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A SEQUENCE SPACE DEFINED BY MUSIELAK-ORLICZ FUNCTION

Kuldip Raj[1 §], Ajay K. Sharma[2], Sunil K. Sharma[3]
[1,2,3]School of Mathematics
Shri Mata Vaishno Devi University
Katra, 182320, INDIA
[1]e-mail: kuldeepraj68@hotmail.com
[2]e-mail: aksju_76@yahoo.com
[3]e-mail: sunilksharma42@yahoo.com

Abstract: In this paper we introduce a new sequence space $BV_\sigma(\mathcal{M}, p, r)$ defined by Musielak-Orlicz function $\mathcal{M} = (M_k)$ and study some topological properties of this sequence space.

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1. Introduction and Preliminaries

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if:

1. $p(x) \geq 0$, for all $x \in X$,
2. $p(-x) = p(x)$, for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [12], Theorem

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§Correspondence author

10.4.2, p. 183).

Let l_∞ and c denotes the Banach spaces of bounded and convergent sequences $x = (x_k)_{k=1}^{\infty}$ respectively. Let σ be an injection of the set of positive integers $\mathbb{N}$ into itself having no finite orbits and T be the operator defined on l_∞ by $T((x_n)_{n=1}^{\infty}) = (x_{\sigma(n)})_{n=1}^{\infty}$.

A positive linear functional φ, with $||\varphi|| = 1$, is called a σ-mean or an invariant mean if $\varphi(x) = \varphi(Tx)$ for all $x \in l_\infty$.

A sequence x is said to be σ-convergent, denoted by $x \in V_\sigma$, if $\varphi(x)$ takes the same value, called σ-$\lim x$, for all σ-means φ. We have

$$V_\sigma = \Big\{x = (x_n) : \sum_{m=1}^{\infty} t_{m,n}(x) = L \text{ uniformly in } n,\ L = \sigma - \lim x\Big\},$$

for $m \geq 0$, $n > 0$, where, $t_{m,n}(x) = \frac{x_n + x_{\sigma(n)} + \cdots + x_{\sigma^m(n)}}{m+1}$, and $t_{-1,n} = 0$ (see Schafer [11]), where $\sigma^m(n)$ denotes the m-th iterate of σ at n. In particular, if σ is the translation, a σ-mean is often called a Banach limit and V_σ reduces to f, the set of almost convergent sequences (see Lorentz [4]). Subsequently invariant mean have been studied by Ahmad and Mursaleen [1] and many others.

A sequence space E is said to be solid (or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ and for all $k \in \mathbb{N}$.

A sequence space E is said to be monotone if it contains the canonical preimages of all its step spaces.

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [3] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$l_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\}$$

which is called a Orlicz sequence space. Also l_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

Also, it was shown in [3] that every Orlicz sequence space l_M contains a subspace isomorphic to l_p $(p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$, for all values of $x \geq 0$, and for $L > 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt,$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$, note that an Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$, for all λ with $0 < \lambda < 1$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see (see [6], [9]). A sequence $\mathcal{N} = (N_k)$ is called a complementary function of a Musielak-Orlicz function $\mathcal{M}$ if

$$N_k(v) = \sup\{|v|u - (M_k) : u \geq 0\}, \ k = 1, 2, \dots .$$

For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_M(cx) < \infty, \quad \text{for some } \ c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty, \quad \text{for all } \ c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf \Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf \Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Mursaleen [8] defined the sequence space

$$BV_\sigma = \Big\{x \in l_\infty : \sum_m |\varphi_{m,n}(x)| < \infty, \ \text{uniformly in} \quad n\Big\},$$

where $\varphi_{m,n}(x) = t_{m,n}(x) - t_{m-1,n}(x)$, assuming that $t_{m,n}(x) = 0$, for $m = -1$. Note that for any sequences x, y and scalar λ we have

$$\varphi_{m,n}(x + y) = \varphi_{m,n}(x) + \varphi_{m,n}(y)$$

and

$$\varphi_{m,n}(\lambda x) = \lambda \varphi_{m,n}(x).$$

Let M be an Orlicz function, $p = (p_m)$ be any sequence of strictly positive real numbers and $r \geq 0$ the sequence space $BV_\sigma(M, p, r)$ defined by Khan [2].

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_m)$ be any sequence of strictly positive real numbers. In the present paper we define the sequence space

$$BV_\sigma(\mathcal{M}, p, r) = \Big\{x = (x_k) : \sum_{k=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} M_k\Big(\frac{|\varphi_{m,n}(x)|}{\rho}\Big)\Big]^{p_m} < \infty,$$
$$\text{uniformly in } n \text{ and for some } \rho > 0\Big\}.$$

For $\mathcal{M}(x) = x$, we get

$$BV_\sigma(p, r) = \Big\{x = (x_k) : \sum_{m=1}^{\infty} |\varphi_{m,n}(x)|^{p_m} < \infty, \quad \text{uniformly in } n \ \Big\}.$$

For $p_m = 1$, for all m, we get

$$BV_\sigma(\mathcal{M}, r) = \Big\{x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} M_k\Big(\frac{|\varphi m, n(x)|}{\rho}\Big)\Big] < \infty,$$
$$\text{uniformly in } n \text{ and for some } \rho > 0\Big\}.$$

For $r = 0$, we get

$$BV_\sigma(\mathcal{M}, p) = \Big\{x = (x_k) : \sum_{m=1}^{\infty} \Big[\sup_{k\geq 0} M_k\Big(\frac{|\varphi_{m,n}(x)|}{\rho}\Big)\Big]^{p_m} < \infty,$$
$$\text{uniformly in } n \text{ and for some } \rho > 0\Big\}.$$

For $\mathcal{M}(x) = x$ and $r = 0$, we get

$$BV_\sigma(p) = \Big\{x = (x_k) : \sum_{m=1}^{\infty} |\varphi_{m,n}(x)|^{p_m} < \infty, \quad \text{uniformly in } n\Big\}.$$

For $p_m = 1$, for all m and $r = o$, we get

$$BV_\sigma(\mathcal{M}) = \Big\{x = (x_k) : \sum_{m=1}^{\infty} \Big[\sup_{k\geq 0} M_k\Big(\frac{\varphi_{m,n}(x)}{\rho}\Big)\Big] < \infty, \ \ r \geq 0,$$
$$\text{uniformly in } n \text{ and for some } \rho > 0\Big\}.$$

For $\mathcal{M}(x) = x$, $p_m = 1$, for all m and $r = 0$, we get,

$$BV_\sigma = \Big\{x = (x_k) : \sum_{m=1}^{\infty} |\varphi_{m,n}(x)| < \infty, \quad \text{uniformly in } \ n\Big\}.$$

In this paper we examine some topological properties of the sequence space $BV_\sigma(\mathcal{M}, p, r)$.

2. Some Properties of Sequence Space $BV_\sigma(\mathcal{M}, p, r)$

Theorem 2.1. *The sequence space $BV_\sigma(\mathcal{M}, p, r)$ is a linear space over the field $\mathbb{C}$ of complex numbers.*

Proof. Let $x, y \in BV_\sigma(\mathcal{M}, p, r)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sum_{m=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} M_k\Big(\frac{|\varphi_{m,n}(x)|}{\rho_1}\Big)\Big]^{p_m} < \infty$$

and

$$\sum_{m=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} M_k\Big(\frac{|\varphi_{m,n}(y)|}{\rho_2}\Big)\Big]^{p_m} < \infty, \quad \text{uniformly in } \ n.$$

Define

$$\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2).$$

Since $\mathcal{M} = (M_k)$ is non decreasing and convex, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} M_k\Big(\frac{|\alpha\varphi_{m,n}(x) + \beta\varphi_{m,n}(y)|}{\rho_3}\Big)\Big]^{p_m} \\
&\quad \leq \sum_{m=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} M_k\Big(\frac{|\alpha\varphi_{m,n}(x)|}{\rho_3} + \frac{|\beta\varphi_{m,n}(y)|}{\rho_3}\Big)\Big]^{p_m} \\
&\quad \leq \sum_{m=1}^{\infty} \frac{1}{m^r}\frac{1}{2}\Big[\sup_{k\geq 0} M_k\Big(\frac{\varphi_{m,n}(x)}{\rho_1}\Big) + \sup_{k\geq 0} M_k\Big(\frac{\varphi_{m,n}(y)}{\rho_2}\Big)\Big] \\
&\quad < \infty, \quad \text{uniformly in } \ n.
\end{aligned}$$

This proves that $BV_\sigma(\mathcal{M}, p, r)$ is a linear space over the field $\mathbb{C}$ of complex numbers. □

Theorem 2.2. *For any Musielak-Orlicz function $\mathcal{M} = (M_k)$ and a bounded sequence $p = (p_m)$ of strictly positive real numbers, $BV_\sigma(\mathcal{M}, p, r)$ is a paranormed space with*

$$g(x) = \inf_{n\geq 1}\Big\{\rho^{\frac{p_n}{H}} : \Big(\sum_{m=1}^{\infty}\frac{1}{m^r}\Big[\sup_{k\geq 0} M_k\frac{(|\varphi_{m,n}(x)|)}{\rho}\Big]^{p_m}\Big)^{\frac{1}{H}} \leq 1 \quad \textit{uniformly in } n\Big\},$$

where $H = \max(1, \sup p_m)$.

Proof. It is clear that $g(x) = g(-x)$. Since $M(0) = 0$, we get $\inf\Big\{\rho^{\frac{p_n}{H}}\Big\} = 0$, for $x = 0$. By using Theorem 1, for $\alpha = \beta = 1$, we get

$$g(x + y) \leq g(x) + g(y).$$

For the continuity of scalar multiplication, let $l \neq 0$ be any complex numbers, then by definition, we have

$$g(lx) = \inf_{n\geq 1}\Big\{\rho^{\frac{p_n}{H}} : \Big(\sum_{m=1}^{\infty}\frac{1}{m^r}\Big[\sup_{k\geq 0} M_k\frac{(|\varphi_{m,n}(lx)|)}{\rho}\Big]^{p_m}\Big)^{\frac{1}{H}} \leq 1, \ \text{uniformly in } n\Big\}.$$

$$g(lx) = \inf_{n\geq 1}\Big\{(|l|s)^{\frac{p_n}{H}} : \Big(\sum_{m=1}^{\infty}\frac{1}{m^r}\Big[\sup_{k\geq 0} M_k\frac{(|\varphi_{m,n}(lx)|)}{s|l|}\Big]^{p_m}\Big)^{\frac{1}{H}} \leq 1, \ \text{uniformly in } n\Big\},$$

where $s = \frac{\rho}{|l|}$. Since $|l|^{p_m} \leq \max(1, |l|^q)$, we have

$$g(lx) \leq \max(1, |l|^q) \times \inf_{n\geq 1}\Big\{s^{\frac{p_n}{H}} : \Big(\sum_{m=1}^{\infty}\frac{1}{m_r}\Big[\sup_{k\geq 0} M_k\frac{(|\varphi_{m,n}|)}{s}^{p_m}\Big]^{\frac{1}{H}}\Big) \leq 1, \ \text{uniformly in } n\Big\} = \max(1, |l|^q)g(x),$$

and therefore $g(lx)$ converges to zero in $BV_\sigma(\mathcal{M}, p, r)$. Now let x be fixed element in $BV_\sigma(\mathcal{M}, p, r)$, there exist $\rho > 0$ such that

$$g(x) = \inf_{n\geq 1}\Big\{\rho^{\frac{p_n}{H}} : \Big(\sum_{m=1}^{\infty}\frac{1}{m_r}\Big[\sup_{k\geq 0} M_k\frac{(|\varphi_{m,n}(x)|)}{\rho}\Big]^{p_m}\Big)^{\frac{1}{H}} \leq 1, \ \text{uniformly in } n\Big\}.$$

Now

$$g(lx) = \inf_{n\geq 1}\Big\{\rho^{\frac{p_n}{H}} : \Big(\sum_{m=1}^{\infty}\frac{1}{m_r}\Big[\sup_{k\geq 0} M_k\frac{(|\varphi_{m,n}(lx)|)}{\rho}\Big]^{p_m}\Big)^{\frac{1}{H}} \leq 1, \text{ uniformly in } n\Big\}$$

$$\longrightarrow 0 \text{ as } l \longrightarrow 0.$$

This completes the proof. □

Theorem 2.3. *Suppose that* $0 < p_m \leq q_m < \infty$, *for each* $m \in \mathbb{N}$ & $r \geq 0$. *Then:*

(i) $BV_\sigma(\mathcal{M}, p) \subseteq BV_\sigma(\mathcal{M}, q)$,

(ii) $BV_\sigma\mathcal{M} \subseteq BV_\sigma(\mathcal{M}, r)$.

Proof. (i) Suppose that $x \in BV_\sigma(\mathcal{M}, p)$. This implies that

$$\Big[\sup_{k\geq 0} M_k\frac{(|\varphi_{i,n}(x)|)}{\rho}\Big]^{p_m} \leq 1,$$

for sufficiently large value of i, say $i \geq m_0$ for some fixed $m_0 \in \mathbb{N}$. Since M_k is non decreasing, we have

$$\sum_{m=m_0}^{\infty}\Big[\sup_{k\geq 0} M_k\frac{(|\varphi_{i,n}(x)|)}{\rho}\Big]^{q_m} \leq \sum_{m=m_0}^{\infty}\Big[\sup_{k\geq 0} M_k\frac{(|\varphi_{i,n}(x)|)}{\rho}\Big]^{p_m} < \infty.$$

Hence $x \in BV_\sigma(\mathcal{M}, q)$.

(ii) Suppose that $x \in BV_\sigma(\mathcal{M})$. This implies that $\Big[\sup_{k\geq 0} M_k\frac{(|\varphi_{i,n}(x)|)}{\rho}\Big] \leq 1$, for sufficiently large value of i, say $i = m_0$ for fixed $m_0 \in \mathbb{N}$. Since M_k is non decreasing, we have

$$\sum_{m=m_0}^{\infty}\frac{1}{m^r}\Big[\sup_{k\geq 0} M_k\frac{(|\varphi_{i,n}(x)|)}{\rho}\Big] \leq \sum_{m=m_0}^{\infty}\Big[\sup_{k\geq 0} M_k\frac{(|\varphi_{i,n}(x)|)}{\rho}\Big] < \infty.$$

Hence $x \in BV_\sigma(\mathcal{M}, r)$. □

Corollary 2.4. *If* $0 < p_m \leq 1$ *for each* m, *then* $BV_\sigma(\mathcal{M}, p) \subseteq BV_\sigma(\mathcal{M})$. *If* $p_m \geq 1$ *for all* m, *then* $BV_\sigma(\mathcal{M}) \subseteq BV_\sigma(\mathcal{M}, p)$.

Proof. It follows from the above theorem. □

Theorem 2.5. *The sequence space* $BV_\sigma(\mathcal{M}, p, r)$ *is solid.*

Proof. Let $x \in BV_\sigma(\mathcal{M}, p, r)$. This implies that

$$\sum_{m=1}^{\infty} m^{-r}\Big[\sup_{k\geq 0} M_k\Big(\frac{|\varphi_{k,n}(x)|}{\rho}\Big)\Big]^{p_m} < \infty.$$

Let (α_m) be the sequence of scalars such that $|\alpha_m| \le 1$ for all $m \in \mathbb{N}$. Then the result follows from the following inequality

$$\sum_{m=1}^{\infty} m^{-r}\Big[\sup_{k\ge 0} M_k\Big(\frac{|\alpha_m \varphi_{k,n}(x)|}{\rho}\Big)\Big]^{p_m} \le \sum_{m=1}^{\infty} m^{-r}\Big[\sup_{k\ge 0} M_k\Big(\frac{|\varphi_{k,n}(x)|}{\rho}\Big)\Big]^{p_m} < \infty.$$

Hence $\alpha x \in BV_\sigma(\mathcal{M}, p, r)$ for all sequences of scalars (α_m) with $|\alpha_m| \le 1$ for all $m \in \mathbb{N}$ whenever $x \in BV_\sigma(\mathcal{M}, p, r)$. □

Corollary 2.6. *The sequence space $BV_\sigma(\mathcal{M}, p, r)$ is monotone.*

Proof. It follows from the above theorem and following lemma. □

Lemma. *A sequence space E is solid implies E is monotone.*

Theorem 2.7. *Let $\mathcal{M}' = (M_k')$, $\mathcal{M}'' = (M_k'')$ be Musielak-Orlicz functions satisfying Δ_2-conditions and $r, r_1, r_2 \ge 0$. Then we have:*
(i) If $r > 1$ then $BV_\sigma(\mathcal{M}, p, r) \subseteq BV_\sigma(\mathcal{M} \circ \mathcal{M}', p, r)$.
(ii)$BV_\sigma(\mathcal{M}', p, r) \bigcap BV_\sigma(\mathcal{M}'', p, r) \subseteq BV_\sigma(\mathcal{M}' + \mathcal{M}'', p, r)$.
(iii) If $r_1 \le r_2$ then $BV_\sigma(\mathcal{M}, p, r_1) \subseteq BV_\sigma(\mathcal{M}, p, r_2)$.

Proof. (i) Since $\mathcal{M} = (M_k)$ is continuous at O from right, for $\epsilon > 0$ there exists $0 < \delta < 1$ such that $0 \le C \le \delta$ implies $M_k(C) < \epsilon$. If we define

$$I_1 = \Big\{m \in \mathbb{N} : \sup_{k\ge 0} M_k\Big(\frac{|\varphi_{m,n}(x)|}{\rho}\Big) \le \delta, \ \text{ for some } \ \rho > 0\Big\},$$

$$I_2 = \Big\{m \in \mathbb{N} : \sup_{k\ge 0} M_k\Big(\frac{|\varphi_{m,n}(x)|}{\rho}\Big) > \delta, \ \text{ for some } \ \rho > 0\Big\},$$

when $\sup_{k\ge 0} M_k\Big(\frac{|\varphi_{m,n}(x)|}{\rho}\Big) > \delta$, we get

$$\sup_{k\ge 0} M_k\Big(\sup_{k\ge 0} M_k'\Big(\frac{|\varphi_{m,n}(x)|}{\rho}\Big)\Big) \le \Big\{2\sup_{k\ge 0} M_k(1)/\delta\Big\} \sup_{k\ge 0} M_k'\Big(\frac{|\varphi_{m,n}(x)|}{\rho}\Big).$$

Hence for $x \in BV_\sigma(\mathcal{M}', p, r)$ and $r > 1$, then

$$\sum_{m=1}^{\infty} m^{-r}\Big[\sup_{k\ge 0}(M_k \circ M_k')\Big(\frac{|\varphi_{m,n}(x)|}{\rho}\Big)\Big]^{p_m}$$
$$= \sum_{m=I_1}^{\infty} m^{-r}\Big[\sup_{k\ge 0}(M_k \circ M_k')\Big(\frac{|\varphi_{m,n}(x)|}{\rho}\Big)\Big]^{p_m}$$

$$+\sum_{m=I_1}^{\infty} m^{-r}\Big[\sup_{k\geq 0}(M_k \circ M_k')\Big(\frac{|\varphi_{m,n}(x)|}{\rho}\Big)\Big]^{p_m}$$

$$\leq \sum_{m\in I_1} m^{-r}[\epsilon]^{p_m} + \sum_{m\in I_2} m^{-r}\Big[\Big\{\sup_{k\geq 0} 2M_k(1)/\delta\Big\}\sup_{k\geq 0} M_k'\Big(\frac{|\varphi_{m,n}(x)|}{\rho}\Big)\Big]^{p_m}$$

$$\leq \max(\epsilon^h, \epsilon^H)\sum_{m=1}^{\infty} m^{-r} + \max\Big(\Big\{\frac{2M_k(1)}{\delta}\Big\}^h, \Big\{\frac{2M_k(1)}{\delta}\Big\}^H\Big),$$

where $0 < h = \inf p_m \leq p_m \leq H = \sup_n p_m < \infty$.

(ii) The proof follows from the following inequality

$$m^{-r}\Big[\sup_{k\geq 0}(M_k' + M_k'')\Big(\frac{|\varphi_{m,n}(x)|}{\rho}\Big)\Big]^{p_m}$$
$$\leq Cm^{-r}\Big[M_k'\Big(\frac{|\varphi_{m,n}(x)|}{\rho}\Big)\Big]^{p_m} + Cm^{-r}\Big[M_k''\Big(\frac{|\varphi_{m,n}(x)|}{\rho}\Big)\Big]^{p_m}.$$

(iii) The proof is straight forward. □

Corollary 2.8. *Let $\mathcal{M} = (M_k)$ be an Musielak-Orlicz function satisfying Δ_2-condition. Then we have:*

(i) If $r > 1$, then $BV_\sigma(p, r) \subset BV_\sigma(\mathcal{M}, p, r)$.

(ii) $BV_\sigma(\mathcal{M}, p) \subseteq BV_\sigma(\mathcal{M}, p, r)$.

(iii) $BV_\sigma(p) \subseteq BV_\sigma(p, r)$.

(iv) $BV_\sigma(\mathcal{M}) \subseteq BV_\sigma(\mathcal{M}, r)$.

Proof. The proof follows from the above theorem. □

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