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EFFECT OF HEAT GENERATION ON THE ONSET OF MARANGONI CONVECTION IN SUPERPOSED LAYERS OF FLUID AND SATURATED POROUS MEDIUM

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Abstract: Linear stability analysis is applied to investigate the effect of internal heat generation on Marangoni convection in a two-layer system comprising an incompressible fluid-saturated porous layer over which lies a layer of the same fluid. The lower rigid surface and the upper non-deformable surface are assumed to be perfectly insulating. The critical eigenvalues are solved exactly and the asymptotic solution of the long wavelength is also obtained using regular perturbation technique. The effect of variation of different physical parameters on the onset of Marangoni convection in the presence of heat generation is investigated in details. The internal heating in the fluid layer and the porous layer are found to decrease the critical conditions.

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1. Introduction

Surface tension driven convection or known as Marangoni convection has been a subject of interest mainly because of its importance in many branches of science, engineering and technology. The problem of convection due to temperature-dependent surface tension was first theoretically analysed by Pearson [10] where he assumed the upper infinite fluid layer to be non-deformable and neglected the effect of gravity. He showed that thermocapillary forces can caused convection when the Marangoni number exceeds a critical value in the absence of buoyancy forces. The effect of the internal heat generation on Marangoni instability of a horizontal fluid layer when the lower boundary is conducting and insulating to temperature perturbations has been investigated by Wilson [15]. He found that the effect of increasing the internal heat generation is always to destabilize the layer.

Since the early analysis by Horton and Rogers [4] and Lapwood [5], the theory of instability of convective flow through porous media heated from below has been extensively studied. They discussed a porous medium saturated by a wetting liquid, heated from below and they concluded that the filtration Rayleigh number has a critical value equal to $4\pi^2$. Gasser and Kazimi [2] studied the onset of convection in a porous medium with internal heat generation by employing a rigid lower surface with a free upper surface and isothermal conditions at the upper and lower surfaces. The combination of critical Rayleigh numbers presented in their paper was expected to hold true for a bed with a rigid isothermal upper boundary as well as a free isothermal surface upper boundary. Hennenberg et al [3] considered a very coarse porous medium which can be described in terms of the Brinkman model, to justify the existence of a sharp boundary between a liquid saturated porous medium and the external gaseous phase. The lower boundary is assumed to be perfectly conducting heated from below. They have developed the model that can be described in terms of the Brinkman model. They solved the Brinkman approach over the whole saturated porous matrix and obtained a critical wave number which was highly dependent on the Darcy number. Recently, Mokhtar et al [6] repeated the linear stability analysis of Hennenberg et al [3] with isothermal lower boundary condition replaced by the adiabatic lower boundary condition. They also proved that the regular perturbation method with wave number a as a perturbation parameter can conveniently be used in solving this convective instability problems. Very recently, Shivakumara et al [11] have obtained numerically and closed form solution for Darcy-Benard-Marangoni convection in a sparsely packed porous medium by employing the Brinkman-Forchheimer-

Lapwood-extended-Darcy flow model with effective viscosity different from fluid viscosity.

The convective instability of a fluid overlying a porous region saturated with the same fluid subject to a uniform temperature gradient has been investigated by several authors (Nield [7], [8]; Nield and Bejan [9]; Straughan [13]; Taslim and Narusawa [14]). The penetrative convection in a two-layer system in which a layer of fluid overlies and saturates a porous medium is simulated via internal heating has been studied by Carr [1]. It is found that a heat source/sink Q in the fluid layer has a destabilizing effect on the porous layer whereas one in the porous medium Q_m has a stabilizing influence on the fluid. The behaviour is explained and illustrated with a range of streamlines. The effect of variation of different physical parameters on the onset of Marangoni convection in a two-layer system was investigated in detail by Shivakumara et al [12] in the absence of heat generation. They found that the ratio of the thickness of the fluid to the porous layer has a profound effect on the stability of the system.

The purpose of this paper is to study the effect of the internal heat generation on the Marangoni instability in a two-layer system comprising an incompressible fluid-saturated porous layer over which lies a layer of the same fluid. We assumed that the both boundaries are flat and undeformable with lower and upper boundaries are perfectly insulating and neglecting the effect of buoyancy forces. The linear stability theory and the normal mode analysis are applied and the resulting eigenvalues problems are solved exactly and by regular perturbation technique.

2. Problem Formulation

Consider an infinite horizontal incompressible fluid-saturated porous layer of thickness d_p underlying a layer of the same fluid of thickness d, heated from below as shown in Figure 1. The upper free surface, which is at $z = 1$ is assumed to be non-deformable, the interface between the saturated porous medium and the fluid is at $z = 0$ and the bottom boundary at $z_p = -1$ is consider to be rigid. The temperature on the lower and upper boundaries are held fixed and equal to T_l and T_u respectively with $T_l > T_u$. The surface tension σ is assumed to vary linearly in the form $\sigma = \sigma_0 - \sigma^*(T - T_0)$ following Pearson [1], where σ_0 is a constant reference value and $-\sigma^*$ is the rate of change of the surface tension with temperature.

Employing the Navier-Stokes equations with Boussinesq approximation, the

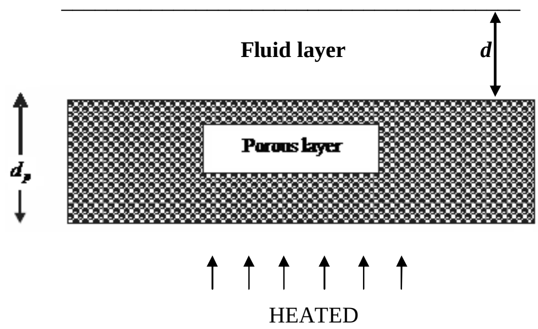

Figure 1: The saturated porous layer heated from below

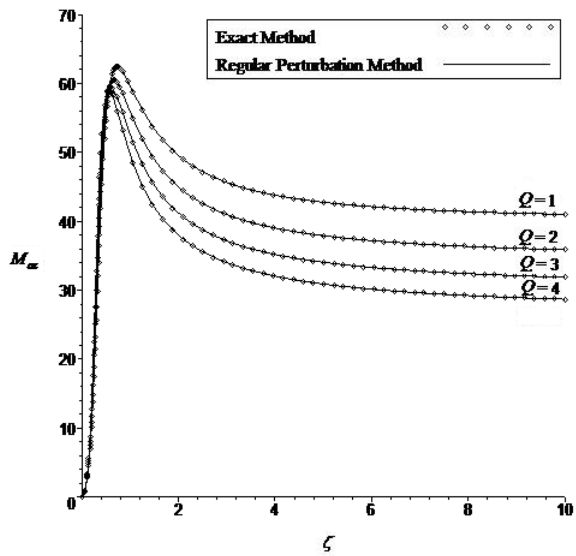

Figure 2: Critical Marangoni number M_{ac} at the onset of convection as a function of ζ in the case $Q_p = 0$ and $Da = 0.003$

governing equations for the fluid can be expressed as

$$\nabla \cdot \boldsymbol{V} = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = -\frac{\nabla P}{\rho_0} + \frac{\mu \nabla^2 \boldsymbol{V}}{\rho_0}, \tag{2}$$

$$\frac{\partial T}{\partial t} + (\boldsymbol{V} \cdot \nabla) T = \kappa \nabla^2 T + q. \tag{3}$$

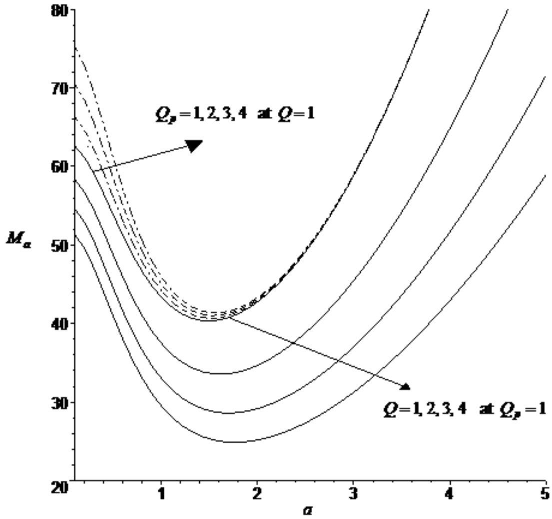

Figure 3: Critical Marangoni number M_{ac} at the onset of convection as a function of wavenumber, a for a ranges of values of Q and Q_p in the case $\alpha = 0.1$, $\zeta = 1$ and $Da = 0.003$

In the porous layer the motion of the fluid is described by Darcy flow that are:

$$\nabla_p \cdot \boldsymbol{V}_p = 0, \tag{4}$$

$$\frac{\rho_0}{\phi}\frac{\partial \boldsymbol{V}_p}{\partial t} = -\nabla_p P_p - \frac{\mu}{K}\boldsymbol{V}_p, \tag{5}$$

$$S\frac{\partial T_p}{\partial t} + (\boldsymbol{V}_p \cdot \nabla_p)T_p = \kappa_p \nabla_p^2 T_p + q_p, \tag{6}$$

where $\boldsymbol{V}$ is the velocity vector, T is the temperature, q is the uniformly distributed volumetric internal heat generation, P is the pressure, μ is the dynamic viscosity, ρ_0 is the fluid density, K is the permeability of the porous medium, κ is the thermal diffusivity of the fluid, ϕ is the porosity of the porous medium, S is the ratio of heat capacities of the fluid saturated porous medium to that of the fluid and the subscript p refer to the porous medium.

Under steady-state conditions, we seek the form of $(u,\ v,\ w,\ P,\ T) = [0,\ 0,\ 0,\ P_b(z),\ T_b(z)]$ in the fluid layer and $(u_p,\ v_p,\ w_p,\ T_p) = [0,\ 0,\ 0,\ P_{pb}(z_p),\ T_{pb}(z_p)]$ in the porous layer, where the subscript b denotes the basic state. The

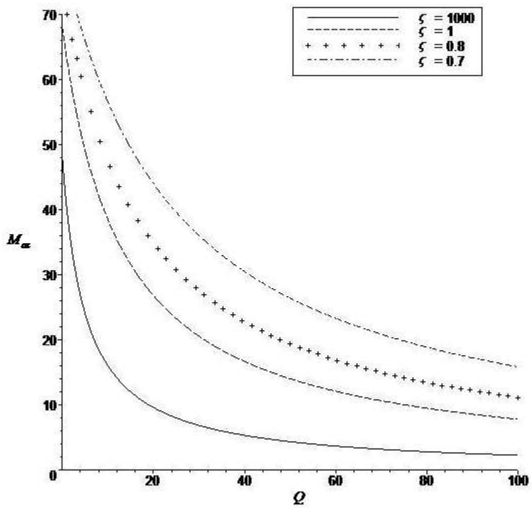

Figure 4: Critical Marangoni number M_{ac} at the onset of convection as a function of Q for a ranges of values of ζ in the case $\alpha = 0.1$, $Q_p = 1$ and $Da = 0.003$

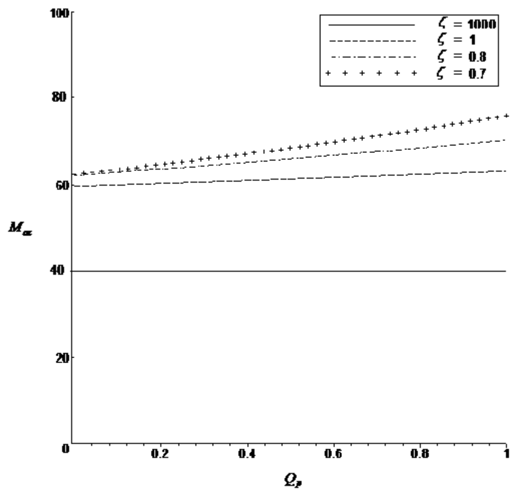

Figure 5: Critical Marangoni number M_{ac} at the onset of convection as a function of Q_p for a ranges of values of ζ in the case $\alpha = 0.1$, $Q = 1$ and $Da = 0.003$

governing equations (1)-(6) admit a steady-state solution in which the velocity

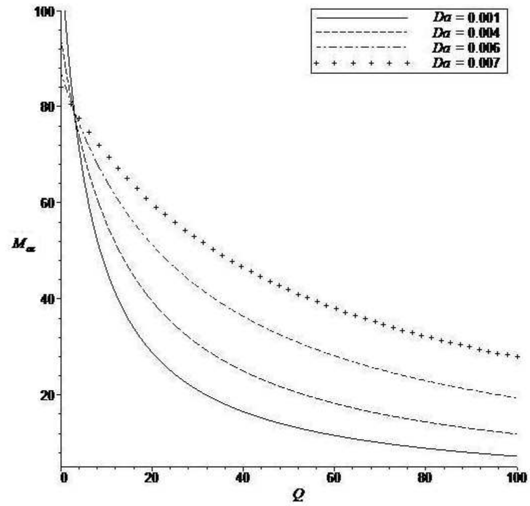

Figure 6: Critical Marangoni number M_{ac} at the onset of convection as a function of Q for a ranges of values of Da in the case $Q_p = 1$

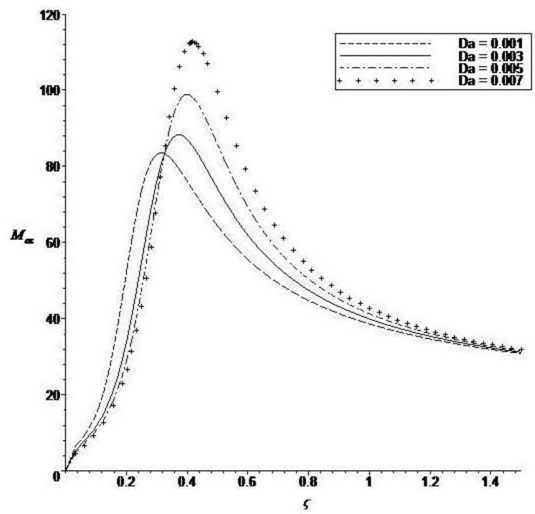

Figure 7: Critical Marangoni number M_{ac} at the onset of convection as a function of ζ for a ranges of values of Da in the case $Q = 10$ and $Q_p = 0$

field is zero and the unperturbed temperature profiles are

$$T_b(z) = T_0 - \left[\left(\frac{(T_0 - T_u)}{d} - \frac{qd}{2\kappa} \right) z + \frac{q}{2\kappa} z^2 \right], \quad \text{at} \quad 0 \leq z \leq d, \qquad (7)$$

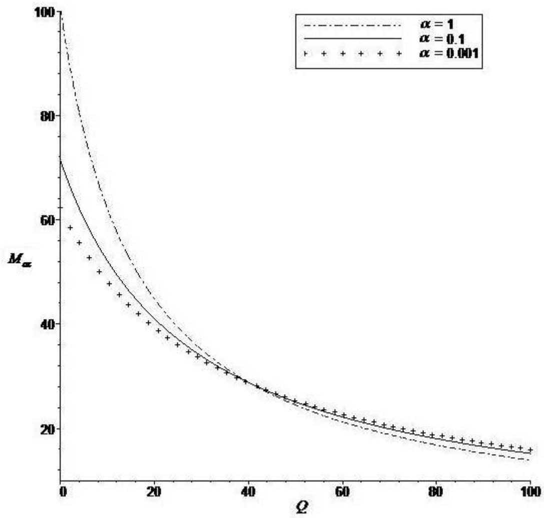

Figure 8: Critical Marangoni number M_{ac} at the onset of convection as a function of Q for a ranges of values of α in the case $Q_p = 1$

$$T_{pb}(z_p) = T_0 - \left[\left(\frac{(T_l - T_0)}{d_p} - \frac{q_p d_p}{2\kappa_p}\right) z_p + \frac{q_p}{2\kappa_p} z_p^2\right], \quad \text{at} \quad -d_p \le z_p \le 0, \quad (8)$$

where $T_0 = [2(\kappa_p T_l d + \kappa T_u d_p) + d(qd + q_p d_p)d_p]/2(\kappa_p d + \kappa d_p)$ is the interface temperature. Infinitesimal disturbances are introduced in the form of

$$(u, v, w, P, T) = [0, 0, 0, P_b(z), T_b(z)] + \left(u^{'}, v^{'}, w^{'}, P^{'}, T^{'}\right), \quad (9)$$

in the fluid layer and in the porous layer we have

$$(u_p, v_p, w_p, P_p, T_p) = [0, 0, 0, P_{pb}(z_p), T_{pb}(z_p)] + \left(u_p^{'}, v_p^{'}, w_p^{'}, P_p^{'}, T_p^{'}\right), \quad (10)$$

where the primed quantities are the perturbed ones over their equilibrium counterparts. Equations (9) and (10) are substituted in equations (1)-(6) and linearized in the usual manner. The variables are then non-dimensionalized using the scales d, d^2/κ , κ/d and $T_0 - T_u$ as the units of length, time, velocity and temperature in the fluid layer and d_p, d_p^2/κ_p, κ_p/d_p and $T_l - T_0$ as the corresponding characteristic quantities in the porous layer. The perturbed governing equations of the liquid layer and the porous layer in the non-dimensional form can be obtained as

$$\frac{1}{Pr}\frac{\partial}{\partial t}\left(\nabla^2 w\right) - \nabla^4 w = 0, \quad (11)$$

$$\frac{\partial T}{\partial t} + [Q(1-2z) - 1]\, w = \nabla^2 T, \tag{12}$$

$$\frac{Da}{Pr_p}\frac{\partial}{\partial t}\left(\nabla_p^2 w_p\right) + \left(\nabla_p^2 w_p\right) = 0, \tag{13}$$

$$S\frac{\partial T_p}{\partial t} + [Q_p(1-2z_p) - 1]\, w_p = \nabla_p^2 T_p. \tag{14}$$

For the fluid layer, $Pr = \nu/\kappa$ is the Prandtl number, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian operator and for the porous layer $Pr_p = \nu/\phi\kappa_p$ is the Prandtl number, $Da = K/d_p^2$ is the Darcy number and $\nabla_p^2 = \partial^2/\partial x_p^2 + \partial^2/\partial y_p^2 + \partial^2/\partial z_p^2$. Q and Q_p are the dimensionless heat source strength which are defined as $Q = qd^2/2\kappa(T_0 - T_u)$ and $Q_p = q_p d_p^2/2\kappa_p(T_l - T_0)$. The dimensionless perturbed boundary conditions at $z = 1$ where the surface tension forces is allowed are given by

$$w = \frac{\partial T}{\partial z} = 0, \tag{15}$$

$$\frac{\partial^2 w}{\partial z^2} = M_a \nabla_h^2 T, \tag{16}$$

where $M_a = \sigma^*(T_0 - T_u)d/\mu\kappa$ is the Marangoni number and $\nabla_h^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian operator. At the interface; $z = 0$, we assumed that the normal component of velocity, temperature and heat flux are continuous and adding two conditions which is derived from the Beavers-Joseph boundary conditions (for details, see [15]), we have

$$w = \frac{\zeta}{\varepsilon} w_p, \tag{17}$$

$$T = \frac{\varepsilon}{\zeta} T_p, \tag{18}$$

$$\frac{\partial T}{\partial z} = \frac{\partial T_p}{\partial z_p}, \tag{19}$$

$$\frac{\partial^2 w}{\partial z^2} + \frac{1}{\sqrt{Da}}\left[\frac{\alpha\zeta^3}{\varepsilon}\right]\frac{\partial w_p}{\partial z_p} = \frac{\alpha\zeta}{\sqrt{Da}}\frac{\partial w}{\partial z}, \tag{20}$$

$$\left(3\nabla_h^2 + \frac{\partial^2}{\partial z^2}\right)\frac{\partial w}{\partial z} - \frac{1}{Pr}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z}\right) + \frac{\zeta^4}{\varepsilon Da}\left(\frac{\partial w_p}{\partial z_p}\right) + \frac{1}{Pr}\frac{\partial}{\partial t}\left(\frac{\partial w_p}{\partial z_p}\right) = 0, \tag{21}$$

and at the bottom boundary of the porous layer ($z_p = -1$) is assumed to be rigid and insulating

$$w_p = \frac{\partial T_p}{\partial z_p} = 0. \tag{22}$$

Here, $\zeta = d/d_p$, α is the slip parameter and $\varepsilon = \kappa/\kappa_p$ is the ratio of thermal diffusivities.

A normal-mode representation is introduced of the form

$$\begin{pmatrix} w \\ T \end{pmatrix} = \begin{pmatrix} W(z) \\ \theta(z) \end{pmatrix} f(x,y), \begin{pmatrix} w_p \\ T_p \end{pmatrix} = \begin{pmatrix} W_p(z_p) \\ \theta_p(z_p) \end{pmatrix} f_p(x_p, y_p), \tag{23}$$

where $\nabla_2^2 f + a^2 f = 0$ and $\nabla_{p2}^2 f_p + a_p^2 f_p = 0$ with a and a_p are non-dimensional horizontal wavenumbers. Substituting equation (23) into equations (11)-(22), matching the solutions in the fluid and porous layer with $a/d = a_p/d_p$, we obtained

$$\left(D^2 - a^2\right)^2 W = 0, \tag{24}$$

$$\left(D^2 - a^2\right)\theta = \left[Q\left(1 - 2z\right) - 1\right]W, \tag{25}$$

$$\left(D_p^2 - a_p^2\right)W_p = 0, \tag{26}$$

$$\left(D_p^2 - a_p^2\right)\theta_p = \left[Q_p\left(1 - 2z_p\right) - 1\right]W_p. \tag{27}$$

The boundary conditions at the upper free surface ($z = 1$) are ;

$$W = 0, \tag{28}$$

$$D\theta = 0, \tag{29}$$

$$D^2 W + M_a\left(a^2\theta\right) = 0, \tag{30}$$

at the interface ($z = 0$);

$$W - \frac{\zeta}{\varepsilon}W_p = 0, \tag{31}$$

$$D\theta - D_p\theta_p = 0, \tag{32}$$

$$\theta - \frac{\varepsilon}{\zeta}\theta_p = 0, \tag{33}$$

$$D^2 W - \frac{\alpha\zeta}{\sqrt{Da}}DW + \frac{\alpha\zeta^3}{\varepsilon\sqrt{Da}}D_p W_p = 0, \tag{34}$$

$$D^3 W - 3a^2 DW + \frac{\zeta^4}{\varepsilon Da}D_p W_p = 0, \tag{35}$$

and those at the bottom boundary ($z_p = -1$) are;

$$W_p = 0, \tag{36}$$

$$D_p\theta_p = 0, \tag{37}$$

where $D = d/dz$ and $D_p = d/dz_p$.

3. Analytical Solution

The resulting eigenvalue problem is solved exactly, in general, with M_a as an eigenvalue. Since equation (24) and (26) are independent of θ and θ_p, they can be directly solved to get the general solution in the form

$$W = A_1 \cosh(az) + A_2 \sinh(az) + A_3 z \cosh(az) + A_4 z \sinh(az), \tag{38}$$

$$W_p = A_{p1} \sinh(a_p z_p) + A_{p2} \cosh(a_p z_p), \tag{39}$$

where A_1-A_4, A_{p1} and A_{p2} are the constants to be determined. Using the boundary conditions (28), (31), (34), (35) and (36), we obtained

$$W = A\left[\cosh(az) + \Delta_1 \sinh(az) + \Delta_2 z \cosh(az) + \Delta_3 z \sinh(az)\right], \tag{40}$$

$$W_p = A\frac{\varepsilon}{\zeta}\left[\coth(a_p)\sinh(a_p z_p) + \cosh(a_p z_p)\right], \tag{41}$$

where

$$\Delta_1 = \frac{\zeta^3 a_p \coth(a_p)}{2a^3 Da},$$

$$\Delta_2 = \frac{\sqrt{Da}\left(-2a[\cosh(a) + \Delta_1 \sinh(a)] - \sinh(a)\left[-a^2 + \frac{\alpha\zeta\alpha\Delta_1}{\sqrt{Da}} - \frac{2a^3\alpha\Delta_1\sqrt{Da}}{\zeta}\right]\right)}{2a\sqrt{Da}\cosh(a) + \alpha\zeta\sinh(a)},$$

$$\Delta_3 = \frac{-\alpha\zeta[\cosh(a) + \Delta_1\sinh(a)] - \sqrt{Da}\cosh(a)\left[a^2 - \frac{\alpha\zeta\alpha\Delta_1}{\sqrt{Da}} + \frac{2a^3\alpha\Delta_1\sqrt{Da}}{\zeta}\right]}{2a\sqrt{Da}\cosh(a) + \alpha\zeta\sinh(a)}.$$

The heat equations (25) and (27) have now to be solved defining their right-hand side expressions given by (40) and (41) respectively. The solutions for θ and θ_p using the boundary conditions (29), (32), (33) and (37) are found to be

$$\theta = \frac{A}{4a^2}\left[\delta_1 \sinh(az) + \delta_2 \cosh(az) + \delta(z)\right], \tag{42}$$

$$\theta_p = A\left[\left(\frac{a\delta_1 - \lambda_4}{a_p} - \frac{\varepsilon z_p}{2\zeta a_p}\right)\sinh(a_p z_p) + \left(\frac{\varepsilon\delta_2}{\zeta} - \frac{\varepsilon\coth(a_p z_p)}{2\zeta a_p}\right)\cosh(a_p z_p)\right], \tag{43}$$

where

$$\delta_1 = \frac{[\lambda_1 \cosh(a) + \lambda_2 \sinh(a) - a\delta_2 \sinh(a)]}{a \cosh(a)},$$

$$\delta_2 = \frac{\varepsilon\{[\lambda_3 - \lambda_4 \cosh(a_p)] \cosh(a) + [\lambda_1 \cosh(a) + \lambda_2 \sinh(a)] \cosh(a_p)\}}{a\varepsilon \cosh(a_p) \sinh(a) + a_p\zeta \cosh(a) \sinh(a_p)},$$

$$\delta(z) = [(\Delta_3 - 2a)z - a\Delta_2 z^2 + Q(2a + 2\Delta_1 - (2/a)\Delta_2 - \Delta_3)z - (2a - a\Delta_2 - 2\Delta_3)z^2$$
$$-(4/3)a\Delta_2 z^3] \sinh(az) + [(-2a\Delta_1 + \Delta_2)z - a\Delta_3 z^2 + Q(2 + 2a\Delta_1 - \Delta_2$$
$$-(2/a)\Delta_3)z - (2a\Delta_1 - 2\Delta_2 - a\Delta_3)z^2 - (4/3)a\Delta_3 z^3] \cosh(az),$$

with

$$\lambda_1 = \left[2a^2 + 2a\Delta_1 + \left(a^2 - 1\right)\Delta_2 + a\Delta_3\right] + Q\left[-2 + \left(a^2/3 - 1\right)\Delta_2 + (a + 2/a)\Delta_3\right],$$

$$\lambda_2 = \left[2a + 2a^2\Delta_1 + \left(a^2 - 1\right)\Delta_3 + a\Delta_2\right] + Q\left[-2\Delta_1 + \left(a^2/3 - 1\right)\Delta_3 + (a + 2/a)\Delta_2\right],$$

$$\lambda_3 = \frac{2a^2\varepsilon}{a_p\zeta \sinh(a_p)}\left[\sinh^2(a_p) - Q_p\left(\cosh^2(a_p) - 2\right)\right],$$

$$\lambda_4 = 2a\Delta_1 - \Delta_2 - Q\left[2 + 2a\Delta_1 - \Delta_2 - (2/a)\Delta_3\right] + Q_p\left[\frac{2\varepsilon a^2}{\zeta(a_p)^2}\right].$$

Substituting equations (40) and (42) in the boundary condition (30), we obtained an analytically expression for the Marangoni number, M_a which can be conveniently written as

$$M_a = \frac{-4a\left[(a + a\Delta_2 + 2\Delta_3)\cosh(a) + (a\Delta_1 + 2\Delta_2 + a\Delta_3)\sinh(a)\right]}{\delta_1 \sinh(a) + \delta_2 \cosh(a) + \delta(1)}. \quad (44)$$

4. Asymptotic Solution by Regular Perturbation Technique

To study the validity of the small wave number analysis, the dependent variables in the porous layers are now expanded in powers of a^2 in the form

$$(W, \Theta) = \sum_{i=0}^{N} \left(a^2\right)^i (W_i, \Theta_i), \tag{45}$$

$$(W_p, \Theta_p) = \sum_{i=0}^{N} \left(\frac{a^2}{\zeta^2}\right)^i (W_{pi}, \Theta_{pi}). \tag{46}$$

Substituting equations (45) and (46) into equations (24)-(37) yields a sequence of equations for the unknown functions $W_i(z)$, $\Theta_i(z)$, $W_{pi}(z_p)$ and $\Theta_{pi}(z_p)$ for $i = 0, 1, 2, \ldots$

At the zeroth order, equations (24)-(37) become, respectively,

$$D^4 W_0 = 0, \tag{47}$$

$$D^2 \Theta_0 = [Q (1 - 2z) - 1] W_0, \tag{48}$$

$$D^2 W_{p0} = 0, \tag{49}$$

$$D^2 \Theta_{p0} = [Q_p (1 - 2z_p) - 1] W_{p0}, \tag{50}$$

at $z = 1$,

$$W_0 = 0, \tag{51}$$

$$D\Theta_0 = 0, \tag{52}$$

$$D^2 W_0 = 0, \tag{53}$$

at $z = 0$,

$$W_0 - \left(\frac{\zeta}{\varepsilon}\right) W_{p0} = 0, \tag{54}$$

$$D\Theta_0 - D\Theta_{p0} = 0, \tag{55}$$

$$\Theta_0 - \frac{\varepsilon}{\zeta}\Theta_{p0} = 0, \tag{56}$$

$$D^2 W_0 - \frac{\alpha\zeta}{\sqrt{Da}} DW_0 + \frac{\alpha\zeta^3}{\varepsilon\sqrt{Da}} DW_{p0} = 0, \tag{57}$$

$$D^3 W_0 + \frac{\zeta^4}{\varepsilon Da} DW_{p0} = 0, \tag{58}$$

at $z_p = -1$,

$$W_{p0} = D\Theta_{p0} = 0. \tag{59}$$

The solution to the zeroth order for the equations (47)-(59) are given by

$$W_0 = 0, \qquad \Theta_0 = \frac{\varepsilon}{\zeta}, \qquad W_{p0} = 0 \qquad \text{and} \qquad \Theta_{p0} = 1. \tag{60}$$

The terms of order a^2 are

$$D^4 W_1 = 0, \tag{61}$$

$$D^2\Theta_1 - W_1\left[Q\left(1-2z\right)-1\right] = \frac{\varepsilon}{\zeta}, \tag{62}$$

$$D^2 W_{p1} = 0, \tag{63}$$

$$D^2\Theta_{p1} - W_{p1}\left[Q_p\left(1-2z_p\right)-1\right] = 1, \tag{64}$$

at $z = 1$,

$$W_1 = 0, \tag{65}$$

$$D\Theta_1 = 0, \tag{66}$$

$$D^2 W_1 + \frac{\varepsilon}{\zeta} M_a = 0, \tag{67}$$

at $z = 0$,

$$W_1 - \frac{W_{p1}}{\varepsilon\zeta} = 0, \tag{68}$$

$$D\Theta_1 - \frac{D\Theta_{p1}}{\zeta^2} = 0, \tag{69}$$

$$\Theta_1 - \frac{\varepsilon\Theta_{p1}}{\zeta^3} = 0, \tag{70}$$

$$D^2 W_1 - \frac{\alpha\zeta}{\sqrt{Da}} DW_1 + \frac{\alpha\zeta}{\varepsilon\sqrt{Da}} DW_{p1} = 0, \tag{71}$$

$$D^3 W_1 + \frac{\zeta^2}{\varepsilon Da} DW_{p1} = 0, \tag{72}$$

at $z_p = -1$,

$$W_{p1} = 0, \tag{73}$$

$$D\Theta_{p1} = 0. \tag{74}$$

Using the symbolic algebra package *Maple 12*, we obtained the critical Marangoni number, M_{ac}; as given below

$$M_{ac} = \frac{240 r_1 (1 + \varepsilon\zeta)}{\varepsilon [5r_2 + Qr_3 - 300\varepsilon Da Q_p r_4]}, \tag{75}$$

where

$$r_1 = \alpha\zeta^3 + 3\sqrt{Da}\left[\zeta^2 + \alpha\sqrt{Da}\,(1+\zeta)\right],$$
$$r_2 = 6\sqrt{Da}\left\{\zeta^3 + 2\sqrt{Da}\left(\alpha\zeta[\zeta+4] + 3\varepsilon\alpha + 6\sqrt{Da}[1+\varepsilon/\zeta]\right)\right\} - \alpha\zeta^4,$$
$$r_3 = 2\sqrt{Da}\left[\zeta^3 - 30\alpha\zeta\sqrt{Da} - 60Da\right] + \alpha\zeta^4,$$
$$r_4 = \alpha + 2/\zeta\sqrt{Da}.$$

5. Result and Discussion

The criterion for the onset of Marangoni convection with internal heat generation in two-layer system comprising an incompressible fluid-saturated porous layer over which lies a layer of the same fluid is investigated theoretically. In the absence of internal heat generation (i.e. $Q = 0$ and $Q_p = 0$), the equations (44) and (75) reduce to the expression given by Shivakumara et al [12]. In each case investigated in this paper, we use equation (44) to obtain the critical Marangoni number. To verify our numerical results, test computations have been performed and the critical Marangoni number allocation shows a good agreement with the results given in [12] which are listed in Table 1. To compare with [12], we have choose $Da = 0.001$, $\varepsilon = 0.725$ and $\alpha = 0.1$ to calculate the critical Marangoni number. From the table, the effect of Q plays a different role in the stability of the system which is depends on the ratio thickness of the system. As we can see, the critical Marangoni number, M_{ac} decreases as Q increases at $\zeta = 0.4, 0.6, 0.8$. However, for $\zeta = 0.2, 0.1, 0.01$ which shows the dominant of the porous layer, the M_{ac} increases as Q increases and promotes the stability of the system.

Table 2 shows the critical values of Marangoni number M_{ac}, for different values of ζ, Q and Q_p on the stability of the Marangoni convection when $\varepsilon = 0.725$, $\alpha = 0.1$ and $Da = 0.003$. Our next main interest is to look at the effect of internal heating in the porous medium, that is Q_p at varies values of Q. Since

	Ref. [12]	Present				
ζ	$Q = 0$	$Q = 0$	$Q = 2$	$Q = 4$	$Q = 6$	$Q = 8$
0.01	-	1.046	1.055	1.065	1.075	1.085
0.1	-	7.440	8.060	8.792	9.671	10.746
0.2	23.421	23.42098	26.31238	30.01824	34.93909	41.78961
0.4	77.940	77.94011	77.31293	76.69576	76.08837	75.49052
0.6	103.148	103.1475	87.69264	76.26554	67.47321	60.49858
0.8	103.683	103.6833	81.79576	67.53839	57.51353	50.08005

Table 1: Critical values of Marangoni number M_{ac}, for different values of ζ and Q at $Q_p = 0$

$\zeta \geq 0.6$ has been chosen, an increased of Q decreased the value of Ma_c for every ζ stated which is the same pattern as can be seen in the previous table. However, inspection of Q_p reveals that the critical Marangoni number; M_{ac} increased when the strength of Q_p become stronger and thus it has a stabilizing effect on the system. Further investigation for ζ below than 0.6 has been made and the critical Marangoni number become negative as Q_p increases.

Figure 2 shows the critical Marangoni numbers obtained by the asymptotic solution, that is equation (75) and analytical solution, equation (44) when $Da = 0.003$, $\alpha = 0.1$, $\varepsilon = 0.725$ and $Q_p = 0$. It can been seen that there is an excellent agreement between the results of the analytical solution and asymptotic solution and this also proved that the regular perturbation method with wave number a as a perturbation parameter can conveniently be used in solving this convective instability problems.

Figure 3 shows the critical Marangoni number, M_a as a function of wavenumber, a with $\alpha = 0.1$, $\zeta = 1$ and $Da = 0.003$ for different values of Q and Q_p. We observed that the onset of Marangoni convection happened earlier at fixed value of $Q_p = 1$ and varies values of Q, rather than at fixed value of $Q = 1$ and varies values of Q_p. If we look precisely the dotted lines only appear 3 times as if there is no $Q_p = 4$, but actually the dotted lines for $Q_p = 1$ appear combined with $Q = 1$ for through out of the wave number. From overall observation from this figure, internal heating in porous media; Q_p can delayed the process of the onset of convection in the system.

The effect of heat generation Q and Q_p on the stability of Marangoni convection for different values of ζ when $Da = 0.003$, $\varepsilon = 0.725$ and $\alpha = 0.1$ are presented in Figures 4 and 5. As discussed earlier, when the internal heating in the fluid layer become stronger, the system become more unstable rapidly

		M_{ac}					
Q_p	Q	$\zeta = 50$	$\zeta = 10$	$\zeta = 1$	$\zeta = 0.8$	$\zeta = 0.7$	$\zeta = 0.6$
	0	47.803	47.851	64.001	65.210	64.336	61.086
	1	40.110	41.006	59.682	62.125	62.316	60.508
0	5	24.403	26.082	46.996	52.239	55.360	58.302
	50	4.514	5.120	13.858	18.722	24.542	41.345
	100	2.369	2.705	7.777	10.930	15.163	31.247
	0	47.803	47.851	67.912	74.272	78.984	85.850
	1	40.110	41.006	63.069	70.296	75.960	84.713
1	5	24.403	26.082	49.071	57.898	65.872	80.451
	50	4.514	5.120	14.033	19.401	26.411	51.376
	100	2.369	2.705	7.825	11.158	15.856	36.656
	0	47.803	47.852	77.367	102.858	145.019	453.763
	1	40.110	41.007	71.143	95.386	135.140	423.708
3	5	24.403	26.083	53.824	73.910	106.203	334.963
	50	4.514	5.120	14.396	20.920	31.154	99.801
	100	2.369	2.705	7.937	11.644	17.452	56.066

Table 2: Critical values of Marangoni number M_{ac}, for different values of ζ, Q and Q_p

for certain value of ζ. Contrast with the effect of Q, an increased in the Q_p, increase the value of the critical Marangoni number and the system is prone to the stability. Besides that, as the ζ value decreases the critical Marangoni number increases and it helps to delay the onset of convection.

To analyze the influence of permeability of the porous layer in the presence of the heat generation, the variation of the Darcy number, Da with Q for $\zeta = 0.8$, $\varepsilon = 0.725$, $Q_p = 1$ and $\alpha = 0.1$, is plotted in Figure 6. At the range of $0 \leq Q \leq 5$, a decreased of Da delayed the onset of convection in the system. However, as Q increasing the pattern switched where an increase of Da, promotes stability of the system. The effect of Q, still destabilizing the system.

The variation of M_{ac} with ζ for different values of Da can be found in the Figure 7 at the fixed value of $Q = 10$, $Q_p = 0$, $\varepsilon = 0.725$ and $\alpha = 0.1$. We find out that, when the ratio of the thickness of the fluid layer to that of the porous layer is less than 0.3, a decrease of the Da number stabilize the system. This may be due to low permeable porous layer dampens the fluid motion and requiring an increased in critical Marangoni number in order to stabilize the system. On the other hand, when ζ increases ($\zeta > 0.3$), the scenario changing where an increase in the Da number increase the critical Marangoni number and thus making the system more stable. Besides that, the M_{ac} value is attained as $\zeta \geq 1$. Figure 8 shows the variation of M_{ac} with Q for different values of slip parameter α when $\zeta = 1$, $Q_p = 1$, $\varepsilon = 0.725$ and $Da = 0.003$. As α increases, it is obviously seen that the critical Marangoni, M_{ac} also increases for $Q \leq 20$. However, as Q increases rapidly especially at the range $Q \geq 20$, the critical value of M_{ac} decreases as α increase.

6. Conclusions

The stability analysis of the Marangoni convection in two-layer system comprising an incompressible fluid-saturated porous layer over which lies a layer of the same fluid is investigated theoretically. For the case of the absence of heat generation, the solutions are in a good agreement with other results published in the literature. Furthermore, the internal heat generation in the two-layer saturated porous matrix has a significant influence on the Marangoni convection which is either stabilized or destabilized depending on the various parameter considered especially the ratio depth of the system.

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