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FIBONACCI AND LUCAS POLYNOMIALS

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Abstract: The paper is devoted to discussing some selected properties of the Fibonacci and Lucas polynomials. On the ground of the appropriate identities for those polynomials a series of new combinatoric identities, connected with the Fibonacci and Lucas numbers, are received. The approach to the analytical problems, presented in this paper, is for the Fibonacci and Lucas polynomials absolutely original.

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1. Basic Identities

Let us set $\xi := \exp(2\pi i/5)$,

$$\alpha := 2\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \beta := -2\cos\left(\frac{2}{5}\pi\right) = \frac{1-\sqrt{5}}{2}.$$

Then, we have

$$\alpha + \beta = 1, \qquad \alpha\beta = -1, \tag{1.1}$$

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad n \in \mathbb{Z}, \tag{1.2}$$

$$L_n = \alpha^n + \beta^n, \qquad n \in \mathbb{Z}. \tag{1.3}$$

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The next two identities make up a basic tool for discussing and providing the proof of properties of the Fibonacci and Lucas numbers

$$(1+\xi+\xi^4)^n = F_{n+1} + F_n(\xi+\xi^4) \tag{1.4}$$

$$\iff \quad \alpha^n = F_n\,\alpha + F_{n-1}, \tag{1.5}$$

and

$$(1+\xi^2+\xi^3)^n = F_{n+1} + F_n(\xi^2+\xi^3) \tag{1.6}$$

$$\iff \quad \beta^n = F_n\,\beta + F_{n-1}. \tag{1.7}$$

Proof. Only the proof of identity (1.4) will be presented in the paper. It follows by induction

$$\begin{aligned}(1+\xi+&\xi^4)(F_{n+1}+F_n(\xi+\xi^4))\\ &= F_{n+1} + F_{n+1}(\xi+\xi^4) + F_n(1+\xi+\xi^4)(\xi+\xi^4)\\ &= F_{n+1} + F_n + F_{n+1}(\xi+\xi^4) + F_n(1+\xi+\xi^2+\xi^3+\xi^4)\\ &= F_{n+2} + F_{n+1}(\xi+\xi^4).\end{aligned}$$

□

Remark 1.1. Identities (1.5) and (1.7) are known (see S. Rabinowitz [4], R. Wituła [2] and also [3]).

Corollary 1.2. *The following identities hold*

$$(1+\xi+\xi^2)^n = \xi^{n-1}(F_n + F_{n+1}\xi + F_n\xi^2), \tag{1.8}$$

$$(1+\xi+\xi^3)^n = \xi^{3n}(F_{n+1} + F_n\xi^2 + F_n\xi^3) \tag{1.9}$$

and

$$(1+\xi^2+\xi^4)^n = \xi^{2n}(F_{n+1} + F_n\xi^2 + F_n\xi^3). \tag{1.10}$$

Corollary 1.3. *We have*

$$(1+\xi+\xi^4)^{n+1} + (1+\xi+\xi^4)^{n-1} = L_{n+1} + L_n(\xi+\xi^4) \tag{1.11}$$

and

$$(1+\xi+\xi^4)^{n+1} - (1+\xi+\xi^4)^{n-1} = F_{n+1} + F_n(\xi+\xi^4). \tag{1.12}$$

Corollary 1.4. *By (1.11) and by the equality* $(1+\xi+\xi^4)^{-1} = \xi+\xi^4$ *we obtain*

$$\begin{aligned}L_{n+1} + L_n(\xi+\xi^4) &= \big[(1+\xi+\xi^4) + (1+\xi+\xi^4)^{-1}\big](1+\xi+\xi^4)^n\\ &= \big[2(\xi+\xi^4)+1\big](1+\xi+\xi^4)^n = \sqrt{5}(1+\xi+\xi^4)^n.\end{aligned} \tag{1.13}$$

Another formula can be also deduced

$$L_{n+1} + L_n(\xi^2+\xi^3) = -\sqrt{5}(1+\xi^2+\xi^3)^n. \tag{1.14}$$

2. Fibonacci Polynomials

First, let us note that by (1.1), (1.5) and (1.7) we obtain

$$(x-\alpha)\,(x-\beta) = (x^2-x-1)\big|(x^n - F_n x - F_{n-1}), \qquad n=2,3,\dots. \tag{2.1}$$

More precisely, the stronger result, written below, can be generated.

Theorem 2.1. *The following identities hold*

$$(x^2-x-1)\left(\sum_{k=0}^{n-1} F_{n-k}x^k\right) = x^{n+1} - F_{n+1}x - F_n \tag{2.2}$$

and

$$(x^2-x-1)\left(\sum_{k=0}^{n-1} L_{n-k}x^k\right) = x^{n+1} + 2x^n - L_{n+1}x - L_n, \tag{2.3}$$

which arises from a more general identity

$$(x^2-x-1)\left(\sum_{k=0}^{n} q_{n-k}x^k\right) = q_0x^{n+2} + (q_1-q_0)x^{n+1} - q_{n+1}x - q_n, \tag{2.4}$$

for any recurrent sequence of complex numbers: $q_{n+1} = q_n + q_{n-1}$, $n=1,2,\dots$.

Corollary 2.2. *For* $x=1$ *we obtain*

$$\sum_{k=1}^{n} F_k = F_{n+1} + F_n - 1 = F_{n+2} - 1, \tag{2.5}$$

$$\sum_{k=1}^{n} L_k = L_{n+2} - 3, \tag{2.6}$$

and generally

$$\sum_{k=0}^{n} q_k = q_{n+2} - q_1. \tag{2.7}$$

Corollary 2.3. *We have*

$$\frac{x^{n+1}}{x^2-x-1} = \sum_{k=0}^{n-1} F_{n-k}x^k + \frac{F_{n+1}x + F_n}{x^2-x-1}, \tag{2.8}$$

or in general (by (2.5)):

$$\frac{x^{n+1}+x^{n+2}+\ldots+x^{n+r}}{x^2-x-1} = \sum_{k=0}^{n+r-2}\big(F_{n+r+1-k}-F_{\max\{1,n+1-k\}}\big)x^k + \frac{(F_{n+r+2}-F_{n+2})x+F_{n+r+1}-F_{n+1}}{x^2-x-1}, \quad (2.9)$$

$$\frac{x^{n+1}+2x^n}{x^2-x-1} = \sum_{k=0}^{n-1} L_{n-k}x^k + \frac{L_{n+1}x+L_n}{x^2-x-1}, \quad (2.10)$$

$$\frac{q_0x^{n+2}+(q_1-q_0)x^{n+1}}{x^2-x-1} = \sum_{k=0}^{n} q_{n-k}x^k + \frac{q_{n+1}x+q_n}{x^2-x-1}, \quad (2.11)$$

or, in the form of more general formula,

$$\frac{(q_1-q_0)x^{n+1}+q_1x^{n+2}+\ldots+q_1x^{n+r-1}+q_0x^{n+r}}{x^2-x-1} = \sum_{k=0}^{n+r-2}\big(q_{n+r-k}-q_{\max\{1,n+1-k\}}\big)x^k + \frac{(q_{n+r+1}-q_{n+2})x+q_{n+r}-q_{n+1}}{x^2-x-1}. \quad (2.12)$$

Remark 2.4. In book [3] the following two formulas (the generating functions of $\{F_{kn+r}\}_{n=0}^{\infty}$ and $\{L_{kn+r}\}_{n=0}^{\infty}$) are given

$$\sum_{n=0}^{\infty} F_{kn+r}x^n = \frac{F_r+(-1)^r\,F_{k-r}\,x}{1-L_k\,x-(-1)^k\,x^2}$$

and

$$\sum_{n=0}^{\infty} L_{kn+r}x^n = \frac{L_r+(-1)^{r-1}\,L_{k-r}\,x}{1-L_k\,x-(-1)^k\,x^2}.$$

Corollary 2.5. *By differentiating the identity (2.2) we get*

$$(2x-1)\left(\sum_{k=0}^{n-1} F_{n-k}x^k\right)+(x^2-x-1)\left(\sum_{k=1}^{n-1} kF_{n-k}x^{k-1}\right) = (n+1)x^n - F_{n+1},$$

which, again by (2.2), implies the identity

$$(x^2-x-1)^2\left(\sum_{k=1}^{n-1} kF_{n-k}x^{k-1}\right)$$

$$= (1-2x)(x^2-x-1)\Bigl(\sum_{k=0}^{n-1} F_{n-k}x^k\Bigr) + (x^2-x-1)\bigl((n+1)x^n - F_{n+1}\bigr)$$

$$= \bigl((n+1)x^n - F_{n+1}\bigr)(x^2-x-1) - (2x-1)(x^{n+1} - F_{n+1}x - F_n). \quad (2.13)$$

Hence, for $x = 1$ we obtain

$$\sum_{k=1}^{n-1} kF_{n-k} = (n+1-F_{n+1})(-1) - (1 - F_{n+1} - F_n)$$
$$= F_{n+3} - n - 2. \quad (2.14)$$

On the other hand, for $x = -1$ we obtain

$$\sum_{k=1}^{n-1}(-1)^{k-1}kF_{n-k} = (n+1)(-1)^n - F_{n+1} + 3\bigl((-1)^{n+1} + F_{n+1} - F_n\bigr)$$
$$= F_{n-3} + (-1)^n(n-2). \quad (2.15)$$

Corollary 2.6. *By differentiating the identity (2.3) we obtain*

$$(2x-1)\Bigl(\sum_{k=0}^{n-1} L_{n-k}x^k\Bigr) + (x^2-x-1)\Bigl(\sum_{k=1}^{n-1} kL_{n-k}x^{k-1}\Bigr)$$
$$= (n+1)x^n + 2nx^{n-1} - L_{n+1},$$

which, again by (2.3), implies the identity:

$$(x^2-x-1)^2\Bigl(\sum_{k=1}^{n-1} kL_{n-k}x^{k-1}\Bigr)$$
$$= \bigl((n+1)x^n + 2nx^{n-1} - L_{n+1}\bigr)(x^2-x-1)$$
$$- (x^{n+1} + 2x^n - L_{n+1}x - L_n)(2x-1)$$
$$= (n-1)x^{n+2} + (n-4)x^{n+1} - (3n-1)x^n - 2nx^{n-1}$$
$$+ L_{n+1}(x^2+1) + L_n(2x-1).$$

If we take $x = 1$ then we receive the identity

$$\sum_{k=1}^{n-1} kL_{n-k} = L_{n+3} - 3n - 4,$$

whereas, if we take $x = -1$ then we obtain

$$\sum_{k=1}^{n-1}(-1)^{k-1}kL_{n-k} = L_{n-3} + (-1)^n(4-n).$$

Corollary 2.7. *By differentiating the identity (2.13), multiplied first by x, we get*

$$2(2x-1)(x^2-x-1)\Big(\sum_{k=1}^{n-1} kF_{n-k}x^k\Big)+(x^2-x-1)^2\Big(\sum_{k=1}^{n-1} k^2F_{n-k}x^{k-1}\Big)$$
$$=\big(-F_{n+1}+(n+1)^2x^n\big)(x^2-x-1)+\big((n+1)x^{n+1}-F_{n+1}x\big)(2x-1)$$
$$-(4x-1)(x^{n+1}-F_{n+1}x-F_n)-(2x^2-x)\big((n+1)x^n-F_{n+1}\big)$$
$$=\big((n+1)^2x^n-F_{n+1}\big)(x^2-x-1)-(4x-1)(x^{n+1}-F_{n+1}x-F_n),$$

which, after multiplying by (x^2-x-1) and applying again the formula (2.13), leads to the relation

$$(x^2-x-1)^3\Big(\sum_{k=1}^{n-1} k^2F_{n-k}x^{k-1}\Big)=\big((n+1)^2x^n-F_{n+1}\big)(x^2-x-1)^2$$
$$-(4x-1)(x^{n+1}-F_{n+1}x-F_n)(x^2-x-1)$$
$$-2(2x^2-x)\big[\big((n+1)x^n-F_{n+1}\big)(x^2-x-1)-(2x-1)(x^{n+1}-F_{n+1}x-F_n)\big].$$

The polynomials $\mathcal{F}_{n-1}(x)=\sum\limits_{k=0}^{n-1} F_{n-k}x^k$, denoted as "*the Fibbonacci polynomials*", possess some interesting properties (see [1] and Remark 2.10 written below).

Theorem 2.8. *The polynomials $\mathcal{F}_{2r}(x)$ have no real roots. Whereas, the polynomials $\mathcal{F}_{2r-1}(x)$ have precisely one single real root. More precisely, $\mathcal{F}_{2r-1}(x)$ is an increasing function for every $r\in\mathbb{N}$.*

Proof. First, the following decompositions are observed

$$\mathcal{F}_{2r}(x)=\sum_{k=0}^{2r} F_{2r+1-k}x^k=x^{2r-2}(x^2+x+1)+\Big(\frac{1}{2}F_{2r-1}x^2+F_{2r}x+F_{2r+1}\Big)$$
$$+\sum_{l=1}^{r-2}\frac{1}{2}x^{2r-2l-2}(F_{2l+1}x^2+2F_{2l+2}x+F_{2l+3}) \tag{2.16}$$

and all the quadratic trinomials in brackets, by the Cassini's identity, have the negative discriminant. On the other hand, another decomposition can be obtained (for the derivative of $\mathcal{F}_{2r-1}(x)$):

$$\mathcal{F}'_{2r-1}(x)=\Big(\sum_{k=0}^{2r-1} F_{2r-k}x^k\Big)'=\sum_{k=1}^{2r-1} kF_{2r-k}x^{k-1}$$

$$
\begin{aligned}
&= (2r-1)x^{2r-2} + (2r-2)x^{2r-3} + (2r-2)x^{2r-4} \\
&+ \sum_{l=2}^{r-2} \big((r-l)F_{2l-1}x^{2r-2l} + (2r-2l)F_{2l}x^{2r-2l-1} + (r-l)F_{2l+1}x^{2r-2l-2}\big) \\
&+ F_{2r-3}x^2 + F_{2r-2}x + F_{2r-1} \qquad (2.17) \\
&= x^{2r-4}\big((2r-1)x^2 + (2r-2)x + (2r-2)\big) \\
&+ \sum_{l=2}^{r-2} x^{2r-2l-2}\big[(r-l)F_{2l-1}x^2 + (2r-2l)F_{2l}x + (r-l)F_{2l+1}\big] \\
&+ (F_{2r-3}x^2 + F_{2r-2}x + F_{2r-1}).
\end{aligned}
$$

In this case similarly, all the quadratic trinomials have negative discriminant. Thus, $\mathcal{F}'_{2r-1}(x) > 0$ and $\mathcal{F}_{2r}(x)$ is an increasing function. □

By using decompositions (2.16) and (2.17) of polynomials $\mathcal{F}_n(x)$, the following result can be proved.

Theorem 2.9. *Let us take*

$$\mathcal{F}_{n,m}(x) := \mathcal{F}_n(x) - x^{n-m}\mathcal{F}_m(x) = \sum_{k=0}^{n-m-1} F_{n+1-k}x^k,$$

for $m, n \in \mathbb{N}$, $m < n$. If $n \in 2\mathbb{N}$ and $m \in 2\mathbb{N}-1$ then $\mathcal{F}_{n,m}(x) > 0$, for $x \in \mathbb{R}$. If $n, m \in 2\mathbb{N}-1$ then $\mathcal{F}_{n,m}$ is an increasing function on $\mathbb{R}$. Moreover, we have the following estimation from below (which is asymptotic strict):

$$\mathcal{F}_{n,m}(x) - \frac{1}{2}\Big(F_{m+2}x^{n-m-1} + F_{n+1}\Big) \geq -\frac{1}{2}\sum_{k=1}^{\frac{n-m+1}{2}} \frac{1}{F_{m+2k}}\alpha^{n-m-2k+1}, \quad (2.18)$$

for $x \geq -\alpha$, $n, m \in 2\mathbb{N}-1$.

Proof. (Only the estimation (2.18) will be proven here.) It is easy to check that

$$\frac{1}{2}F_{2k-2}x^2 + F_{2k-1}x + \frac{1}{2}F_{2k} \geq -\frac{1}{2F_{2k-2}}, \qquad x \in \mathbb{R}. \quad (2.19)$$

Hence, for $x \in [-\alpha, \alpha]$ we obtain

$$
\begin{aligned}
&\mathcal{F}_{n,m}(x) - \frac{1}{2}\Big(F_{m+2}x^{n-m-1} + F_{n+1}\Big) \geq \\
&\quad \geq -\frac{1}{2}\sum_{k=1}^{\frac{n-m+1}{2}} \frac{1}{F_{m+2k}}x^{n-m-2k+1} \geq -\frac{1}{2}\sum_{k=1}^{\frac{n-m+1}{2}} \frac{1}{F_{m+2k}}\alpha^{n-m-2k+1}. \quad (2.20)
\end{aligned}
$$

Since in (2.19) the equality holds for $x = -\frac{F_{2k-1}}{F_{2k-2}} \xrightarrow[k\to\infty]{} -\alpha$, whereas the relation (2.20) was obtained in course of replacing x by α, one can state that the estimation (2.18) is asymptotic strict. In particular we have

$$\lim_{n-2m\to\infty} \frac{-2\mathcal{F}_{n,m}(-\alpha) - F_{m+2}\alpha^{n-m-1} + F_{n+1}}{\sum\limits_{k=1}^{(n-m+1)/2} \frac{1}{F_{m+2k}}\alpha^{n-m-2k+1}} = 1. \quad \square$$

Remark 2.10. The authors in paper [1] have received the result, which is analogical with the above Theorem 2.8, but in the absolutely non-elementary way (by applying some techniques of complex analysis).

Lemma 2.11. *We have the following decomposition*

$$\begin{aligned}\mathcal{F}_{n-1}(x) = \left(\sum_{k=0}^{n-1} x^k\right) + \sum_{r=1}^{n-2} F_r \left(\sum_{k=0}^{n-r-2} x^k\right)& \\ = \frac{x^n - 1}{x-1} + \sum_{r=1}^{n-2} F_r \frac{x^{n-r-1}-1}{x-1}&. \end{aligned} \tag{2.21}$$

Corollary 2.12. *By (2.21) we obtain*

$$\mathcal{F}_{n-1}(1) = \sum_{k=1}^{n} F_k = n + \sum_{r=1}^{n-2}(n-r-1)F_r. \tag{2.22}$$

From (2.22) we derive the following identity (compare with the identity (2.14)):

$$\begin{aligned}\sum_{r=1}^{n-2} rF_r &= n - F_{n+1} + (n-2)\sum_{r=1}^{n-2} F_r \\ &= n - F_{n+1} + (n-2)(F_n - 1) \\ &= 2 + n\,F_n - L_{n+1}. \end{aligned} \tag{2.23}$$

Corollary 2.13. *By (2.21) it can be easily observed that* $\mathcal{F}_{n-1}(x) > 0$, *for* $x > -1$.

3. Lucas Polynomials

Let $a \in \mathbb{R}$ and let us define

$$L_{2n}(x;a) := ax^{2n} + \sum_{k=1}^{2n} L_k x^{2n-k}.$$

Theorem 3.1. *If $a \geq 7/52$ then $L_{2n}(x;a) > 0$, for $x \in \mathbb{R}$.*

Proof. Consider the following decomposition of $L_{2n}(x,a)$:

$$\begin{aligned} L_{2n}(x,a) &= x^{2n-2}(ax^2 + x + \delta_1 L_2) \\ &\quad + \sum_{k=1}^{n-2} x^{2n-2k-2}\big((1-\delta_k)L_{2k}x^2 + L_{2k+1}x + \delta_{k+1}L_{2k+2}\big) \\ &\quad + (1-\delta_{n-1})L_{2n-2}x^2 + L_{2n-1}x + L_{2n}, \end{aligned} \tag{3.1}$$

where

$$\delta_{k+1} := \frac{L_{2k+1}^2}{4(1-\delta_k)L_{2k}L_{2k+2}}, \qquad k = 1, 2, \ldots, n-2.$$

Hence, by the identities

$$L_{k-1}^2 = L_{2(k-1)} + 2(-1)^{k-1} \qquad \text{and} \qquad L_k L_{k-2} = L_{2(k-1)} + 3(-1)^k$$

we get

$$\delta_{k+1} = \frac{g_k}{4(1-\delta_k)},$$

where

$$g_k = \frac{L_{4k+2} - 2}{L_{4k+2} + 3}.$$

If $\delta_1 = \frac{13}{21}$ then $\delta_2 = \frac{1}{2}$ and by the identity

$$\delta_{k+1} - \frac{1}{2} = \frac{g_k - 1 - 2(\frac{1}{2} - \delta_k)}{4(1-\delta_k)} \tag{3.2}$$

we deduce that $\delta_k \leq \frac{1}{2}$, for all $k \geq 2$. Moreover, from (3.1) we obtain

$$\begin{aligned} L_{2n}(x,a) &= x^{2n-2}\left(ax^2 + x + \frac{13}{7}\right) \\ &\quad + \sum_{k=1}^{n-2}(1-\delta_k)L_{2k}x^{2n-2k-2}\left(x + \frac{L_{2k+1}}{2(1-\delta_k)L_{2k}}\right)^2 \\ &\quad + \Big((1-\delta_{n-1})L_{2n-2}x^2 + L_{2n-1}x + L_{2n}\Big), \end{aligned}$$

which implies that $L_{2n}(x,a) > 0$, for $x \in \mathbb{R}$. ∎

Corollary 3.2. *Let $a, b \in \mathbb{R}$, $a \geq 7/52$. Then the polynomial*

$$p_{2n+1}(x; a, b) := b + \int\limits_0^x L_{2n}(x; a)dx = a\frac{x^{2n+1}}{2n+1} + b + \sum_{k=1}^{2n} L_k \frac{x^{2n-k+1}}{2n-k+1}$$

is an increasing function.

The polynomials

$$\mathcal{L}_n(x) := \sum_{k=0}^{n} L_{n-k}x^k, \quad n \in \mathbb{N},$$

will be denoted as *the Lucas polynomials.*

Lemma 3.3. *The following decompositions hold*

$$\mathcal{L}_n(x) = L_0x^n + L_1x^{n-1} + L_2x^{n-2} + \frac{x^{n-2}-1}{x-1} + L_2\left(\frac{x^{n-2}-1}{x-1} + \frac{x^{n-3}-1}{x-1}\right) + \sum_{k=3}^{n-2} L_k \frac{x^{n-k-1}-1}{x-1} \tag{3.3}$$

and

$$\mathcal{L}_n(x) = 2x^n + x^{n-1} + \frac{x^{n-2}-1}{x-1} + L_2\left(\frac{x^{n-3}-1}{x-1} + \frac{x^{n-1}-1}{x-1}\right) + \sum_{k=3}^{n-2} L_k \frac{x^{n-k-1}-1}{x-1}. \tag{3.4}$$

Proof. We get

$$\begin{aligned}(x-1)\mathcal{L}_n(x) &= \sum_{k=0}^{n} L_{n-k}x^{k+1} - \sum_{k=0}^{n} L_{n-k}x^k \\ &= (2x^n + x^{n-1})(x-1) + x^{n-2} - 1 \\ &\quad + L_2(x^{n-3} + x^{n-1} - 2) + \sum_{k=3}^{n-2} L_k x^{n-k-1} - \sum_{k=3}^{n-2} L_k. \qquad \square\end{aligned}$$

Corollary 3.4. *We have*

$$\sum_{k=1}^{n-2} kL_k = 4 + (n-2)L_n - L_{n+1} = 4 + n\,L_n - 5\,F_{n+1}.$$

Proof. From the definition of $\mathcal{L}_n(x)$ and from (3.4), by D'Hôspital rule for $x=1$, we obtain

$$\mathcal{L}_n(1) = 2 + \sum_{k=1}^{n} L_k = 2 + 1 + (n-2) + L_2\big((n-3)+(n-1)\big) + \sum_{k=3}^{n-2}(n-k-1)L_k = n+2+\sum_{k=0}^{n-2}(n-k-1)L_k.$$

Hence, by (2.6) we receive

$$\begin{aligned}\sum_{k=1}^{n-2} kL_k &= n+2-L_{n+1}+(n-2)\sum_{k=0}^{n-2} L_k\\ &= n+2-L_{n+1}+(n-2)(L_n-1)\\ &= 4+(n-2)L_n-L_{n+1} = 4+n\,L_n-5\,F_{n+1}.\end{aligned}$$

□

Corollary 3.5. *We have*

$$\mathcal{L}_n(x) > 0, \qquad \textit{for} \qquad x\in(-1,0).$$

Proof. If $n\in 2\mathbb{N}$ then

$$L_0x^n + L_1x^{n-1} + L_2x^{n-2} = x^{n-2}(2x^2+x+3) > 0, \qquad \text{for} \qquad x\in\mathbb{R}\setminus\{0\}.$$

Moreover, we have

$$\frac{x^{n-k-1}-1}{x-1} > 0, \qquad \text{for} \qquad x\in(-1,0). \tag{3.5}$$

Hence, by (3.3) we obtain $\mathcal{L}_{2n}(x)>0$, for $x\in(-1,0)$. On the other hand, if $n\in 2\mathbb{N}-1$ then

$$x^n + x^{n-1} > 0, \qquad \text{for} \qquad x\in(-1,0)$$

and by (3.4) together with (3.5) we receive $\mathcal{L}_{2n-1}(x)>0$, for $x\in(-1,0)$. □

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