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ON NEW CLASSES INVOLVING MEROMORPHIC FUNCTIONS

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Abstract: A certain generalized differential operator $D^k_{\mu,\nu,\lambda}$, $\mu,\nu \in \mathbf{R}$, $\lambda \geq 0$ and $k \in \mathbf{N}_0 = 0, 1, ...$, is introduced. The main object of this paper is to give an application of the above operator $D^k_{\mu,\nu,\lambda}$ to the differential inequalities.

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1. Introduction

Let $\Sigma(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n z^n \quad (p \in \mathbf{N} = \{1, 2, ...\}) \tag{1}$$

which are analytic in the open unit disk $U^* = \{z : z \in \mathbf{C} \text{ and } 0 < |z| < 1\}$. In the present paper we define a generalized differential operator as follows: $D^0 f(z) = f(z)$,

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$$D^1_{\mu,\nu,\lambda}f(z) = \frac{1}{z^p} + \frac{(p+\mu)(1-\lambda)}{(p+\nu)[1-\lambda(p+1)]}a_0 + \frac{(p+2+\mu)(1+\lambda)}{(p+2+\nu)[1-\lambda(p+1)]}a_2z^2 + ...,$$
$$D^2_{\mu,\nu,\lambda}f(z) = D(D^1_{\mu,\nu,\lambda}f(z)),$$
$$\vdots$$

and for $k = 1, 2, ...(p \in \mathbf{N} = \{1, 2, ...\}, \mu, \nu \in \mathbf{R},\ \lambda \geq 0,\ z \in U^*)$

$$D^k_{\mu,\nu,\lambda}f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty}\Big[\frac{(p+n+\mu)(1+(n-1)\lambda)}{(p+n+\nu)(1-(p+1)\lambda)}\Big]^k a_n z^n, \tag{2}$$

Clearly, (2) yields $f \in \Sigma(p) \Rightarrow D^k_{\mu,\nu,\lambda}f(z) \in \Sigma(p)$.

Remark 1.1. Operator (2) involving the differential operator of meromorphic functions is defined and studied by many authors (see [10], [2], [4], [1]).

Given two functions $f, g \in \Sigma(p)$, $f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} b_n z^n$. Their convolution or Hadamard product $f(z) * g(z)$ is defined by $f(z) * g(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n b_n z^n$ $(z \in U^*)$.

The object of the present paper is to investigate some new properties of meromorphic p-valent functions defined by the above operator.

A function $f(z) \in \Sigma(p)$ belongs to the class $\mathcal{S}_p(\rho)$ the class of meromorphically p-valent starlike functions of order ρ where $0 \leq \rho < p$, if and only if $f \neq 0$, and $-\Re\{\frac{zf'(z)}{f(z)}\} > \rho$ $(z \in U)$.

A function $f \in \Sigma(p)$ belongs to the class $\mathcal{C}_p(\rho)$, the class of meromorphically p-valent convex functions of order μ where $0 \leq \rho < p$, if and only if $f' \neq 0$, and $-\Re\{1 + \frac{zf''(z)}{f'(z)}\} > \rho$ $(z \in U)$. Thus for $f \in \Sigma(p)$ it yields

$$f(z) \in \mathcal{C}_p(\rho) \Leftrightarrow -\frac{zf'(z)}{p} \in \mathcal{S}_p(\rho).$$

A function $f \in \Sigma(p)$ belongs to the class $\mathcal{S}_{\mu,\nu,\lambda,p}(\rho)$, the class of meromorphically p-valent starlike functions of order ρ where $0 \leq \rho < p$, if and only if $f \neq 0$, and $-\Re\{\frac{z[D^k_{\mu,\nu,\lambda}f(z)]'}{[D^k_{\mu,\nu,\lambda}f(z)]}\} > \rho$ $(z \in U)$.

A function $f \in \Sigma(p)$ belongs to the class $\mathcal{C}_{\mu,\nu,\lambda,p}(\rho)$, the class of meromorphically p-valent convex functions of order ρ where $0 \leq \rho < p$, if and only if $f' \neq 0$, and $-\Re\{1 + \frac{z[D^k_{\mu,\nu,\lambda}f(z)]''}{[D^k_{\mu,\nu,\lambda}f(z)]'}\} > \rho$ $(z \in U)$. Classes of meromorphic p-valent functions, including the classes the above classes were studied rather extensively by many authors (see [9], [8], [7], [5], [3], [6]).

Now for $0 \leq \alpha < p$ and $\beta \geq 0$, we let $\mathcal{S}_{\mu,\nu,\lambda,p}(\alpha,\beta)$ be the subclass of $\Sigma(p)$ consisting of functions of the form (1) and satisfying the analytic criterion

$$-\Re\Big\{\frac{z[D^k_{\mu,\nu,\lambda}f(z)]'}{[D^k_{\mu,\nu,\lambda}f(z)]}+\alpha\Big\} > \beta\Big|\frac{z[D^k_{\mu,\nu,\lambda}f(z)]'}{[D^k_{\mu,\nu,\lambda}f(z)]}+p\Big|, \tag{3}$$

where $D^k_{\mu,\nu,\lambda}f(z)$ is the operator defined in (2). The main aim of this work is to study the general class $\mathcal{S}_{\mu,\nu,\lambda,p}(\alpha,\beta)$ of functions which belong to the class $\Sigma(p)$ defined in (1).

2. Main Results

In this section, we obtain a sufficient condition for functions f in the class $\mathcal{S}_{\mu,\nu,\lambda,p}(\alpha,\beta)$.

Theorem 2.1. *A sufficient condition for function $f(z)$ of the form (1) to be in $\mathcal{S}_{\mu,\nu,\lambda,p}(\alpha,\beta)$ is that*

$$\sum_{n=0}^{\infty}\frac{[n(1+\beta)+p(2+\beta)-\alpha]|B^k_{n,p}(\mu,\nu,\lambda)||a_n|}{p-\alpha} \leq 1 \ \ (0\leq\alpha<p,\ \beta\geq 0), \tag{4}$$

where

$$B^k_{n,p}(\mu,\nu,\lambda)=\Big[\frac{(p+n+\mu)(1+(n-1)\lambda)}{(p+n+\nu)(1-(p+1)\lambda)}\Big]^k \ \ (\lambda\geq 0).$$

Proof. It suffices to show that

$$\beta\Big|\frac{z[D^k_{\mu,\nu,\lambda}f(z)]'}{[D^k_{\mu,\nu,\lambda}f(z)]}+p\Big|+\Re\Big\{\frac{z[D^k_{\mu,\nu,\lambda}f(z)]'}{[D^k_{\mu,\nu,\lambda}f(z)]}+p\Big\}\leq p-\alpha \ \ (z\in U^*).$$

We receive

$$\beta\Big|\frac{z[D^k_{\mu,\nu,\lambda}f(z)]'}{[D^k_{\mu,\nu,\lambda}f(z)]}+p\Big|+\Re\Big\{\frac{z[D^k_{\mu,\nu,\lambda}f(z)]'}{[D^k_{\mu,\nu,\lambda}f(z)]}+p\Big\}\leq(1+\beta)\Big|\frac{z[D^k_{\mu,\nu,\lambda}f(z)]'}{[D^k_{\mu,\nu,\lambda}f(z)]}+p\Big|$$

$$\leq\frac{(1+\beta)\sum_{n=0}^{\infty}(n+p)|B^k_{n,p}(\mu,\nu,\lambda)||a_n||z|^{n+p}}{1-\sum_{n=0}^{\infty}|B^k_{n,p}(\mu,\nu,\lambda)||a_n||z|^{n+p}}$$

$$\leq \frac{(1+\beta)\sum_{n=0}^{\infty}(n+p)|B_{n,p}^{k}(\mu,\nu,\lambda)||a_n|}{1-\sum_{n=0}^{\infty}|B_{n,p}^{k}(\mu,\nu,\lambda)||a_n|}.$$

This last expression is bounded by $p-\alpha$ if

$$\sum_{n=0}^{\infty}[n(1+\beta)+p(2+\beta)-\alpha]|B_{n,p}^{k}(\mu,\nu,\lambda)||a_n| \leq p-\alpha.$$

The result is attained for a function f given by

$$f(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty}\frac{p-\alpha}{[n(1+\beta)+p(2+\beta)-\alpha]|B_{n,p}^{k}(\mu,\nu,\lambda)|}z^n \qquad (p\in \mathbf{N}=\{1,2,...\}) \quad (5)$$

and this completes the proof of the theorem. □

Corollary 2.1. *Let the assumptions of Theorem 2.1 hold. Then*

$$|a_n| \leq \frac{p-\alpha}{[n(1+\beta)+p(2+\beta)-\alpha]|B_{n,p}^{k}(\mu,\nu,\lambda)|}, \quad n \geq 0.$$

The result is sharp for functions f given by (5).

Corollary 2.2. *Let the assumptions of Theorem 2.1 hold. Then*

$$-\Re\Big\{\frac{z\big(D_{\mu,\nu,\lambda}^{k}f(z)\big)'}{D_{\mu,\nu,\lambda}^{k}f(z)}\Big\} > \alpha \ \ (z\in U^*).$$

Next we introduce some well known results which were studied by different authors.

Corollary 2.3. *Let the assumptions of Theorem 2.1 hold. Then*

$$-\Re\Big\{\frac{z\big(f(z)\big)'}{f(z)}\Big\} > \alpha \ \ (z\in U^*).$$

Corollary 2.4. *Let the assumptions of Theorem 2.1 hold. Then*

$$-\Re\Big\{\frac{z\big(D_{\mu,\mu,\lambda}^{k}f(z)\big)'}{D_{\mu,\mu,\lambda}^{k}f(z)}\Big\} > \alpha \ \ (z\in U^*).$$

Proof. By assuming $\mu = \nu, \beta = 0$. □

Note that $D^k_{\mu,\mu,\lambda} f(z)$ is the differential operator for meromorphic functions introduced by Al-Dihan [1].

Corollary 2.5. *Let the assumptions of Theorem 2.1 hold. Then*

$$-\Re\Big\{\frac{zf'(z)}{f(z)}\Big\} > 0 \ \ (z \in U^*).$$

A distortion property for functions f to be in the class $\mathcal{S}_{\mu,\nu,\lambda,p}(\alpha, \beta)$, is given as follows:

Theorem 2.2. *Let the assumptions of Theorem 2.1 be satisfied. Then for* $0 < |z| = r < 1$, *we have*

$$\frac{1}{r^p} - \frac{p-\alpha}{[p(2+\beta)-\alpha]|B^k_{0,p}(\mu,\nu,\lambda)|} \le |f(z)| \le \frac{1}{r^p} + \frac{p-\alpha}{[p(2+\beta)-\alpha]|B^k_{0,p}(\mu,\nu,\lambda)|}$$

with equality holds for

$$f(z) = \frac{1}{z^p} + \frac{p-\alpha}{[p(2+\beta)-\alpha]|B^k_{0,p}(\mu,\nu,\lambda)|}.$$

And

$$pr^{-p-1} + \frac{(p-\alpha)}{[p(2+\beta)-\alpha]|B^k_{1,p}(\mu,\nu,\lambda)|} r^{p-1} \le |f'(z)|$$
$$\le pr^{-p-1} - \frac{(p-\alpha)}{[p(2+\beta)-\alpha]|B^k_{1,p}(\mu,\nu,\lambda)|} r^{p-1}$$

with equality holds for

$$f(z) = \frac{1}{z^{p-1}} + \frac{p-\alpha}{[p(2+\beta)-\alpha]|B^k_{0,p}(\mu,\nu,\lambda)|} z^{p-1},$$

where

$$B^k_{n,p}(\mu,\nu,\lambda) = \Big[\frac{(p+n+\mu)(1+(n-1)\lambda)}{(p+n+\nu)(1-(p+1)\lambda)}\Big]^k \ \ (\lambda \ge 0).$$

Proof. Since f satisfies the conditions of Theorem 2.1 yields $f \in \mathcal{S}_{\mu,\nu,\lambda,p}(\alpha, \beta)$. Then we receive

$$\sum_{n=0}^{\infty} |a_n| \le \frac{p-\alpha}{[n(1+\beta)+p(2+\beta)-\alpha]|B^k_{n,p}(\mu,\nu,\lambda)|} \ \ (0 \le \alpha < p, \ n \ge 0).$$

Thus for $0 < |z| = r < 1$, we have

$$|f(z)| \le |z|^{-p} + \sum_{n=0}^{\infty} |a_n||z|^n \le r^{-p} + \frac{p-\alpha}{[p(2+\beta)-\alpha]|B_{0,p}^k(\mu,\nu,\lambda)|}.$$

On the other hand we have

$$|f(z)| \ge |z|^{-p} - \sum_{n=0}^{\infty} |a_n||z|^n \ge r^{-p} - \frac{p-\alpha}{[p(2+\beta)-\alpha]|B_{0,p}^k(\mu,\nu,\lambda)|}.$$

Also from Theorem 2.1, we have

$$\sum_{n=0}^{\infty} |a_n| \le \frac{p-\alpha}{[p(2+\beta)-\alpha]|B_{0,p}^k(\mu,\nu,\lambda)|} \quad (0 \le \alpha < p,\ n \ge 0).$$

Hence

$$|f'(z)| \le p|z|^{-p-1} + \sum_{n=1}^{\infty} n|a_n||z|^{n-1} \le pr^{-p-1} + \frac{(p-\alpha)}{[p(2+\beta)-\alpha]|B_{1,p}^k(\mu,\nu,\lambda)|} r^{p-1}.$$

On the other hand we have

$$|f'(z)| \ge |z|^{-p} - \sum_{n=0}^{\infty} |a_n||z|^{n-1} \ge pr^{-p-1} - \frac{(p-\alpha)}{[p(2+\beta)-\alpha]|B_{1,p}^k(\mu,\nu,\lambda)|} r^{p-1}.$$

Hence, this completes the proof of Theorem 2.2. □

Theorem 2.3. *Let the assumptions of Theorem 2.1 be satisfied. Then*

$$r^{-p} - \frac{p-\alpha}{[p(2+\beta)-\alpha]} \le |D_{\mu,\nu,\lambda}^k f(z)| \le r^{-p} + \frac{p-\alpha}{[p(2+\beta)-\alpha]}.$$

Proof. Since f satisfies the conditions of Theorem 2.1 yields $f \in \mathcal{S}_{\mu,\nu,\lambda,p}(\alpha,\beta)$. Then we receive

$$|a_n| \le \frac{p-\alpha}{[n(1+\beta)+p(2+\beta)-\alpha]|B_{n,p}^k(\mu,\nu,\lambda)|} \quad (0 \le \alpha < p,\ n \ge 0).$$

Thus we have

$$\begin{aligned}|D_{\mu,\nu,\lambda}^k f(z)| &\le |z|^{-p} + \sum_{n=0}^{\infty} \Big[\frac{(p+n+\mu)(1+(n-1)\lambda)}{(p+n+\nu)(1-(p+1)\lambda)}\Big]^k |a_n||z|^n \\ &\le r^{-p} + \frac{p-\alpha}{[p(2+\beta)-\alpha]}.\end{aligned}$$

On the other hand we have

$$|D^k_{\mu,\nu,\lambda}f(z)| \geq |z|^{-p} - \sum_{n=0}^{\infty}\Big[\frac{(p+n+\mu)(1+(n-1)\lambda)}{(p+n+\nu)(1-(p+1)\lambda)}\Big]^k |a_n||z|^n$$
$$\leq r^{-p} - \frac{p-\alpha}{[p(2+\beta)-\alpha]}. \quad \square$$

Next the radii of starlikeness and convexity for the class $\mathcal{S}_{\mu,\nu,\lambda,p}(\alpha,\beta)$, is given by the following theorem:

Theorem 2.4. *Let the assumptions of Theorem 2.1 be satisfied. Then f is starlike of order ρ, $0 \leq \rho < p$ in the unit disc $0 < |z| < r_1 < 1$, where r_1 is the largest value*

$$r_1 = \inf_{n\geq 0}\Big(\frac{p-\rho}{[n+2-\rho]H_n}\Big)^{\frac{1}{n+p}}, \quad n \geq 0,$$

where

$$H_n := \frac{p-\alpha}{[n(1+\beta)+p(2+\beta)-\alpha]}.$$

The result is sharp.

Proof. It suffices to show that

$$\Big|\frac{z[D^k_{\mu,\nu,\lambda}f(z)]'}{[D^k_{\mu,\nu,\lambda}f(z)]} + 1\Big| \leq 1-\rho$$

for $|z| < r_1$ we receive

$$\Big|\frac{z[D^k_{\mu,\nu,\lambda}f(z)]'}{[D^k_{\mu,\nu,\lambda}f(z)]} + 1\Big| = \Big|\frac{(1-p)+\sum_{n=0}^{\infty}(n+1)B^k_{n,p}(\mu,\nu,\lambda)a_n z^{n+p}}{1+\sum_{n=0}^{\infty}B^k_{n,p}(\mu,\nu,\lambda)a_n z^{n+p}}\Big|$$

$$\leq \frac{|1-p|+\sum_{n=0}^{\infty}(n+1)\frac{p-\alpha}{[n(1+\beta)+p(2+\beta)-\alpha]}|z|^{n+p}}{1-\sum_{n=0}^{\infty}\frac{p-\alpha}{[n(1+\beta)+p(2+\beta)-\alpha]}|z|^{n+p}}$$
$$\leq 1-\rho$$

when

$$\sum_{n=0}^{\infty}[n+2-\rho]H_n|z|^{n+p} \leq p-\rho, \quad n \geq 0.$$

And it follows that

$$|z| \leq \Big(\frac{p-\rho}{[n+2-\rho]H_n}\Big)^{\frac{1}{n+p}}, \quad n \geq 0.$$

Then

$$r_1 = \inf_{n\geq 0} \Big(\frac{p-\rho}{[n+2-\rho]H_n}\Big)^{\frac{1}{n+p}}, \quad n \geq 0.$$

Theorem 2.5. *Let the assumptions of Theorem 2.1 be satisfied. Then f is convex of order ρ, $0 \leq \rho < p$ in the unit disc $0 < |z| < r_2 < 1$, where r_2 is the largest value*

$$r_2 = \inf_{n\geq 1} \Big(\frac{p(1-\rho)}{n[n+2-\rho]H_n}\Big)^{\frac{1}{n+p-2}}, \quad n \geq 1,$$

where

$$H_n := \frac{p-\alpha}{[n(1+\beta)+p(2+\beta)-\alpha]}.$$

The result is sharp.

Proof. It suffices to show that

$$\Big|\frac{z[D^k_{\mu,\nu,\lambda}f(z)]''}{[D^k_{\mu,\nu,\lambda}f(z)]'} + p + 1\Big| \leq 1-\rho.$$

For $|z| < r_2$ we receive

$$\Big|\frac{z[D^k_{\mu,\nu,\lambda}f(z)]''}{[D^k_{\mu,\nu,\lambda}f(z)]'} + p + 1\Big| = \Big|\frac{\sum_{n=0}^{\infty} n(n+1)B^k_{n,p}(\mu,\nu,\lambda)a_n z^{n+p-2}}{-p+\sum_{n=0}^{\infty} nB^k_{n,p}(\mu,\nu,\lambda)a_n z^{n+p-2}}\Big|$$

$$\leq \frac{\sum_{n=0}^{\infty} n(n+1)\frac{p-\alpha}{[n(1+\beta)+p(2+\beta)-\alpha]}|z|^{n+p-2}}{p-\sum_{n=0}^{\infty}\frac{n(p-\alpha)}{[n(1+\beta)+p(2+\beta)-\alpha]}|z|^{n+p-2}}$$

$$\leq 1-\rho$$

when

$$\sum_{n=0}^{\infty} n[n+2-\rho]H_n|z|^{n+p-2} \leq p(1-\rho), \quad n \geq 0.$$

And it follows that

$$|z| \leq \Big(\frac{p(1-\rho)}{n[n+2-\rho]H_n}\Big)^{\frac{1}{n+p-2}}, \quad n \geq 1.$$

Then

$$r_2 = \inf_{n\geq 1} \Big(\frac{p(1-\rho)}{n[n+2-\rho]H_n}\Big)^{\frac{1}{n+p-2}}, \quad n \geq 1. \qquad \square$$

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