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# ON CERTAIN SMALL SUBSETS OF $l^p$

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**Abstract:** In this note certain subsets of the space $l^p$ are discussed, simultanously from the point of view of the category theory, as well as from purely existential point.

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## 1. Introduction

Our note is focused on the discussion of certain subsets of the space $l^p = l^p(\mathbb{K})$, $p > 1$, $\mathbb{K} = \mathbb{R} \vee \mathbb{C}$ from the point of view of the category theory (the first two theorems), as well as from purely existential approach (the next two theorems). It was proved (Theorem 2.1) that the set

$$l^{=p} := l^p \setminus \bigcup_{1 \le q < p} l^q$$

called "exactly $l^p$", is a residual subset of $l^p$, yet, it is not easy to indicate a specific element of $l^{=p}$. For example we have:

$$\left\{k^{-1-(\ln\ln k)^{-\alpha}}\right\} \in l^{=1}, \qquad \alpha > 0.$$

For more details on families $l^{=p}$, $p > 1$ and the families

$$l^{>p} := \bigcap_{q > p} l^q \setminus l^p$$

called "almost $l^p$", see the paper [1]. Surely $l^{>p} \neq \emptyset$, because, for example $\{k^{-1/p}\} \in l^{>p}$.

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In this paper we also analyze the so called $(\mathbf{a}, p)$-subfamilies of $l^p$. Let us fix $p > 1$ and a sequence $\mathbf{a} = \{a_n\} \subset \mathbb{R}_+$ such that $\limsup a_n = \infty$.

**Definition 1.1.** A sequence $\{b_n\} \in l^p$ is called $(\mathbf{a}, p)$-summable if the following condition is fulfilled:

$$\Big(\forall\ q > 0\Big)\ \Big(\sum_{n=1}^{\infty} a_n |b_n|^q < \infty \quad \Longleftrightarrow \quad q \geq p\Big).$$

**Definition 1.2.** The $(\mathbf{a}, p)$-subfamily of $l^p$, is defined to be the set of all $(\mathbf{a}, p)$-summable sequences $\{b_n\} \in l^p$.

In Theorem 2.2 it is proved that each $(\mathbf{a}, p)$-summable family is a set of the first category in $l^p$. Two other results, Theorems 2.3 and 2.4, on $(\mathbf{a}, p)$-summable sequences have an existential nature.

## 2. Main Results

**Theorem 2.1.** *For each $p > 1$, the set $\bigcup_{1 \leq q < p} l^q$ is the first category in the space $l^p$.*

*Proof.* Let us suppose that $p > 1$. First we introduce auxiliary functions and sets:

$$\|b\|_{qm} = \left(\sum_{n=1}^{m} |b_n|^q\right)^{1/q}, \quad \text{where} \quad b \in l^p, \quad b = \{b_n\},$$

$$B_{qmk} = \big\{b \in l^p : \|b\|_{qm} \leq k\big\} \quad \text{and} \quad B_{qk} = \bigcap_{r=1}^{\infty} B_{qrk}$$

for every $k, m \in \mathbb{N}$ and $1 \leq q < p$.

From Minkowski inequality for finite sums it follows that function $\|\ \|_{qm}$ is semi-norm on linear space $l^p$ (for all $1 \leq q < p$ and $m \in \mathbb{N}$). Furthermore, we easily check that sets $B_{qmk}$ are closed in $l^p$, thus, sets $B_{qk}$ are also closed in $l^p$. We shall prove that sets $B_{qk}$ are nowhere dense in $l^p$. To achieve this, let us suppose that: $1 \leq q < p$, $k \in \mathbb{N}$, $b \in B_{qk}$ and let $\mathbb{K} \subset l^p$ be a ball with the centre in $b$. Additionally, set $g = \{(n \ln^2 n)^{-1/p}\}$. Surely, $g \in l^p$, so, there is $\varepsilon > 0$ such that $(b + \varepsilon g) \in \mathbb{K}$. Because $\lim_{m \to \infty} \|g\|_{qm} = \infty$ and $\|b\|_{qm} \leq k$ for each $m \in \mathbb{N}$, from inequality $\|b + \varepsilon g\|_{qm} \geq \varepsilon \|g\|_{qm} - \|b\|_{qm}$ we derive that $(b + \varepsilon g) \notin B_{qk}$. Accordingly, set $B_{qk}$ is nowhere dense set, as it is a closed set.

Let us also fix a sequence $\{q_n\} \subset [1, p)$, which is convergent to $p$. Then, we have

$$\bigcup_{1 \le q < p} l^q = \bigcup_{k,n \in \mathbb{N}} B_{q_n k}$$

which means that the set $\bigcup\limits_{1 \le q < p} l^q$ is really the first category in the space $l^p$. □

**Theorem 2.2.** *Each $(\mathbf{a}, p)$-subfamily of $l^p$ is a set of the first category in $l^p$.*

*Proof.* Let us fix $p > 1$ and a sequence $\mathbf{a} = \{a_n\} \subset \mathbb{R}_+$, $\limsup a_n = +\infty$. Let us assume that $B_k = \big\{\{b_n\} \in l^p : \sum\limits_{n=1}^{\infty} a_n |b_n|^p \le k\big\}$, $k = 1, 2, \ldots$, and $B = \bigcup\limits_{k=1}^{\infty} B_k$. Surely, $B$ is a family of all $(\mathbf{a}, p)$-summable sequences belonging to $l^p$. We easily check that all sets $B_k$, $k \in \mathbb{N}$, are closed in $l^p$. We shall prove that sets $B_k$, $k \in \mathbb{N}$, are also nowhere dense in $l^p$. To achieve this, let us assume, conversely, that a certain set $B_k$ contains sphere $\mathbb{K} \subset l^p$; let $\mathbf{b} \in B_k$ be the centre of $\mathbb{K}$. As the sequence $\{a_n\}$ is unbounded, thus, there is a subsequence $\{a_{r(n)}\}$ such that $\sum\limits_{n=1}^{\infty} (a_{r(n)})^{-1} < \infty$; we define auxilliary sequence $\mathbf{c} = \{c_n\}$ in the following way: $c_n = (a_{r(k)})^{-1/p}$ when $n = r(k)$, $k \in \mathbb{N}$ and $c_n = 0$ for other indices $n \in \mathbb{N}$. We note that $\mathbf{c}$ belongs to $l^p$. Thus, there is $\varepsilon > 0$ such that $(\mathbf{b} + \varepsilon \mathbf{c}) \in \mathbb{K}$. From the convexity of the function $x \mapsto x^p$ it is inferred that

$$\left|\frac{x}{2}\right|^p = \left|\frac{(x+y) - y}{2}\right|^p \le \left(\frac{|x+y| + |y|}{2}\right)^p \le \frac{1}{2}\big(|x+y|^p + |y|^p\big),$$

thus

$$2^{1-p}|x|^p - |y|^p \le |x+y|^p, \tag{2.1}$$

for any $x, y \in \mathbb{R}$. From (2.1) and from the relation:

$$\sum_{n=1}^{\infty} a_n (c_n)^p = \sum_{n=1}^{\infty} a_{r(n)} \big((a_{r(n)})^{-1/p}\big)^p = \infty,$$

we derive

$$\sum_{n=1}^{\infty} a_n |b_n + \varepsilon c_n|^p \ge 2^{1-p} \varepsilon^p \sum_{n=1}^{\infty} a_n (c_n)^p - \sum_{n=1}^{\infty} a_n |b_n|^p = \infty$$

because, let us remind once again, that $\mathbf{b} \in B_k$. Thus, $(\mathbf{b} + \varepsilon\mathbf{c}) \notin B_k$, which contradicts the assumption that $\mathbb{K} \subset B_k$. As a result we demonstrated that every set $B_k$, $k \in \mathbb{N}$, is nowhere dense in $l^p$, so, $B$ is a set of the first category in $l^p$. □

**Theorem 2.3.** *Let us assume* $\{a_n\} \subset \mathbb{R}_+$, $\sum a_n < \infty$. *Then:*

(i) *there is nondecreasing sequence* $\{\beta_n\} \subset \mathbb{R}_+$ *such that*

$$\sum_{n=1}^{\infty} a_n(\beta_n)^q < \infty \quad \Leftrightarrow \quad q < 1;$$

(ii) *there exist also nondecreasing sequence* $\{\gamma_n\} \subset \mathbb{R}_+$ *such that*

$$\sum_{n=1}^{\infty} a_n(\gamma_n)^q < \infty \quad \Leftrightarrow \quad q \le 1.$$

*Proof.* (i) Let us establish a decreasing sequence $\{p(n)\}$ of positive integers with the following properties:

$$\sum_{n=p(k)}^{\infty} a_n < 2^{-k}, \tag{2.2}$$

$$\sum_{n\in N(k)} a_n > \sum_{n\in N(k+1)} 2a_n, \tag{2.3}$$

for every $k \in \mathbb{N}$, where $N(k) := \{n \in \mathbb{N} : p(k) \le n < p(k+1)\}$. We determine auxiliary sequence $\{x_n\} \subset \mathbb{R}_+$ by the following relations:

$$\sum_{n\in N(k)} a_n = (kx_k)^{-1}, \qquad k = 1, 2, \dots . \tag{2.4}$$

It results from (2.3) and (2.4) that $(kx_k)^{-1} > 2((k+1)x_{k+1})^{-1}$, so, consequently, the sequence $\{x_k\}$ is increasing. On the other hand, from (2.2) we derive $x_k > 2^k k^{-1}$, and, subsequently,

$$(x_k)^q \sum_{n\in N(k)} a_n = k^{-1}(x_k)^{q-1} < k^{-q}2^{(q-1)k}, \tag{2.5}$$

for $q \in (0, 1)$ and $k \in \mathbb{N}$. Now, let us set $\beta_n = x_k$ for $n \in N(k)$, $k \in \mathbb{N}$, and $\beta_n = x_1$ for $n < p(1)$. Then, it can be inferred from (2.4) and (2.5) that

$$\sum_{n=p(1)}^{\infty} a_n(\beta_n)^q \le \sum_{k=1}^{\infty} k^{-q}2^{(q-1)k} < \infty,$$

for every $q \in (0,1)$, and $\sum_{n=1}^{\infty} a_n\beta_n \geq \sum_{k=p(1)}^{\infty} k^{-1} = \infty$.

(ii) The procedure of the construction of a sequence $\{\gamma_n\}$ is a similar to the procedure of the construction of the sequence $\{\beta_n\}$; in consideration, however, of the two changes are needed:

– in (2.3) we additionally multiply the right side of the inequality by $\frac{\ln^2 3}{\ln^2 2}$ (the function $x \mapsto \ln^2(x+1)/\ln^2 x$ is decreasing on $(1,\infty)$); and,

– in (2.4) we introduce $\sum_{n\in N(k)} a_n = \big(kx_k \ln^2(k+1)\big)^{-1}$, $k \in \mathbb{N}$.

Let us notice that

$$(x_k)^q \sum_{n\in N(k)} a_n = k^{-1}(x_k)^{q-1}\ln^{-2}(k+1) > k^{-q}2^{(q-1)k}\ln^{-2q}(k+1) \xrightarrow{k\to\infty} \infty$$

for every $q > 1$. □

**Theorem 2.4.** *Let us assume $\{a_n\} \subset \mathbb{R}_+$, $\sum a_n = \infty$. Then:*

*(i) there is nonincreasing sequence $\{\beta_n\} \subset \mathbb{R}_+$ such that*

$$\sum_{n=1}^{\infty} a_n(\beta_n)^q < \infty \quad \Leftrightarrow \quad q \geq 1;$$

*(ii) moreover, there is also nonincreasing sequence $\{\gamma_n\} \subset \mathbb{R}_+$ such that*

$$\sum_{n=1}^{\infty} a_n(\gamma_n)^q < \infty \quad \Leftrightarrow \quad q > 1.$$

*Proof.* (i) Let us fix an increasing sequence $\{p(n)\}$ of positive integers, which satisfies the following conditions:

$$\sum_{n\in N(k)} a_n > 2^k, \tag{2.6}$$

$$\sum_{n\in N(k+1)} a_n > \sum_{n\in N(k)} a_n, \tag{2.7}$$

for each $k \in \mathbb{N}$, where $N(k) := \{n \in \mathbb{N} : p(k) \le n < p(k+1)\}$. We define the auxiliary sequence $\{x_n\} \subset \mathbb{R}_+$ by the relations:

$$\sum_{n\in N(k)} a_n = \big(kx_k \ln^2(k+1)\big)^{-1}, \qquad k = 1, 2, \ldots \tag{2.8}$$

Thus, from (2.7) and (2.8) we obtain:

$$\big((k+1)x_{k+1} \ln^2(k+1)\big)^{-1} > \big(kx_k \ln^2(k+1)\big)^{-1},$$

so, the sequence $\{x_k\}$ is an increasing one. Furthermore, it results from (2.6) and (2.8) that

$$(x_k)^q \sum_{n\in N(k)} a_n = k^{-1}(x_k)^{q-1} \ln^{-2}(k+1) > k^{-q} 2^{(1-q)k} \ln^{-2q}(k+1) \tag{2.9}$$

for $q \in (0, 1)$ and $k \in \mathbb{N}$. Accordingly, it is sufficient to define $\beta_n = x_k$ for $n \in N(k)$, $k \in \mathbb{N}$, and $\beta_n = x_1$ for $n < p(1)$.

(ii) The construction of a sequence $\{\gamma_n\}$ is almost the same as the construction of the sequence $\{\beta_n\}$ above; in considerations two following changes are needed:

- in (2.7) we additionally multiply the right side of the inequality by $\frac{\ln^2 3}{\ln^2 2}$ (the function $x \mapsto \ln^2(x+1)/\ln^2 x$ is decreasing on $(1, \infty)$);
- in (2.8) we introduce $\sum_{n\in N(k)} a_n = \big(kx_k \ln^2(k+1)\big)^{-1}$, $k \in \mathbb{N}$. □

## References

[1] R. Wituła, D. Słota, Families $l^{>p}$ and $l^{=p}$, *Tatra Mountains Publ.*, Under Review.