H\} = 0. \tag{3.9}$$

According to the supersymmetric quantum mechanics [5], h represents a super-Hamiltonian and Q a supercharge. Therefore, the quantum mechanics built using $P_{(f)i}$ contains the usual supersymmetric quantum mechanics.

Moreover, $P_{(f)}$ allows us to generalize supersymmetric quantum mechanics. In fact, we can define the matrices

$$Q_1 = \begin{pmatrix} 0 & \alpha P_{(f)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta P^\dagger_{(f)} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha P^\dagger_{(f)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta P^\dagger_{(f)} & 0 \end{pmatrix}, \tag{3.10}$$

and then $Q_1^2 = Q_2^2 = 0$. Using $Q_1^2 = Q_2^2 = 0$ we can construct the Hamiltonian

$$H = \{Q_1, Q_2\} = \begin{pmatrix} H_1 & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 \\ 0 & 0 & H_3 & 0 \\ 0 & 0 & 0 & H_3 \end{pmatrix} \tag{3.11}$$

with the conserved charges

$$\dot{Q}_1 = [Q_1, H] = 0, \qquad \dot{Q}_2 = [Q_2, H] = 0; \tag{3.12}$$

now, if $\beta = 0$, we have $Q_1 = Q_2 = Q$ and this quantum mechanics reduces to the usual supersymmetric quantum mechanics.

Besides, we have another quantum mechanics that reduces to the usual supersymmetric quantum mechanics. In fact, if

$$Q_3 = \begin{pmatrix} 0 & \alpha P^\dagger_{(f)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta P_{(f)} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha P_{(f)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta P_{(f)} & 0 \end{pmatrix}, \tag{3.13}$$

then $Q_3^2 = Q_4^2 = 0$ and we can construct the Hamiltonian

$$\tilde{H} \;=\; \{Q_3, Q_4\} = \begin{pmatrix} H_2 & 0 & 0 & 0 \\ 0 & H_1 & 0 & 0 \\ 0 & 0 & H_4 & 0 \\ 0 & 0 & 0 & H_4 \end{pmatrix}. \tag{3.14}$$

Note that if $\beta = 0$ this Hamiltonian is just the usual super-Hamiltonian h.

If we make the transformation $f \to -f$, we have

$$(H_1, H_2, H_3, H_4) \qquad \to \qquad (H_2, H_1, H_4, H_3)\,, \tag{3.15}$$

i.e

$$H \to \tilde{H}. \tag{3.16}$$

Then, there is a duality transformation between Hamiltonians H and $\tilde{H}$. Therefore these generalized quantum mechanics is duals.

Now, if we consider the functions $\psi_0 = A_1 e^f, \phi_0 = A_2 e^{-f}$ that satisfy

$$P_{(f)}\psi_0 = 0, \qquad P^{\dagger}_{(f)}\phi_0 = 0. \tag{3.17}$$

Then, the wave function

$$\psi \;=\; \begin{pmatrix} A_1 e^{-f} \\ A_2 e^{f} \\ A_3 e^{-f} \\ A_3 e^{-f} \end{pmatrix} \tag{3.18}$$

satisfies

$$H\psi = 0. \tag{3.19}$$

Moreover, the wave function

$$\tilde{\psi} \;=\; \begin{pmatrix} e^{f} \\ e^{-f} \\ e^{f} \\ e^{f} \end{pmatrix} \tag{3.20}$$

satisfies

$$\tilde{H}\tilde{\psi} = 0. \tag{3.21}$$

Thus, ψ is the ground state of H and $\tilde{\psi}$ is the ground state of $\tilde{H}$.

4. The Black-Scholes Model

This model is a partial differential equation whose solution describes the value of an European Option (see [2], [8]). Nowadays, it is widely used to estimate the pricing of options other than the European ones. Let $(\Omega, \mathcal{F}, P, \mathcal{F}_{t\geq 0})$ be a filtered probability space and let W_t be a Brownian motion in $\mathbb{R}$. We will consider the stochastic differential equation (s.d.e.)

$$dX(t) = a(t, X(t))dt + \sigma(t, X(t))dW(t),$$

with a and σ continuous in (t, x) and Lipschitz in x. The price processes given by the geometric Brownian motion $S(t)$, $S(0) = x_0$, solution of the s.d.e.

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

with μ and σ constants. It is well know the solution of this s.d.e. it is given by:

$$dS(t) = x_0 \exp\{\sigma(W(t) - W(t_0)) + (r - \frac{1}{2}\sigma^2)(t - t_0)\}.$$

Let $0 \leq t < T$ and h be a Borel measurable function, $h(X(T))$ denote the contingent claim, let $E^{x,t}h(X(T))$ be the expectation of $h(X(T))$, with the initial condition $X(t) = x$.

Now we recall the Feynman-Kac Theorem, see [9]. Let $v(t, x) = E^{x,t}h(X(T))$ be, $0 \leq t < T$, where $dX(t) = a(X(t))dt + \sigma(X(t))dW(t)$. Then

$$v_t(t, x) + a(x)v_x(t, x) + \frac{1}{2}\sigma^2(x)v_{xx}(t, x) = 0, \text{ and } v(T, x) = h(x). \tag{4.1}$$

Now, we consider the discounted value

$$u(t, x) = e^{-r(T-t)}E^{x,t}h(X(T)) = e^{-r(T-t)}v(t, x).$$

Then if at time t, $S(t) = x$, if we proceed in standard way,

$$\begin{aligned} v(t, x) &= e^{r(T-t)}u(t, x), \\ v_t(t, x) &= -re^{r(T-t)}u(t, x) + e^{r(T-t)}u_t(t, x), \\ v_x(t, x) &= e^{r(T-t)}u_x(t, x), \\ v_{xx}(t, x) &= e^{r(T-t)}u_{xx}(t, x). \end{aligned}$$

The Black-Scholes equation is obtained substituting the above equalities in the equation (4.1) and multiplying by the factor $e^{-r(T-t)}$:

$$-ru(t, x) + u_t(t, x) + rxu_x(t, x) + \tfrac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = 0, \tag{4.2}$$
$$0 \leq t < T, \quad x \geq 0.$$

5. The Relation with the Black-Scholes Equation

The operators $\vec{P}_f \cdot \vec{P}_f$ and $\vec{P}^\dagger_f \cdot \vec{P}^\dagger_f$ are non-Hermitians, using them only non-Hermitians Hamiltonians such as H_3 and H_4, can be constructed. However, we will show that these operators may have applications in some other areas such as quantum finance. In order to see this, we define the potentials

$$U_1(x,y,z) = -\beta^2\left(\nabla^2 f - \left(\vec{\nabla} f\right)^2\right) + V_1(x,y,z), \tag{5.1}$$

$$U_2(x,y,z) = \beta^2\left(\nabla^2 f + \left(\vec{\nabla} f\right)^2\right) + V_2(x,y,z), \tag{5.2}$$

and the non-Hermitians Hamiltonians

$$\begin{aligned} H_I &= \beta^2 \vec{P}_{(f)} \cdot \vec{P}_{(f)} + U_1(x,y,z) \\ &= \beta^2\left(\vec{P}^{\,2} + 2i\vec{\nabla} f \cdot \vec{P}\right) + V_1(x,y,z), \end{aligned} \tag{5.3}$$

$$\begin{aligned} H_{II} &= \beta^2 \vec{P}^\dagger_{(f)} \cdot \vec{P}^\dagger_{(f)} + U_2(x,y,z) \\ &= \beta^2\left(\vec{P}^{\,2} - 2i\vec{\nabla} f \cdot \vec{P}\right) + V_2(x,y,z). \end{aligned} \tag{5.4}$$

On the other hand, let us consider the fundamental equation in quantum finance, the so-called Black-Scholes equation (4.2)

$$\frac{\partial C}{\partial t} = -\frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} - rS\frac{\partial C}{\partial S} + rC, \tag{5.5}$$

where C is the option price, σ is a constant called the volatility and r is the interest rate [1]. With the change of variable $S = e^x$ we obtain

$$\begin{aligned} \frac{\partial C}{\partial t} &= H_{BS} C, \\ H_{BS} &= -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial}{\partial x} + r. \end{aligned} \tag{5.6}$$

This non-Hermitian Hamiltonian is called Black-Scholes Hamiltonian. Now, considering the one dimensional case of (5.4) and identifying

$$\beta^2 = \frac{\sigma^2}{2}, \quad f(x) = \frac{1}{\sigma^2}\left(\frac{\sigma^2}{2} - r\right)x, \quad V_2(x) = r,$$

we obtain $H_{II} = H_{BS}$.

One generalized Black-Scholes equation (see [1]) is given by

$$H_{BSG} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - V(x)\right)\frac{\partial}{\partial x} + V(x). \tag{5.7}$$

In this case, considering again the one dimensional case of equation (5.4) and with

$$\beta^2 = \frac{\sigma^2}{2}, \quad f(x) = \int_0^x du \frac{1}{\sigma^2}\left(\frac{\sigma^2}{2} - V(u)\right), \quad V_2(x) = V(x),$$

we have $H_{II} = H_{BSG}$.

Moreover, the so-called barrier option case has Hamiltonian

$$H_{BSB} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial}{\partial x} + V(x). \tag{5.8}$$

Again, considering (5.4) in one dimension and with

$$\beta^2 = \frac{\sigma^2}{2}, \quad f(x) = \frac{1}{\sigma^2}\left(\frac{\sigma^2}{2} - r\right)x, \quad V_2(x) = V(x)$$

we have $H_{II} = H_{BSB}$.

As we have seen, several important Hamiltonians appearing in quantum finance are particular cases of this new version of quantum mechanics.

6. Summary

A quantum mechanics is built by means of a non-Hermitian momentum operator. Moreover, it is shown that using this momentum operator it is possible to construct two Hermitian and two non-Hermitian type of Hamiltonians. Using these Hermitian Hamiltonians we have built, a generalized supersymmetric quantum mechanics with a dual that can be constructed. It also shown that, the non-Hermitian Hamiltonians of this theory may be related to so-called quantum finance Hamiltonian.

References

[1] B.E. Baaquie, *Quantum Finance,* Cambridge University Press (2004).

[2] F. Black, M. Scholes, The pricing options and corporate liabilities, *Journal of Political Economy*, **81** (1973), 637-659.

[3] C.M. Bender, Introduction to PT-symmetric quantum theory, *Contemp. Phys.*, **46** (2005), 277.

[4] C.M. Bender, P.D. Mannheim, No-ghost theorem for the fourth-order derivative Pais-Uhlenbeck oscillator model, *Phys. Rev. Lett.*, **100** (2008), 110402.

[5] F. Cooper, A. Khare, U. Sukhatme, Supersymmetry and quantum mechanics, *Phys. Rept.*, **251** (2995), 267-385.

[6] P.A.M. Dirac, *The Principles of Quantum Mechanics*, Oxford (1930).

[7] P.K. Ghosh, On the construction of a pseudo-Hermitian quantum system with a pre-determined metric in the Hilbert space, *J. Phys.* **A**, *Math. Theor.*, **43** (2010), 125203.

[8] R.C. Merton, Theory of rational options pricing, *Bell Journal of Economic and Management Science*, **4** (1973), 141-183.

[9] S. Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer Finance (2004).

[10] T. Tanaka, General aspects of PT-symmetric and P-self-adjoint quantum theory in a Krein space, *J. Phys. A*, **39** (2006), 14175.