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PRETOPOLOGY, MATROÏDES AND HYPERGRAPHS

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Abstract: In this paper, we present the theoretical links between pretopology and matroïdes on one hand and pretopology and hypergraphs on the other hand. We show how the pretopology recovers these two domains.

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1. Introduction

We have already shown in this reviews [5] and [6], that the pretopology recovers at the same time the graphs and the topology and, that consequently, it widens their field of applications.

In this paper, we wish to go farther still by showing the theoretical equivalences and the possible passages between pretopology and matroïdes on one hand, between pretopology and hypergraphs on the other hand. Indeed, matroïdes and hypergraphs are theoretical frames allowing the resolution of difficult problems even impossible to deal within the framework of the graphs. To show that the pretopology can, it too, to model and to resolve these problems

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is to show all the interest which there is to work in this at once simple only theoretical frame of approach and conciliatory.

2. Different Types of Pretopological Spaces, see [2], [4]

Definition 1. Let X be a non empty set. P(X) denotes the family of subsets of X. We call pseudoclosure on X any mapping a from P(X) onto P(X) such that:

$$a\ (\phi) = \phi,$$
$$\forall\ \mathrm{A} \subset \mathrm{X},\ \mathrm{A} \subset a\ (\mathrm{A}).$$

(X, a) is then called pretopological space.

We can define 4 different types of pretopological spaces.

1 – (X, a) is a V type pretopological space if and only if

$$\forall\ \mathrm{A} \subset \mathrm{X}, \forall\ \mathrm{B} \subset \mathrm{X},\ \mathrm{A} \subset \mathrm{B} \Rightarrow\ a\ (\mathrm{A}) \subset a\ (\mathrm{B}).$$

2 – (X, a) is a V_D type pretopological space if and only if

$$\forall\ \mathrm{A} \subset \mathrm{X}, \forall\ \mathrm{B} \subset \mathrm{X},\ a\ (\mathrm{A} \cup \mathrm{B}) =\ a\ (\mathrm{A}) \cup a\ (\mathrm{B}).$$

3 – (X, a) is a V_S type pretopological space if and only if

$$\forall\ \mathrm{A} \subset \mathrm{X},\ a(A) = \bigcup_{x \in A} a(\{x\}).$$

4 – (X, a) a V_D type pretopological space, is a topological space if and only if

$$\forall\ \mathrm{A} \subset \mathrm{X},\ a\ (a\ (\mathrm{A})) = a\ (\mathrm{A}).$$

Remark. That last definition is equivalent to the definition of Kuratowski [9]. It shows that any topological space is a particular pretopological space.

Property 1. If (X, a) is a V_S space then (X, a) is a V_D space. If (X, a) is a V_D space then (X, a) is a V space.

Examples. Let X be a non empty set and R be a binary relationship defined on X. The pretopology of descendants, noted a_d, is defined by the following pseudoclosure:

$\forall\ \mathrm{A} \subset \mathrm{X},\ a_d(\mathrm{A}) = \{\ \mathrm{x} \in \mathrm{X} /\ R(\mathrm{x}) \cap \mathrm{A} \neq \phi\ \} \cup \mathrm{A}$ with $R(\mathrm{x}) = \{\ \mathrm{y} \in \mathrm{X} /\ \mathrm{x}\ R\ \mathrm{y}\ \}$.

The pretopology of ascendants, noted a_a, is defined by:

$$\forall\ \mathrm{A} \subset \mathrm{X},\ a_a(\mathrm{A}) = \{\ \mathrm{x} \in \mathrm{X} /\ R^{-1}(\mathrm{x}) \cap \mathrm{A} \neq \phi\ \} \cup \mathrm{A}$$

with $R^{-1}(\mathrm{x}) = \{\ \mathrm{y} \in \mathrm{X} /\ \mathrm{y}\ R\ \mathrm{x}\}$.

The pretopology of ascendant-descendants, noted a_{ad}, is defined by:

$$\forall\ \mathrm{A} \subset \mathrm{X},\ a_{ad}(\mathrm{A}) = \{\ \mathrm{x} \in \mathrm{X} /\ R^{-1}(\mathrm{x}) \cap \mathrm{A} \neq \phi \text{ and } R(\mathrm{x}) \cap \mathrm{A} \neq \phi\ \} \cup \mathrm{A}.$$

The pretopology of descendants and the pretopology of ascendants are V_S ones. The pretopology of ascendant-descendants is only V one.

Remark. Pretopology of descendants enables us to model all applications relevant of the graph theory. In particular, the concepts of path, chain, connectivity are extended in pretopology while remaining compatible with the graph theory, see [2].

Definition 2. (see [1]) Let X be a non empty set. We call weak pretopology on X any mapping a from P(X) onto P(X) such that:

$$\forall \text{ A} \subset \text{X}, \quad \text{A} \subset a \text{ (A)}.$$

(X, a) is called weak pretopological space.

(X, a), weak pretopological space, is a V type space if and only if

$$\forall \text{ A} \subset \text{X}, \forall \text{ B} \subset \text{X}, \text{A} \subset \text{B} \Rightarrow a \text{ (A)} \subset a \text{ (B)}.$$

3. Closures and Connectivities in a Pretopological Space, see [2], [4]

Definition 3. Let (X, a) be a V pretopological space. Let A $\subset$ X. A is a closed subset if and only if a (A) = A.

We note $\forall$ A$\subset$X, a^0 (A) = A and $\forall$ n, n$\geq$ 1, a^n(A) = $a(a)^{n-1}$(A).

We name closure of A the subset of X, denoted F(A), which is the smallest closed subset which contains A.

F', the inverse of the closure generated by a, is defined by:

$$\forall \text{A} \subset \text{X}, \text{F'(A)} = \{ \text{ y} \in \text{X/ F(\{y\})} \cap \text{A} \neq \phi \ \}.$$

We note a" = F'F (a" is the composed of the mapping F' and F) and F" the closure according to a".

Let A be a non empty subset of X. Let B be a non empty subset of X. There exists a path in (X, a) from B to A if and only if B $\subset$F(A). There exists a chain in (X, a) from B to A if and only if B $\subset$F"(A).

Remark. F(A) is the intersect of all closed subsets which contain A. In the case where (X, a) is a "general" pretopological space (i.e. is not a V space, nor a V_D space, nor a V_S space, nor a topological space), the closure may not exist.

Proposition 1. *Let (X, a) be a V space. Let A $\subset$ X. If one of the two following conditions is fulfilled:*

— *X is a finite space;*

— *a is a V_S space;*

then F(A) = $\bigcup_{n\geq 0} a^n(A)$.

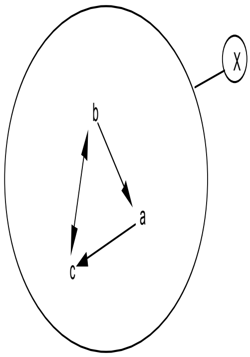

Figure 1

Remark. If a is of V type then a^n, F, a", F" also are of V type and F' is of V_S type. If a is of V_S type then a^n, F, a", F", F' are also V_S type.

Property 2. *Let (X, a) be a V pretopological space. Let x $\in$ X and let y $\in$ X.*

i) If there exists a sequence x_0... x_n of elements of X such as x_0 = x, x_n= y with $\forall$j = 0,..., n-1, $x_{j+1}\in$ a ({ x_j}) or x_j $\in$ a ({ x_{j+1}}) then there exists a chain in (X, a) from {y} to { x }.

ii) If there exists a sequence x_0...x_n of elements of X such as x_0 = x, x_n= y with $\forall$j = 0,..., n-1, $x_{j+1}\in$ a ({ x_j}) then it exists a path in (X, a) from { y } to { x }.

Proof. i) If there exists a sequence x_0...x_n of elements of X such as x_0 = x, x_n= y with $\forall$j = 0,...,n-1, $x_{j+1}\in$ a ({ x_j}) or x_j $\in$ a ({ x_{j+1}}) then $x_n\in$ a ({ x_{n-1}}) or x_{n-1} $\in$ a ({ x_n}) and then $x_{n-1}\in$ a ({ x_{n-2}}) or x_{n-2} $\in$ a ({ x_{n-1}})... At the end $x_1\in$ a ({x_0}) or x_0 $\in$ a ({ x_1}) so $x_n\in$ (a")^n({ x_0}) $\subset$ F"({ x_0}) and then y $\in$ F"({ x}).

ii) If there exists a sequence x_0...x_n of elements of X such as x_0 = x, x_n= y with $\forall$j = 0,..., n-1, $x_{j+1}\in$ a ({ x_j}) then $x_n\in$ a ({ x_{n-1}}) and then $x_{n-1}\in$ a ({x_{n-2}})... At the end $x_1\in$ a ({ x_0}) so $x_n\in$ a^n({ x_0}) and then y$\in$ a^n({ x}) $\subset$ F({x}).

In general, the converse is not true.

Example. Let (X, a) a pretopological space with X = { a, b, c } and a the pretopology of ascendant-descendants defined on the graph given on Figure 1.

i) It exists a chain in (X, a) from {a} to { c } because a $\in$ F"({ c}) but there is not a sequence such as x_0 = c, x_n= a with $\forall$j = 0,..., n-1, $x_{j+1}\in$ a ({ x_j}) or x_j $\in$ a ({ x_{j+1}}).

ii) It exists a path in (X, a) from { a } to { c } because a ∈ F({ c}) but there is not a sequence such as x_0 = c, x_n= a with ∀j = 0,..., n-1, $x_{j+1} \in a$ ({ x_j}).

Property 3. *Let (X, a) be a V_S pretopological space. Let x ∈ X and let y ∈ X.*

i) There exists a chain in (X, a) from {y} to { x } ⇔ it exists a sequence $x_0 ... x_n$ of elements of X such as x_0 = x, x_n= y with ∀j = 0,..., n-1, $x_{j+1} \in a$ ({ x_j}) or $x_j \in a$ ({ x_{j+1}}).

ii) There exists a path in (X, a) from { y } to { x } ⇔ it exists a sequence $x_0 ... x_n$ of elements of X such as x_0 = x, x_n= y with ∀j = 0,..., n-1, $x_{j+1} \in a$ ({ x_j}).

Proof. i) If y ∈ F"({ x}) then y $\in \bigcup_{p \geq 0} (a")^p(\{ x\})$ because F"({ x}) = $\bigcup_{p \geq 0} (a")^p(\{ x\})$ (Proposition 1) and then, it exists p such as y ∈ $(a")^p(\{ x\})$ and then it exists a sequence because a", F and F' are V_S pretopological spaces.

Conversely, according to the Property 2.

ii) If y ∈ F({ x}) then y $\in \bigcup_{p \geq 0} a^p(\{ x\})$ because F({ x}) = $\bigcup_{p \geq 0} a^p(\{ x\})$ (Proposition 1) and then, it exists p such as y ∈ $a^p(\{ x\})$ and then it exists a sequence because ∀A ⊂ X, a (A) =$\bigcup_{x \in A} a(\{ x\})$ (a is a V_S pretopological space).

Conversely, according to the Property 2.

Definition 4. Let (X, a) be a V pretopological space. (X, a) is strongly connected if and only if ∀ C ⊂X, C ≠ ϕ, F(C) = X.

This definition is equivalent to the following one: (X, a) is strongly connected if and only if ∀A ⊂ X, A ≠ ϕ,∀B ⊂ X, B ≠ ϕ, it exists a path in (X, a) from B to A.

(X, a) is connected if and only if ∀C ⊂X, C ≠ ϕ, F(C) = X or F(X-F(C)) ∩ F(C) ≠ ϕ.

Proposition 2. *Let (X, a) be a V pretopological space.*

i) If ∀ x ∈ X and ∀ y ∈ X, it exists a chain in (X, a) from {y} to { x } then (X, a) is connected.

ii) (X, a) is strongly connected ⇔ ∀ x ∈ X and ∀ y ∈ X, it exists a path in (X, a) from { y } to { x }.

Proof. i) If ∀ x ∈ X and ∀ y ∈ X, it exists a chain in (X, a) from {y} to { x } then ∀C ⊂ X, C non empty, we have:

- F(C) = X. In this case, (X, a) is connected (definition 4)

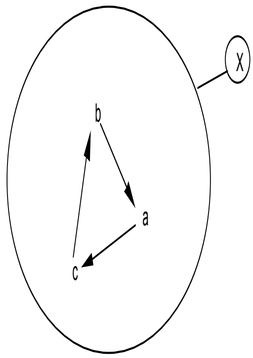

Figure 2

- or F(C) $\subset$ X and F(C) $\neq$ X

F(C) is a closed subset for F, but it is not a closed subset for F'.

Indeed, if F(C) was a closed subset for F', we should have $\forall$ x $\in$ C, F"({x })$\subset$ F'F(C) = F(C) = F"(C). And then, if y $\in$ X-F(C), y does not belong to F(C) and then y does not belong to F"(C) and it doesn't exist a chain from { y } to { x } in (X, a) and then a contradiction. So F(C) is not a closed subset for F'. It exists then z $\in$ X-F(C), z $\in$ F'F(C). We have then F({z }) $\cap$ F(C) $\neq$ ϕ (definition of F'F(C)) but F({z }) $\subset$ F(X-F(C)) then F(X-F(C)) $\cap$ F(C) $\neq$ ϕ.

ii) If (X, a) strongly connected then $\forall$ x $\in$ X and $\forall$ y $\in$ X, it exists a path in (X, a) from { y } to { x } (definition 4).

Conversely, if $\forall$ x $\in$ X and $\forall$ y $\in$ X, it exists a path in (X, a) from { y } to { x } then $\forall$ A $\subset$ X, A $\neq \phi$, $\forall$ B $\subset$ X, B $\neq \phi$, we have $\forall$ y $\in$ B, it exists x $\in$ A such as it exists a path from { y } to { x} in (X, a) then $\forall$ A $\subset$ X, A $\neq \phi$, $\forall$ B $\subset$ X, B $\neq \phi$, $\forall$ y $\in$ B, it exists x $\in$ A, y $\in$ F({ x}) so B $\subset \bigcup_{x \in A} F(\{ x\}) \subset$ F(A) and then it exists a path from B to A in (X, a) and the result according to the definition 4.

In general, the converse of i is not true.

Example. Let (X, a) a pretopological space with X = { a, b, c } and a the pretopology of ascendant-descendants defined on the graph given on Figure 2.

(X, a) is connected but, for example, it doesn't exist a chain between { c } and { a }. Indeed, F({ a }) = { a } and F'F({ a }) = { a } then F" ({ a}) = { a } and then c doesn't belong to F"({a}).

Proposition 3. *Let (X, a) be a* V_S *pretopological space.*

(X, a) is connected ⇔ ∀ x ∈ X *and* ∀ y ∈ X, *it exists a chain from* { y } *to* { x } *in (X, a).*

Proof. (X, a) is connected ⇔ ∀C ⊂ X, C ≠ ϕ, F(C) = X or F(X-F(C)) ∩ F(C) ≠ ϕ. Let x ∈ X, let us show that F(X-F(F" ({ x}))) ∩ F(F"({ x})) = ϕ:

F"({ x}) is a closed subset for F then F(F"({ x})) = F"({ x}). On the other hand, F(X-F"({ x})) =$\bigcup_{y\in X-F"(\{y\})} F(\{y\})$ but y doesn't belong to F"({ x}) then F({ y}) ∩ F"({ x}) = ϕ and then $\bigcup_{y\in X-F"(\{y\})} F(\{y\})$∩ F"({ x}) = ϕ and F(X-F(F"({ x}))) ∩ F(F"({ x})) = ϕ. So we have ∀ x ∈ X, F(F"({ x})) = X according to the connectivity then ∀ x ∈ X, ∀ y ∈ X, y ∈ F(F"({ x})) = F"({ x}) and then ∀ x ∈ X, ∀ y ∈ X, it exists a chain from { y } to { x } in (X, a).

Conversely, according to the Proposition 2.

Proposition 4. *Let (X, a) be a V_S pretopological space.*

i) (X, a) is connected ⇔ ∀ x ∈ X *and* ∀ y ∈ X, *it exists a sequence* $x_0 ... x_n$ *of elements of X such as* x_0 = x, x_n= y *with* ∀j = 0,..., n-1, x_{j+1}∈ a a ({ x_j}) *or* x_j∈ a ({ x_{j+1}}).

ii) (X, a) strongly connected ⇔ ∀ x ∈ X *and* ∀ y ∈ X, *it exists a sequence* $x_0 ... x_n$ *of elements of X such as* x_0= x, x_n= y *with* ∀j = 0,..., n-1, x_{j+1}∈ a ({ x_j}).

Proof. i) (X, a) is connected
⇔ ∀ x ∈ X and ∀ y ∈ X, it exists a chain from { y } to { x } in (X, a) (Proposition 3)
⇔∀ x ∈ X and ∀ y ∈X, it exists a sequence $x_0 ... x_n$ of elements of X such as x_0 = x, x_n= y with ∀j = 0,..., n-1, x_{j+1}∈ a ({ x_j}) or x_j∈ a ({ x_{j+1}}) (Property 3).

ii) (X, a) is strongly connected
⇔ ∀ x ∈ X and ∀ y ∈ X, it exists a path in (X, a) from { y } to { x } (Proposition 2)
⇔ ∀ x ∈ X and ∀ y ∈ X, it exists a sequence $x_0 ... x_n$ of elements of X such as x_0 = x, x_n= y with ∀j = 0,..., n-1, x_{j+1}∈ a ({ x_j}) (Property 3).

Proposition 5. *Let (X, a) be a V_S pretopological space where a is a symmetric pseudoclosure (a is a symmetric pseudoclosure if and only if* ∀ x∈ X, ∀ y∈ X, y∈ a ({x }) ⇔ x∈ a ({y })).

(X, a) is strongly connected ⇔ *(X, a) is connected.*

Proof. (X, a) is strongly connected

⇔ ∀ x ∈ X and ∀ y ∈ X, it exists a sequence $x_0 ... x_n$ of elements of X such as x_0 = x, x_n= y with ∀j = 0,..., n-1, $x_{j+1} \in a$ ({ x_j}) (proposition 4)
⇔ ∀ x ∈ X and ∀ y ∈ X, it exists a sequence $x_0 ... x_n$ of elements of X such as x_0 = x, x_n= y with ∀j = 0,..., n-1, $x_{j+1} \in a$ ({ x_j}) or $x_j \in a$ ({ x_{j+1}}) (according to the symmetry)
⇔ (X, a) is connected (proposition 4).

4. Pretopology and Matroïdes, see [3], [7], [8], [10]

Definition 5. Let X = {x_1, x_2,..., x_n} be a finite set which contains n elements. We denote matroïde on X a couple M = [X, $\mathcal{G}$] where $\mathcal{G}$ is a family of subsets of X which verify the following axioms:

1- $\phi \in \mathcal{G}$
And if F ∈ $\mathcal{G}$ with F' ⊂ F then F' ∈ $\mathcal{G}$.

2- ∀ A⊂ X, if F ∈ $\mathcal{G}$ and F' ∈ $\mathcal{G}$ are two maximal subsets of A then card(F) = card(F').

We remind that a subset S is maximal (for the inclusion) with regard to a property $\mathcal{P}$ if and only if there is no subset S' such as S' contains strictly S and S' possess the property $\mathcal{P}$.

This definition can spell also from a mapping called closure of dependence [7]. The definition is then the following one:

Definition 6. A closure of dependence on a set X is a mapping, denoted FER, from P(X) onto P(X) which verify the following axioms:
(D1) ∀ A ⊂ X, A ⊂ FER(A),
(D2) ∀ A ⊂ X, ∀ B ⊂ X, A ⊂ B ⇒ FER(A) ⊂ FER(B),
(D3) ∀ A ⊂ X, FER(FER(A)) = FER(A),
(D4) ∀ A ⊂ X, ∀ x ∈ X, ∀ y ∈ X, if y ∈ FER(A ∪ {x}) and y ∉ FER(A) then x ∈ FER(A ∪ {y}),
(D5) ∀ A ⊂ X, ∀ x ∈ FER(A), it exists a finite subset Z of A such as x ∈ FER(Z).

The couple constituted by a set X and a closure of dependence defines a matroïde on X.

Remark. The axiom (D5) is useless in case X is a finite set.

Definition 7. A matroïde on a set X defined by a closure of dependence FER is a geometrical matroïde if FER(ϕ) = ϕ and ∀ x ∈X, FER({x}) = {x}.

Remark. It seems very sharply that a closure of dependence (Definition 6) corresponds, in an axiom near, to the definition of a pretopology (axiom (D1)) of Vtype (axiom (D2)) idempotente (axiom (D3)) verifying in more axioms (D4) and (D5). The missing axiom concerns the definition of a pretopology which supposes however that the pseudoclosure of the empty set was empty. Of this remark, we deduct the following Proposition:

Proposition 6. i) (X,FER) is a matroïde on X verifying (D0) FER(ϕ) = ϕ
$\Leftrightarrow$ (X,FER) is a V pretopological space with FER idempotente, symmetric verifying the axioms (D4) and (D5).

ii) (X,FER) is a matroïde on X which not verify (D0) FER(ϕ) = ϕ
$\Leftrightarrow$ (X,FER) is a weak pretopological space of V type with FER idempotente verifying the axioms (D4) and (D5).

Proof. i) (X,FER) is a matroïde on X verifying (D0) FER(ϕ) = ϕ
$\Leftrightarrow$ (X,FER) is a V pretopological space with FER idempotente verifying the axioms (D4) and (D5) (definition).

We remind that FER is symmetric if and only if $\forall$ x$\in$ X, $\forall$ y$\in$ X,
y$\in$ FER({x }) $\Leftrightarrow$ x$\in$ FER({y }).

Let us show that (D4) implies that FER is symmetric:
(D4) $\forall$ A $\subset$ X, $\forall$ x $\in$ X, $\forall$ y $\in$ X, if y $\in$ FER(A $\cup$ {x}) and y $\notin$ FER(A) then x $\in$ FER(A $\cup$ {y})

If A = ϕ, we have $\forall$ x $\in$ X, $\forall$ y $\in$ X, if y $\in$ FER({x}) and y $\notin$ FER(ϕ) then x $\in$ FER({y})then $\forall$ x $\in$ X, $\forall$ y $\in$ X, if y $\in$ FER({x}) then x $\in$ FER({y}) (axiom (D0))

ii) According to the definition.

Remark. (X,FER) is a geometrical matroïde on X
$\Leftrightarrow$ (X,FER) is a V pretopological space with FER idempotente, symmetric verifying the axioms (D4) and (D5) and $\forall$ x $\in$ X, FER({x}) = {x}.

Consequence. *In every case, a matroïde can be likened to a particular pretopological space.*

5. Pretopology and Hypergraphs, see [3]

Definition 8. Let X = $\{x_1, x_2,..., x_n\}$ be a finite set which contains n elements. A hypergraph on X is a family H = $\{E_1, E_2,..., E_m\}$ of subsets of X verifying the following axioms:

1- $E_i \neq \phi$ with 1 $\leq$i $\leq$m

2- $\bigcup_{i=1}^{i=m} E_i$= X.

A hypergraph is connected if the graph of intersection of the edges (an element of H is called an edge) is connected.

This definition can spell in the following way:

Let H be a hypergraph on a set X. Let R be a binary relationship defined on H by: $\forall$ i $\in$ { 1,.., m }, $\forall$ j $\in$ { 1,.., m }, $E_i \; R \; E_j \Leftrightarrow E_i \cap E_j \neq \phi$.

Let G = (H,U) be the graph associated to R with U = { (E_i, E_j) $\in$ H X H/ $E_i \; R \; E_j$ }.

H is connected $\Leftrightarrow$ G = (H,U)is connected (according to the definition)

$\Leftrightarrow \forall$ (E, E') $\in$ H X H, E $\neq$ E', it exists a chain between E and E'

$\Leftrightarrow \forall$ (E, E') $\in$ H X H, E $\neq$ E', it exists a sequence E_1, E_2,..., E_p such as E = E_1, E' = E_p and such as $\forall$ i = 1,.., p-1, $E_i \cap E_{i+1} \neq \phi$.

Remarks. The pretopology is not reduced to the study of finite sets. It concerns also infinite sets. On the contrary, the hypergraphs suppose the knowledge of a finite set X.

In a hypergraph, we can always associate a pretopological space; this last one can be built by various ways. We propose in the following part an example of construction of a particular pretopological space in a hypergraph. Then, we show an equivalence concerning the connectivity.

Example of construction of a particular pretopological space.

Proposition 7. *Let a be a mapping from P(X) into P(X) defined by:* $\forall A \subset X$, a (A) $= \bigcup_{E_i \in I_A} E_i$*with* I_A = { E_i∈H / $E_i \cap$ A $\neq \phi$ }.

i) (X, a) is a V_S *pretopological space.*

ii) a is a symmetric pseudoclosure.

Proof. i) a is a pseudoclosure. Indeed, a (ϕ) $= \bigcup_{E_i \in I\phi} E_i$with $I_\phi = \phi$ then a (ϕ) = ϕ.

In other hand, $\forall$ x$\in$ A, it exists E_i∈H such as x $\in E_i$ (according the definition) then $\forall$ x$\in$ A, it exists E_i∈H such as x $\in E_i$ with $E_i \cap$ A $\neq \phi$ then $\forall$ x$\in$ A, x $\in a$ (A) and then A $\subset a$ (A).

a is V_S type. Indeed, a (A) $= \bigcup_{E_i \in I_A} E_i$ with I_A = { E_i∈H / $E_i \cap$ A $\neq \phi$ } but I_A = { E_i∈H / $E_i \cap$ A $\neq \phi$ } $= \bigcup_{x \in A}$ { $\mathrm{E_i} \in$ H / $\mathrm{E_i} \bigcap \{\mathrm{x}\} \neq \phi$ } $= \bigcup_{x \in A} I_{\{\mathrm{x}\}}$ then

a (A) $= \bigcup_{E_i \in I_A} E_i$with

$$I_A = \bigcup_{x \in A} I_{\{\mathrm{x}\}}$$

then a (A) $=\bigcup_{x \in A}(\bigcup_{E_i \in I_{\{\mathrm{x}\}}} E_i) = \bigcup_{x \in A} a(\{x\})$.

ii) a is a symmetric pseudoclosure if and only if $\forall$ x$\in$X, $\forall$ y$\in$X, y$\in$ a ({x }) $\Leftrightarrow$ x$\in$ a ({y }).

y$\in$ a ({x }) $\Leftrightarrow$ y$\in \bigcup_{E_i \in I_{\{x\}}} E_i$ with $I_{\{x\}} = \{ E_i \in H \, / \, E_i \cap \{x\} \neq \phi \}$

$\Leftrightarrow$ y$\in \bigcup_{E_i \in I_{\{x\}}} E_i$with $I_{\{x\}} = \{ E_i \in H \, / \, x \in E_i \}$

$\Leftrightarrow$ it exists $E_i \in$ H such as y$\in$ E_i and x $\in$ E_i.

In the same way, we have:

x$\in$ a ({y }) $\Leftrightarrow$ it exists $E_i \in$ H such as x$\in E_i$ and y$\in$ E_i

$\Leftrightarrow$ it exists $E_i \in$ H such as y$\in$ E_i and x $\in$ E_i $\Leftrightarrow$ y$\in$ a ({x }).

Consequence. *It is always possible to associate a pretopological space to a hypergraph.*

Proposition 8. *Let H be a hypergraph on a set X. Let R be a binary relationship defined in H by:* $\forall$ *i* $\in$ { *1,.., m* }, $\forall$ *j* $\in$ { *1,.., m* },

$$E_i R E_j \Leftrightarrow E_i \cap E_j \neq \phi.$$

Let G = (H,U) be graph associated to R with U = { *(*E_i*,* E_j*)* $\in$ *H X H /* E_i *R* E_j }.

We associate to R the pretopology of ascendants defined by:

$$\forall A \subset H, \ a(A) = \{ E \in H / \ R(E) \cap A \neq \phi \} \cup A$$

with R(E) = { *E'* $\in$ *H/ E R E'* }.

We have:

i) (H, a) is a V_S *pretopological space.*

ii) a is a symmetric pseudoclosure.

iii) H is a connected hypergraph

$\Leftrightarrow$ *G = (H,U) is a connected graph* $\Leftrightarrow$ *(H, a) is a connected pretopological space*

$\Leftrightarrow$ *(H, a) is a strongly connected pretopological space.*

Proof. i) See [2].

ii) a is a symmetric pseudoclosure if and only if $\forall$ F$\in$ H, $\forall$ F'$\in$ H, F$\in$ a ({F' }) $\Leftrightarrow$ F'$\in$ a ({F })

F'$\in$ a ({F }) $\Leftrightarrow$ F' $\in$ ({ E $\in$ H/ R (E) $\cap$ {F} $\neq \phi$ } $\cup$ {F})

$\Leftrightarrow$ F' = F or F $\in$ R(F') $\Leftrightarrow$ F' = F or F' R F $\Leftrightarrow$ F' = F or F' $\cap$ F $\neq \phi$

$\Leftrightarrow$ F' $\cap$ F $\neq \phi$.

In the same way, we have:

F$\in$ a ({F' }) $\Leftrightarrow$ F $\cap$ F' $\neq \phi$ $\Leftrightarrow$ F' $\cap$ F $\neq \phi$ $\Leftrightarrow$ F'$\in$ a ({F }).

iii) H is a connected hypergraph $\Leftrightarrow$ G = (H,U) is a connected graph (according to the definition of a connected hypergraph).

G = (H,U) is a connected graph $\Leftrightarrow$ (H, a) is a connected pretopological space (Proposition 4 and definition of a).

(H, a) is a connected pretopological space ⇔ (H, a) is a strongly connected pretopological space (according to Proposition 5).

Consequence. *To study the connectivity of a hypergraph H amounts to study the connectivity (or the strong connectivity) of a* V_S *pretopological space (H, a) where a is symmetric.*

6. Conclusion

The pretopology appears, consequently, as a theoretical frame flexible and adapted to model and deals with problems recovering from domains varied as the theory of the graphs, the topology but also the matroïdes or the hypergraphs. It proposes the same language to generalize and create footbridges between all these domains.

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