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# ON THE ADJACENT STRONG EQUITABLE EDGE COLORING OF $C_n \vee C_n$

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**Abstract:** In this paper, we discuss the adjacent strong equitable edge coloring of join-graphs about $C_n \vee C_n$.

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**Key Words:** cycle, join-graph, adjacent strong equitable edge coloring

## 1. Introduction

The coloring problem of graphs is widely applied in practice. In [9], some conditional coloring problems as introduced. Some network problem can be converted to the strong edge coloring (see [6], [4], [1], [3]) and adjacent strong edge coloring, see [11].

**Definition 1.** (see [6], [4], [1], [3]) For a graph $G(V, E)$, if a proper coloring $f$ is satisfied with $C(u) \neq C(v)$ for $\forall u, v \in V(G)(u \neq v)$, then $f$ is called *$k$-strong edge coloring of $G$*, abbreviated $k$-SEC, and

$$\chi_s^{'}(G) = \min\{k|k\text{-SEC of } G\}$$

is called the strong edge chromatic number of $G$. And for $\forall uv \in E(G)$, $C(u) \neq$

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$C(v)$, $f$ is called $k$-adjacent strong edge coloring of $G$, abbreviated $k$-ASEC, and

$$\chi'_{as}(G) = \min\{k|k\text{-ASEC of } G\}$$

is called the adjacent strong edge chromatic number of G, see [11]. Here

$$C(u) = \{f(uv)|uv \in E(G)\}.$$

**Definition 2.** Let $f$ be a $k$-ASEC of $G$ and satisfy

$$||E_i| - |E_j|| \leq 1, \quad i, j = 1, 2, \cdots, k,$$

$f$ is called *the adjacent strong equitable edge coloring of* $G$, and is denoted by $k$-ASEEC of $G$, and

$$\chi'_{ase}(G) = \min\{k|k\text{-ASEEC of } G\}$$

is called *the adjacent strong equitable edge cromatic number* of $G$. Here

$$E_i = \{e|f(e) = i\}, \quad i = 1, 2, \cdots, k.$$

**Conjecture.** (see [11]) *For a connected graph with order* $p \geq 3$, *and* $G \neq C_5$ *(5-cycle),*

$$\chi'_{as}(G) \leq \Delta(G) + 2.$$

*Here* $p = |V(G)|$, $\Delta(G)$ *is maximal degree of* $G$.

There are many proofs of this conjecture, for example [10], [2], for $\Delta(G) \leq 3$. For a connected graph with $|V(G)| \geq 3$:

(1) If $G$ is a bipartite graph with no isolate edges, then

$$\chi'_{as}(G) \leq \Delta(G) + 2.$$

(2) If $G$ is a $k$-chromatic graph with no isolate edges, then

$$\chi'_{as}(G) \leq \Delta(G) + O(\log\ k).$$

**Definition 3.** (see [5]) For graph $G$ and graph $H$, $V(G) \cap V(H) = E(G) \cap E(H) = \emptyset$, and

$$\begin{cases} V(G \vee H) = V(G) \cup V(H), \\ E(G \vee H) = E(G) \cup E(H) \cup \{uv|u \in V(G), v \in V(H)\}, \end{cases}$$

then $G \vee H$ is called join-graph of $G$ and $H$.

**Lemma 1.** (see [11]) *If* $G$ *is a connected graph with* $|V(G)| \geq 3$*, and* $uv \in E(G)$*, d(u)=d(v)=*$\Delta(G)$*. Then*

$$\chi'_{as}(G) \geq \Delta(G) + 1.$$

**Lemma 2.** (see [5]) *If* $k \geq \chi'(G)$*, then k-PEC of* $G$ *exists*

$$||E_i| - |E_j|| \leq 1; \quad i, j = 1, 2, \cdots, k,$$

*where* $e \in E_i, f(e) = i(i = 1, 2, \cdots, k), \chi'(G)$ *is the chromatic number of* $G$*.*

**Lemma 3.** *For* $n \geq 3$*,*

$$|E(C_n \vee C_n)| = n^2 + 2n.$$

For $m > n \geq 1$, there are many adjacent strong chromatic numbers of $C_m \vee C_n$. In this paper we have the adjacent strong equitable chromatic number of $C_n \vee C_n$, for the others terminologies refer to [5], [7], [8].

## 2. Adjacent Strong Edge Coloring of $C_n \vee C_n$

**Theorem 1.** *For* $n \geq 4$*. Then*

$$\chi'_{ase}(C_n \vee C_n) = \begin{cases} n+3, n = 3 \text{ or } n \equiv 0( \text{ mod } 2), \\ n+4, n \geq 5 \text{ or } n \equiv 1( \text{ mod } 2). \end{cases}$$

*Proof.* Supposing the two cycles are $u_1u_2 \cdots u_nu_1$ and $v_1v_2 \cdots v_nv_1$ with separately.

When $n = 3$, $C_3 \vee C_3 = K_6$ (complete graph with order 6), can be seen in appendix.

*Case 1.* $n \equiv 0($ mod $2)$. By Lemmas 1, 2, and 3, $C_n \vee C_n$ there exist three perfect matching $M_1, M_2, M_3$ and $M_1 \cap M_2 = M_2 \cap M_3 = M_3 \cap M_1 = \phi$.

When $n = 4$, let $f$ be as follows:

$$f(u_1u_4) = f(u_2u_3) = f(v_1v_4) = f(v_2v_3) = 7,$$

$$f(u_1v_4) = f(u_2v_1) = f(u_3v_3) = f(u_4v_2) = 6,$$

$$f(u_1v_2) = f(u_2v_4) = f(u_3v_1) = f(u_4v_3) = 5,$$

$$f(u_1v_1) = f(u_2v_3) = f(u_3u_4) = 4,$$

$$f(u_1u_2) = f(u_3v_2) = f(u_4v_4) = 3,$$

$$f(u_1v_3) = f(u_3v_4) = f(v_1v_2) = 2,$$

$$f(u_2v_2) = f(u_4v_1) = f(v_3v_4) = 1.$$

So $f$ is a $7 - ASEEC$ of $C_4 \vee C_4$.

When $n = 6$:

$$(C_6 \vee C_6) \setminus \{u_1, u_3\} \setminus \bigcup_{i=1}^{3} M_i, \text{ there exists a perfect matching } M_4,$$

$$(C_6 \vee C_6) \setminus \{u_2, u_5\} \setminus \bigcup_{i=1}^{4} M_i, \text{ there exists a perfect matching } M_5,$$

$$(C_6 \vee C_6) \setminus \{u_4, u_6\} \setminus \bigcup_{i=1}^{5} M_i, \text{ there exists a perfect matching } M_6,$$

$$(C_6 \vee C_6) \setminus \{v_1, v_3\} \setminus \bigcup_{i=1}^{6} M_i, \text{ there exists a perfect matching } M_7,$$

$$(C_6 \vee C_6) \setminus \{v_2, v_5\} \setminus \bigcup_{i=1}^{7} M_i, \text{ there exists a perfect matching } M_8,$$

$$(C_6 \vee C_6) \setminus \{v_4, v_6\} \setminus \bigcup_{i=1}^{8} M_i, \text{ there exists a perfect matching } M_9.$$

Let

$$\forall e \in M_i : \quad f(e) = i.$$

So, $f$ is a 9-ASEEC of $C_6 \vee C_6$.

Similarly we can proove that $C_n \vee C_n$ exist $(n + 3)$-ASEEC, when $n \equiv 0(\mod 2)$ and $n \geq 8$.

By Lemma 3, $C_n \vee C_n$ there exist three perfect matching $M_1$, $M_2$, $M_3$ and $M_1 \cap M_2 = M_2 \cap M_3 = M_3 \cap M_1 = \phi$. Suppose $n = 2k, k \geq 2$.

$$(C_n \vee C_n) \setminus \{u_1, u_{k+1}\}\} \setminus \bigcup_{i=1}^{3} M_i \text{ there exists a perfect matching } M_4,$$

$$(C_n \vee C_n) \setminus \{u_2, u_{k+2}\}\} \setminus \bigcup_{i=1}^{4} M_i \text{ there exists a perfect matching } M_5,$$

$$\vdots$$

$$(C_n \vee C_n) \setminus \{u_k, u_n\}\} \setminus \bigcup_{i=1}^{k+3} M_i \text{ there exists a perfect matching } M_{k+4},$$

$$(C_n \vee C_n) \setminus \{v_1, v_k\}\} \setminus \bigcup_{i=1}^{k+4} M_i \text{ there exists a perfect matching } M_{k+5},$$

$$\vdots$$

$$(C_n \vee C_n) \setminus \{v_k, v_n\}\} \setminus \bigcup_{i=1}^{n+2} M_i \text{ there exists a perfect matching } M_{n+3}.$$

let

$$\forall e \in M_i : \quad f(e) = i, \quad i = 1, 2, \cdots, n+3.$$

For $f$ we have

$$\overline{C}(u_i) = \overline{C}(u_{k+i}) = \{i\}, \quad i = 1, 2, \cdots, k,$$

$$\overline{C}(v_i) = \overline{C}(v_{k+i}) = \{k+i+1\}, \quad i = 1, 2, \cdots, k,$$

and

$$|E_i| = \begin{cases} n, & i = 1, 2, 3; \\ n-1, & i = 4, 5, \cdots, n+3. \end{cases}$$

So, $f$ is a $(n+3)$-ASEEC of $C_n \vee C_n$.

*Case 2.* $n \equiv 1(\mod 2)$ and $n \geq 5$. We proove $\chi'_{as}(C_n \vee C_n) \geq n+4$.

If $\chi'_{as}(C_n \vee C_n) = n+3$, there exist at least 3 colors represented at every vertex. For $|E(C_n \vee C_n)| = n^2 + 2n$, and each vertex lack just one color and the vertices lack the same color in a same cycle and not adjacent,and $n$ is an odd, so it is not possible that odd number vertices lack same color. For the edge number of $C_n \vee C_n$, the number of vertices lack same color just an even. So there must have two vertices in two different cycle lack same color. It is contradictory. Thus $\chi'_{ase}(C_n \vee C_n) \geq n+4$ when $n \equiv 1(\mod 2)$.

Let $f$ be as follows: the edges of $u_1u_2, u_2u_3, \cdots, u_{n-1}u_n$ are coloring colors with $n+3$, $n+4$ rotate; the edges of $v_1v_2, v_2v_3, \cdots, v_{n-1}v_n$ are coloring colors with n+4,n+3 rotate;

$$f(u_1u_n) = f(v_1v_n) = n+1,$$

$$f(u_1v_j) = j, \quad j = 1, 2\cdots, n,$$

$$f(u_iv_j) = i + j \quad (\text{when } i + j > n + 2, \text{ then } \mod n + 2),$$
$$i = 2, 3, \cdots, n - 1; \quad j = 1, 2, \cdots, n,$$

$$f(u_nv_1) = n + 2; \ f(u_nv_j) = j - 1, j = 2, 3, \cdots, n.$$

For the $f$, we have

$$\overline{C}(u_1) = \{n + 2, n + 4\}; \quad \overline{C}(u_i) = \{i - 1, i\}, \quad i = 2, 3\cdots, n - 1,$$

$$\overline{C}(u_n) = \{n, n + 3\} \quad (n \equiv 1(\mod 2)),$$

$$\overline{C}(v_1) = \{2, n + 3\}; \ \overline{C}(v_i) = \{i + 1, n + i\}(\text{when } i + j > n + 2, \text{ take } \mod n + 2,$$
$$\overline{C}(v_n) = \{n, n + 4\}(n \equiv 1(\mod 2)).$$

So, $f$ is a $(n + 4)$-ASEEC of $C_n \vee C_n$. Theorem 3 is true.
Using the arguments above, it is easy to verify Theorem 1. □

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