International Journal of Pure and Applied Mathematics
Volume 67 No. 3 2011, 327-335

A NOTE ON CONVERGENCE SPEED AND POPULATION IN THE AK MODEL WITH HABIT FORMATION

Massimiliano Ferrara
Department SSGES
Mediterranean University of Reggio Calabria
2, Via dei Bianchi, Reggio Calabria, 89127, ITALY
e-mail: massimiliano.ferrara@unirc.it

Abstract: This paper examines the implications of a logistic population growth hypothesis in the AK model with habit formation of Gomez [11]. We first prove that there exists a steady state equilibrium, which is a saddle point with a two dimensional stable manifold. Then, in contrast to Gomez [11], we show that the asymptotic speed of convergence does not necessarily decrease as the value of some parameter increases.

AMS Subject Classification: 91B62
Key Words: habit formation, logistic population, AK model

1. Introduction

AK-type models with habit formation have been used in the literature to address a wide variety of issues (e.g., Barlevy [1]; Carroll et al [4-5]; Chen [6]; Gomez [11]; Ikefuji [21]; Mansoorian and Michelis [23]; Shie et al [24]; Tsoukis [25]). For example, through numerical simulations, Carroll et al [4-5] have shown that the introduction of habit formation in the standard AK endogenous growth model may cause this model to exhibit transitional dynamics, while Gomez [11] has proved that the convergence speed of the AK model with external habits is higher than that in the AK model with internal habits. It is known that usually standard economic growth theory assumes that labor (population) force grows at a positive constant rate (Malthusian model, see [22]).

Received: January 5, 2011

However, this assumption is not a good approximation to reality as population exponentially grows without limits, which is clearly unrealistic. To remove the prediction of unbounded population size in the very long-run, Verhulst [27] wrote an alternative model, known as the logistic growth model, where the population stock evolves according to an elongated S-curve. Recent forecasts (e.g., United Nations [26]) confirm that the annual growth rate of population is expected to fall gradually until 2100, and that world population will stabilize at a level of about eleven billion people by 2200. Thus, not only theoretically but also empirically, it seems reasonable to model population size as following a logistic process. Bucci and Guerrini [3], Ferrara and Guerrini [7-9], Germanà and Guerrini [10], and Guerrini [12-20], have recently explored the implications of studying some economic growth models within a framework where the change over time of the labor force is non constant but governed by the logistic law or by a bounded population growth rate. In this paper, we wish to investigate the dynamic effects of assuming a logistic population growth hypothesis in Gomez's model [11]. This set-up leads the economy to be described by a four dimensional dynamical system, whose unique non-trivial steady state equilibrium is a saddle point with a two dimensional stable manifold. Two stable roots, rather than only one as in Gomez [11], determine the speed of convergence. Now, the crucial determinant of the asymptotic speed of convergence is the larger of the two negative eigenvalues. As a result, contrary to Gomez [11], the asymptotic speed of convergence may be not necessarily decreasing as the value of some parameter increases.

2. The Model

Consider an economy with a continuum of identical agents, where an agent derives utility from the comparison of his current consumption to the habit stock, as well as from the absolute level of his current consumption. The discounted sum of the intertemporal utility of a representative agent is given by

$$\int_0^\infty \frac{(C/H^\gamma)^{1-\varepsilon}}{1-\varepsilon}\, e^{-\beta t} dt, \tag{1}$$

with $\varepsilon > 1$, $0 < \gamma < 1$, and where C denotes the agent's current consumption, H is the reference consumption level or habits stock, and $\beta > 0$ is the rate of time preference. The habit stock is determined as a weighted average of past consumption according to

$$H = \rho \int_{-\infty}^t e^{\rho(s-t)} C(s)^\phi \bar{C}(s)^{1-\phi} ds, \tag{2}$$

where $\rho > 0$ and $0 \leq \phi \leq 1$. $\bar{C}$ is economy-wide average consumption. Differentiating the habit stock (2) with respect to time yields the evolution of the habit stock, i.e.

$$\dot{H} = \rho\left(C^{\phi}\bar{C}^{1-\phi} - H\right). \tag{3}$$

If $\phi = 0$, habits are formed from the economy-wide average past consumption levels, which corresponds to the model with external habits. If $\phi = 1$, habits arise from own past consumption, namely we have the model with internal habits. Finally, if $0 < \phi < 1$, habits arise from both own and average past consumption. Gross output per capita Y is determined by the AK technology $Y = AK$, $A > 0$, where K is the aggregate capital stock. The model and the notation are so far essentially the same as in Gomez [11]. Contrary to Gomez [11] we now assume that the growth rate of population L is non-constant, but it evolves according to the following law

$$\dot{L} = L(a - bL), \quad a > b > 0, \tag{4}$$

where, for simplicity, the initial population has been normalized to one, $L_0 = 1$. Equation (4) is called Verhulst equation, see [27], and the underlying population model is known as the logistic model. Next, the agent's budget constraint is

$$\dot{K} = [A - n(L)]\,K - C, \tag{5}$$

where $n(L) = \dot{L}/L$ denotes the population growth rate. For convenience, capital stock does not depreciate. As well, we assume that $A > \beta$ in order to ensure that technology is sufficiently productive to allow for endogenous growth.

3. The Equilibrium

The agent's problem is to maximize (1) subject to (3)-(5), taking as given $\bar{C}$ and the initial conditions on capital, $K_0 > 0$, and habits stock, $H_0 > 0$. Solving this continuous-time dynamic problem involves using calculus of variations. The current-value Hamiltonian of the agent's problem is given by

$$\mathcal{H} = \frac{(C/H^{\gamma})^{1-\varepsilon}}{1-\varepsilon} + \lambda\left\{[A - n(L)]\,K - C\right\} + \mu\left[\rho\left(C^{\phi}\bar{C}^{1-\phi} - H\right)\right],$$

where λ and μ are the shadow values of capital and habits stock, respectively. The optimality conditions are $\mathcal{H}_C = 0$, $\dot{\lambda} = \beta\lambda - \mathcal{H}_K$, $\dot{\mu} = \beta\mu - \mathcal{H}_H$, namely

$$C^{-\varepsilon}H^{-\gamma(1-\varepsilon)} - \lambda + \phi\mu\rho C^{\phi-1}\bar{C}^{1-\phi} = 0, \tag{6}$$

$$\dot{\lambda} = \{\beta - [A - n(L)]\}\,\lambda, \tag{7}$$

$$\dot{\mu} = (\beta + \rho)\mu + \gamma C^{1-\varepsilon} H^{-\gamma(1-\varepsilon)-1}, \tag{8}$$

plus equations (3)-(5) with the transversality condition

$$\lim_{t\to\infty} e^{-\beta t}\lambda K = 0, \quad \lim_{t\to\infty} e^{-\beta t}\mu H = 0. \tag{9}$$

Let $q \equiv -\mu/\lambda$. Since $\bar{C} = C$ in a symmetric equilibrium, differentiating (6), using equation (7) and then equations (3), (5), we eventually get

$$\lambda = \frac{C^{-\varepsilon} H^{-\gamma(1-\varepsilon)}}{1 + \rho\phi q}, \quad \mu = -\frac{C^{-\varepsilon} H^{-\gamma(1-\varepsilon)} q}{1 + \rho\phi q}. \tag{10}$$

Set $c \equiv C/H$, $h \equiv H/K$. Since $\dot{q}/q = \dot{\mu}/\mu - \dot{\lambda}/\lambda$, differentiation of (10) with respect to time, and the use of equations (7)-(8), allow us to get the system that drives the dynamics of the economy in terms of the variables c, h, q and L. More precisely, we have

$$\begin{aligned}
\dot{c} &= \frac{c}{\varepsilon}\left[\frac{A - n(L) + \rho}{1 + \rho\phi q} - \rho(1 - \gamma\phi)c + \rho(1 - \gamma)(\varepsilon - 1)(1 - c) - \beta\right], && (11)\\
\dot{h} &= -h\left[A - n(L) + \rho - (h + \rho)c\right], && (12)\\
\dot{q} &= q\left[A - n(L) + \rho(1 - \gamma\phi c)\right] - \gamma c, && (13)\\
\dot{L} &= Ln(L) = L\left(a - bL\right). && (14)
\end{aligned}$$

From here on, we pay attention to the steady-state equilibrium of the dynamical system (11)-(14). We recall that in steady state the variables c, h, q, L are constant. Let us denote the steady state values of these variables by c_*, h_*, q_*, and L_*, respectively. Note that our analysis is restricted to interior steady states only, i.e. we exclude the economically meaningless solutions such as $c_* = 0, h_* = 0, q_* = 0$, or $L_* = 0$.

Lemma 1. *Set $g = (A - \beta)/[\gamma + \varepsilon(1 - \gamma)]$. There exists a unique steady state equilibrium (c_*, h_*, q_*, L_*), where*

$$c_* = 1 + \frac{g}{\rho}, \qquad h_* = \frac{(1 - \gamma)(\varepsilon - 1)g + \beta}{c_*},$$

$$q_* = \frac{\gamma c_*}{\beta + \rho(1 - \gamma\phi) + [\gamma(1 - \phi) + \varepsilon(1 - \gamma)]\, g}, \qquad L_* = \frac{a}{b}.$$

Proof. Imposing the stationary conditions $\dot{c} = \dot{h} = \dot{q} = \dot{L} = 0$, we get

$$\frac{A+\rho}{1+\rho\phi q} + \rho(1-\gamma)(\varepsilon-1)(1-c) - \rho(1-\gamma\phi)c - \beta = 0, \qquad L = \frac{a}{b},$$

$$[A + \rho(1-\gamma\phi c)] - \frac{\gamma c}{q} = 0, \qquad A + \rho - (h+\rho)c = 0.$$

The steady state value can now be determined in a recursive manner. □

Remark 1. $g > 0$ due to the assumption $A > \beta$. Hence, we derive $c_* > 0$, $q_* > 0$, as well as $h_* > 0$ since $\varepsilon > 1$.

The transversality condition (9) associated with the capital stock and the habit stock are both satisfied. In fact, equation (9) implies that $\dot{\lambda}/\lambda + \dot{K}/K - \beta < 0$ and $\dot{\mu}/\mu + \dot{H}/H - \beta < 0$. On the steady state, $\dot{q} = \dot{h} = 0$. As well, we have that (5) yields $\dot{K}/K = A - c_* h_*$, while (7) gives $\dot{\lambda}/\lambda = \beta - A$. Combining these results together, we obtain $\dot{\lambda}/\lambda + \dot{K}/K - \beta = \dot{\mu}/\mu + \dot{H}/H - \beta = -(1-\gamma)(\varepsilon-1)g < 0$.

Proposition 1. *The steady state equilibrium is a saddle point with a two dimensional stable manifold.*

Proof. Linearizing equations (11), (12), (13) and (14) around (c_*, h_*, q_*, L_*) yields

$$\begin{bmatrix} \dot{c} \\ \dot{h} \\ \dot{q} \\ \dot{L} \end{bmatrix} = J^* \begin{bmatrix} c - c_* \\ h - h_* \\ q - q_* \\ L - L_* \end{bmatrix}, \qquad J^* = \begin{bmatrix} J_{11} & 0 & J_{13} & J_{14} \\ J_{21} & J_{22} & 0 & J_{24} \\ J_{31} & 0 & J_{33} & J_{34} \\ 0 & 0 & 0 & J_{44} \end{bmatrix},$$

where

$$J_{11} = -\frac{[\varepsilon(1-\gamma) + \gamma(1-\phi)]\,\rho c_*}{\varepsilon}, \qquad J_{14} = \frac{b c_*}{\varepsilon\,(1+\rho\phi q_*)},$$

$$J_{13} = -\frac{[\beta + \rho c_*(1-\gamma\phi) + \rho(c_*-1)(\varepsilon-1)(1-\gamma)]\,\rho\phi c_*}{\varepsilon(1+\rho\phi q_*)},$$

$$J_{21} = h_*(h_*+\rho), \quad J_{22} = c_* h_*, \quad J_{24} = -b h_*, \quad J_{31} = -\gamma(1+\rho\phi q_*),$$

$$J_{33} = A + \rho(1-\gamma\phi c_*), \quad J_{34} = b q_*, \quad J_{41} = J_{42} = J_{43} = 0, \quad J_{44} = -a.$$

In order to characterize the local stability of the system, we need to compute the four eigenvalues of the Jacobian matrix J^*. One eigenvalue is immediate to be $\xi_1 = -a < 0$. The other three eigenvalues are those of the matrix

$$Q = \begin{bmatrix} J_{11} & 0 & J_{13} \\ J_{21} & J_{22} & 0 \\ J_{31} & 0 & J_{33} \end{bmatrix}.$$

It is now straightforward that $\xi_2 = c_* h_* > 0$ is an eigenvalue of Q. After simplification, we find that the determinant of this matrix is

$$\det(Q) = -\frac{\rho c_*^2 h_* (A - \beta) \left[\beta + g(\varepsilon - 1)(1 - \gamma) + (1 - \gamma\phi)\rho c_*\right]}{\varepsilon g} < 0.$$

Recalling that the determinant of a matrix is also equal to the product of its eigenvalues, we derive that the remaining two eigenvalues of Q must be one negative and one positive. In conclusion, we have found that J^* has two negative ξ_1, ξ_3 (stable) and two positive ξ_2, ξ_4 (unstable) roots. This proves that the steady state is (locally) a saddle point. The stable manifold is the hyperplane generated by the associated eigenvectors, with dimension equal to the number of negative eigenvalues (see, e.g., Blume and Simon [2]) □

4. Speed of Convergence

Many models of growth, including Gomez's model [11], have the property that the transitional dynamics are determined by a one dimensional stable manifold. As a consequence, all the variables converge to their respective steady states at the same constant speed, which is equal to the magnitude of the unique stable eigenvalue. By contrast, in the present model, the stable transitional path is a two dimensional locus, thereby introducing important flexibility to the convergence and transition characteristics. From what done in Section 3, we have that, as the system approaches the stationary state for t tending to infinity, c, h, q, and L converge to their steady states values with a speed of convergence which is determined by the eigenvalue with the smaller absolute value, $\min\{|\xi_1|, |\xi_3|\}$. Finally, notice that over time the weight of the smaller (more negative) eigenvalue declines, so that the larger of the two stable (negative) eigenvalues will describe the asymptotic speed of convergence.

Proposition 2. *The asymptotic speed of convergence decreases as the value of ϕ increases if $|\xi_1| < |\xi_3|$. This does not happen if $|\xi_1| > |\xi_3|$.*

Proof. First, notice that the asymptotic speed of convergence is equal to $|\xi_1|$ if $|\xi_3| < |\xi_1|$, and to $|\xi_3|$ if $|\xi_1| < |\xi_3|$. Next, we prove that $|\xi_3|$ decreases as the value of ϕ increases. Now, ξ_3 is the unique negative eigenvalue of the submatrix

$$Q_{13} = \begin{bmatrix} J_{11} & J_{13} \\ J_{31} & J_{33} \end{bmatrix}.$$

This means that ξ_3 solves the polynomial equation $p(\xi) = \xi^2 - tr(Q_{13})\xi + \det(Q_{13}) = 0$, where tr (resp. det) denotes the trace (resp. the determinant) of Q_{13}. We get the following analytical expression

$$\xi_3 = \frac{tr(Q_{13}) - \sqrt{[tr(Q_{13})]^2 - 4\det(Q_{13})}}{2},$$

with

$$tr(Q_{13}) = -\frac{[\varepsilon(1-\gamma) + \gamma(1-\phi)]\,\rho c_*}{\varepsilon} + A + \rho(1 - \gamma\phi c_*), \quad \det(Q_{13}) = \frac{\det(Q)}{J_{22}}.$$

After a tedious calculation, we arrive to

$$\frac{d\xi_3}{d\phi} = \frac{\gamma(\rho+g)^2\,[\gamma + \varepsilon(1-\gamma)] + \gamma(\rho+g)(\varepsilon-1)\xi_3}{(A-g)\varepsilon + (\rho+g)(\varepsilon-1)\gamma(1-\phi) - 2\varepsilon\xi_3}.$$

The denominator of this expression is clearly positive. To determine the sign of the numerator, we set $M = -(\rho+g)\,[\gamma + \varepsilon(1-\gamma)]\,/(\varepsilon - 1) < 0$. Since

$$p(M) = -\frac{M\,\{[\gamma + \varepsilon(1-\gamma)]\,g + \rho\varepsilon\}}{(\varepsilon-1)\varepsilon} - \frac{M\,[A + \rho(1-\gamma)]}{\varepsilon} > 0 = p(\xi_3),$$

then it must be $M < \xi_3 < 0$. Consequently, the numerator is positive, yielding $d\xi_3/d\phi > 0$. □

5. Conclusion

In this paper, we introduce a logistic-type population growth law in Gomez's [11] AK endogenous growth model with habit formation. Within this setup, we are able to demonstrate that the model has a unique non-trivial steady state equilibrium, which is proved to be a saddle point with a two dimensional stable manifold. As a result, contrary to Gomez [11], the stable transitional path is a two dimensional locus, thereby introducing important flexibility to the convergence and transition characteristics. Consequently, the asymptotic speed of convergence in the model may be not necessarily decreasing as the value of some parameter increases.

References

[1] G. Barlevy, The cost of business cycles under endogenous growth, *American Economic Review*, **94** (2004), 964-990.

[2] L.E. Blume, C.P. Simon, *Mathematics for Economists*, Norton, New York (1994).

[3] A. Bucci, L. Guerrini, Transitional dynamics in the Solow-Swan growth model with AK technology and logistic population change, *B.E. Journal of Macroeconomics*, **9** (2009), 1-16.

[4] C.D. Carroll, J. Overland, D.N. Weil, Comparison utility in a growth model, *Journal of Economic Growth*, **2** (1997), 339-367.

[5] C.D. Carroll, J. Overland, D.N. Weil, Saving and growth with habit formation, *American Economic Review*, **90** (2000), 341-355.

[6] B.L. Chen, Multiple BGPs in a growth model with habit persistence, *Journal of Money, Credit and Banking*, **39** (2007), 25-48.

[7] M. Ferrara, L. Guerrini, On the dynamics of a three sector growth model, *International Review of Economics*, **55** (2008), 275-283.

[8] M. Ferrara, L. Guerrini, The Ramsey model with logistic population growth and Benthamite felicity function revisited, *WSEAS Transactions on Mathematics*, **8** (2009), 41-50.

[9] M. Ferrara, L. Guerrini, More on the Green Solow model with logistic population change, *WSEAS Transactions on Mathematics*, **8** (2009), 97-106.

[10] C. Germanà, L. Guerrini, On the closed-form solution of the improved labor augmented Solow-Swan model, *Applied Sciences*, **7** (2005), 101-106.

[11] M.A. Gomez, Convergence speed in the AK endogenous growth model with habit formation, *Economics Letters*, **100** (2008), 16-21.

[12] L. Guerrini, The Solow-Swan model with a bounded population growth rate, *Journal of Mathematical Economics*, **42** (2006), 14-21.

[13] L. Guerrini, Logistic population change and the Mankiw-Romer-Weil model, *Applied Sciences*, **12** (2010), 96-101.

[14] L. Guerrini, The Ramsey model with a bounded population growth rate, *Journal of Macroeconomics*, **32** (2010), 872-878.

[15] L. Guerrini, The Ramsey model with AK technology and a bonded population growth rate, *Journal of Macroeconomics*, **32** (2010), 1178-1183.

[16] L. Guerrini, A closed-form solution to the Ramsey model with logistic population growth, *Economic Modelling*, **27** (2010), 1178-1182.

[17] L. Guerrini, Transitional dynamics in the Ramsey model with AK technology and logistic population change, *Economics Letters*, **109** (2010), 17-19.

[18] L. Guerrini, The AK Ramsey model with von Bertalanffy population law and Benthamite function, *Far East Journal of Mathematical Sciences*, **45** (2010), 187-192.

[19] L. Guerrini, A note on the Ramsey growth model with the von Bertalanffy population law, *Applied Mathematical Sciences*, **4** (2010), 3233-3238.

[20] L. Guerrini, The AK Ramsey growth model with the von Bertalanffy population law, *Applied Mathematical Sciences*, **4** (2010), 3245-3249.

[21] M. Ikefuji, Habit formation in an endogenous growth model with pollution abatement activities, *Journal of Economics*, **94** (2008), 241-259.

[22] T.R. Malthus, *An Essay on the Principle of Population*, Cambridge University Press, Cambridge (1992).

[23] A. Mansoorian, L. Michelis, Money, habits and growth, *Journal of Economic Dynamics and Control*, **29** (2005), 1267-1285.

[24] J. Shie, C. Lai, W. Chang, Addictive behavior and endogenous growth, *Journal of Economics*, **72** (2000), 263-273.

[25] C. Tsoukis, Keeping up with the Joneses, growth, and distribution, *Scottish Journal of Political Economy*, **54** (2007), 575-600.

[26] United Nations, *World Population Prospects: The 2000 Revision*, New York, United Nations (2000).

[27] P.F. Verhulst, Notice sur la loi que la population suit dans son accroissement, *Correspondance Mathématique et Physique*, **10** (1838), 113-121.