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THE FOURIER TRANSFORM OF THE MULTIDIMENTIONAL GENERALIZED GAUSSIAN DISTRIBUTION

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Abstract: We present expressions for the generalized Gaussian distribution in n dimensions and compute their Fourier transforms. We obtain expressions in terms of Bessel functions and Maclaurin series.

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1. Introduction

The generalized Gaussian (GG) or the generalized normal (GN) distribution [38], also known as the exponential power distribution [6] or the generalized Laplace distribution [18, 20, 22], was first proposed by Subbotin [31] as a generalized error distribution. Statistical problems have been considered for this distribution by many authors. For example, parameter estimation was studied in [2, 9, 17, 25, 34, 36, 37, 39], Bayesian statistical analysis in [4, 5, 6, 7, 10, 35], and other statistical problems in [8, 14, 15, 27, 28, 29, 30, 32, 33]. The GG distribution is also used in several applications related to signal and image analysis, for example [3, 11, 24, 26]. For other information concerning this distribution and its applications see [6, 18, 20, 22].

The Fourier transform (or the characteristic function) of the GG distribution is very often used in the applications. In one dimension (1D), the GG distribution is defined for $x \in \mathbb{R}$ by the formula

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$$f_\lambda(x) = e^{-|x|^\lambda} \tag{1}$$

where λ is a strictly positive real number. Its Fourier transform is defined by

$$\hat{f}_\lambda(\omega) = \int_{-\infty}^{+\infty} e^{-|x|^\lambda} e^{-2\pi i\omega x} dx. \tag{2}$$

Series expansions have been obtained in this case for integer valued parameter (see [12]) and more generally for real valued parameter, see [13].

In this paper we propose some generalizations in n dimensions (nD) of the GG distribution and study their Fourier transforms. We extend some results presented in [15, 23]. Let $x^t = (x_1, x_2, \ldots, x_n)$ and $\omega^t = (\omega_1, \omega_2, \ldots, \omega_n)$ be elements of $\mathbb{R}^n$, and let $\omega^t x$ be the standard scalar product in $\mathbb{R}^n$ defined by

$$\omega^t x = \sum_{i=1}^{n} \omega_i x_i.$$

If $f(x) : \mathbb{R}^n \to \mathbb{R}$, its Fourier transform, $\hat{f}_\lambda(\omega)$, is given by

$$\hat{f}_\lambda(\omega) = \int_{\mathbb{R}^n} f_\lambda(x) e^{-2\pi i\omega^t x} dx, \tag{3}$$

where dx stands for $dx_1 dx_2 \cdots dx_n$.

2. First Generalization

We consider the simplest generalization, the separable case with n possible different parameters. We set

$$F_{\lambda_1,\lambda_2,\ldots,\lambda_n}(x_1, x_2, ..., x_n) = f_{\lambda_1}(x_1) f_{\lambda_2}(x_2) ... f_{\lambda_n}(x_n). \tag{4}$$

This form is not invariant by rotation but is separable, as a consequence its Fourier transform is the product of n 1D Fourier transforms. Indeed we have

$$\hat{F}_{\lambda_1,\lambda_2,\ldots,\lambda_n}(\omega_1, \omega_2, ..., \omega_n) = \hat{f}_{\lambda_1}(\omega_1) \hat{f}_{\lambda_2}(\omega_2) ... \hat{f}_{\lambda_n}(\omega_n), \tag{5}$$

where formulas for $\hat{f}_\lambda(\omega)$ will be given in the next section.

3. Second Generalization

3.1. Definition

The second and natural generalization of the GG distribution is defined for $x \in \mathbb{R}^n$ by the formula

$$G_\lambda(x) = e^{-\|x\|^\lambda} = e^{-(x_1^2+x_2^2+\cdots+x_n^2)^{\lambda/2}}, \tag{6}$$

where $\|x\|$ is the standard norm in $\mathbb{R}^n$ defined by

$$\|x\| = \left(x^t x\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}.$$

So defined, the function $G_\lambda(x)$ is a radial function and is invariant by rotation.

Since $G_\lambda(x)$ is a rapidly decaying function, its Fourier transform and all its derivatives are well defined, see [19, 21]. The Fourier transform of $G_\lambda(\omega)$ is given by

$$\hat{G}_\lambda(\omega) = \int_{\mathbb{R}^n} e^{-\|x\|^\lambda} e^{-2\pi i \omega^t x} dx. \tag{7}$$

3.2. First Formula

Let us observe that for any linear orthogonal transformation on $\mathbb{R}^n$, $V : \mathbb{R}^n \to \mathbb{R}^n$ such that $V^t V = I$, we have $x = Vy$ or $V^t x = y$, and

$$\hat{G}_\lambda(\omega) = \int_{\mathbb{R}^n} e^{-\|y\|^\lambda} e^{-2\pi i \omega^t V y} dy$$

since $|\mathrm{Det}(V)| = 1$. For any $\omega \neq 0$, if we consider a linear orthogonal transformation V such that its first column is $\frac{1}{\|\omega\|}\omega$, then $\omega^t V y = \|\omega\| \, y_1$ because ω is orthogonal to the $(n-1)$ other columns of V. Then, for any ω we can write

$$\hat{G}_\lambda(\omega) = \int_{\mathbb{R}^n} e^{-\|y\|^\lambda} e^{-2\pi i \|\omega\| y_1} dy.$$

Let us now consider the three cases:

(i) $n = 1$: We have

$$\hat{G}_\lambda(\omega) \quad = \quad \int_{-\infty}^{+\infty} e^{-|y|^\lambda} e^{-2\pi i |\omega| y} dy$$

$$= \quad 2\int_0^{+\infty} e^{-y^\lambda}\cos(-2\pi|\omega|y)dy.$$

(ii) $n=2$: We can write

$$\begin{aligned}\hat{G}_\lambda(\omega) &= \int_{\mathbb{R}^2} e^{-(y_1^2+y_2^2)^{\frac{\lambda}{2}}} e^{-2\pi i\|\omega\|y_1} dy_1 dy_2 \\ &= 2\int_{-\infty}^{+\infty} e^{-2\pi i\|\omega\|y} \int_0^{+\infty} e^{-(y^2+\rho^2)^{\frac{\lambda}{2}}} d\rho dy,\end{aligned}$$

where we used $y_1 = y$ and $y_2 = \rho$.

(iii) $n \geq 3$: Let us consider the change of variables

$$\begin{cases} y_1 &= y, \\ y_2 &= \rho\cos(\theta_2), \\ y_3 &= \rho\sin(\theta_2)\cos(\theta_3), \\ \vdots & \quad\vdots \\ y_{n-1} &= \rho\sin(\theta_2)\sin(\theta_3)\cdots\sin(\theta_{n-2})\cos(\theta_{n-1}), \\ y_n &= \rho\sin(\theta_2)\sin(\theta_3)\cdots\sin(\theta_{n-2})\sin(\theta_{n-1}), \end{cases}$$

where $y \in \mathbb{R}$, $\rho \in [0,+\infty)$, $\theta_2, \theta_3, \cdots, \theta_{n-2} \in [0,\pi]$, and $\theta_{n-1} \in [-\pi,\pi]$. We have

$$\begin{aligned} & dy_1 dy_2 \cdots dy_n \\ &\quad = \rho^{n-2}\sin^{n-3}(\theta_2)\sin^{n-2}(\theta_3)\cdots\sin(\theta_{n-2})dyd\rho d\theta_2 d\theta_3\cdots d\theta_{n-2}d\theta_{n-1}, \end{aligned}$$

it follows that

$$\hat{G}_\lambda(\omega) = S_{n-1}\int_{-\infty}^{+\infty} e^{-2\pi i\|\omega\|y}\int_0^{+\infty} e^{-(y^2+\rho^2)^{\frac{\lambda}{2}}}\rho^{n-2}d\rho dy,$$

where S_{n-1} is the area of the unit sphere in $\mathbb{R}^{n-1}$ given by

$$S_{n-1} = \int_{[0,\pi]^{n-3}\times[-\pi,\pi]}\prod_{j=2}^{n-2}\sin^{n-1-j}(\theta_j)d\theta_2 d\theta_3\cdots d\theta_{n-2}d\theta_{n-1} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}.$$

Hence

$$\hat{G}_\lambda(\omega) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\int_{-\infty}^{+\infty} e^{-2\pi i\|\omega\|y}\int_0^{+\infty} e^{-(y^2+\rho^2)^{\frac{\lambda}{2}}}\rho^{n-2}d\rho dy.$$

It follows that for $n \geq 2$

$$\hat{G}_\lambda(\omega) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_{-\infty}^{+\infty} \cos(2\pi \|\omega\| y) \int_0^{+\infty} e^{-(y^2+\rho^2)^{\frac{\lambda}{2}}} \rho^{n-2} d\rho dy. \qquad (8)$$

Using the change of variables

$$\begin{cases} y &= \varsigma \cos(\psi), \\ \rho &= \varsigma \sin(\psi), \end{cases}$$

where $\varsigma \in [0, +\infty)$ and $\psi \in [0, \pi]$, we have $d\rho dy = \varsigma d\varsigma d\psi$, and (8) becomes

$$\hat{G}_\lambda(\omega) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^{+\infty} e^{-\varsigma^\lambda} \varsigma^{n-1} \int_0^{\pi} \sin^{n-2}(\psi) \cos\big(2\pi \|\omega\| \varsigma \cos(\psi)\big) d\psi d\varsigma.$$

From the series expansion of the $\cos(\cdot)$, we have

$$\int_0^{\pi} \sin^{n-2}(\psi) \cos\big(2\pi \|\omega\| \varsigma \cos(\psi)\big) d\psi$$
$$= \sum_{k=0}^{+\infty} (-1)^k \frac{(2\pi \|\omega\| \varsigma)^{2k}}{(2k)!} \int_0^{\pi} \sin^{n-2}(\psi) \cos^{2k}(\psi) d\psi.$$

Then we use the folllowing identities (see [1])

$$\int_0^{\pi} \sin^{n-2}(\psi) \cos^{2k}(\psi) d\psi = \frac{\Gamma\left(\frac{2k+1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{2k+n}{2}\right)},$$

and

$$\Gamma\left(\frac{2k+1}{2}\right) = \frac{(2k)!}{2^{2k} k!} \pi^{\frac{1}{2}},$$

to obtain

$$\hat{G}_\lambda(\omega) = 2\pi^{\frac{n}{2}} \int_0^{+\infty} e^{-\varsigma^\lambda} \varsigma^{n-1} \left[\sum_{k=0}^{+\infty} (-1)^k \frac{(\pi \|\omega\| \varsigma)^{2k}}{\Gamma(k+1)\Gamma\left(\frac{2k+n}{2}\right)} \right] d\varsigma, \qquad (9)$$

which is valid for any $n \geq 1$.

3.3. Fourier Transform and Bessel Function

Introducing the Bessel function

$$\mathrm{J}_\nu(\xi) = \left(\frac{\xi}{2}\right)\sum_{k=0}^{+\infty}(-1)^k \frac{\xi^{2k}}{2^{2k}k!\Gamma(\nu+k+1)},$$

with $\nu = \frac{n}{2} - 1$ and $\xi = 2\pi \|\omega\| \varsigma$, (9) becomes

$$\hat{G}_\lambda(\omega) = \frac{2\pi}{\|\omega\|^{\frac{n}{2}-1}} \int_0^{+\infty} e^{-\varsigma^\lambda} \varsigma^{\frac{n}{2}} \mathrm{J}_{\frac{n}{2}-1}(2\pi \|\omega\| \varsigma) d\varsigma. \tag{10}$$

3.4. Fourier Transform and Maclaurin Series

If we perform the integration term by term in (9), we get

$$\hat{G}_\lambda(\omega) = 2\pi^{\frac{n}{2}} \sum_{k=0}^{+\infty}(-1)^k \frac{(\pi \|\omega\|)^{2k}}{\Gamma(k+1)\Gamma\left(\frac{2k+n}{2}\right)} \int_0^{+\infty} e^{-\varsigma^\lambda} \varsigma^{2k+n-1} d\varsigma.$$

Since

$$\int_0^{+\infty} e^{-\varsigma^\lambda} \varsigma^{2k+n-1} d\varsigma = \frac{1}{\lambda}\Gamma\left(\frac{2k+n}{\lambda}\right),$$

we finally have

$$\hat{G}_\lambda(\omega) = \frac{2\pi^{\frac{n}{2}}}{\lambda} \sum_{k=0}^{+\infty}(-1)^k \frac{\Gamma\left(\frac{2k+n}{\lambda}\right)}{\Gamma(k+1)\Gamma\left(\frac{2k+n}{2}\right)} (\pi \|\omega\|)^{2k}. \tag{11}$$

The interchange of integral and summation symbols in these expressions is allowed as long as the series is uniformly convergent. To obtain the condition on λ, we use the ratio test and let

$$A_k = (-1)^k \frac{\Gamma\left(\frac{2k+n}{\lambda}\right)}{\Gamma(k+1)\Gamma\left(\frac{2k+n}{2}\right)} \pi^{2k}.$$

Using the following asymptotic formula for the Gamma function (see [1])

$$\Gamma(az+b) = \sqrt{2\pi} e^{-az} (az)^{az+b-\frac{1}{2}} \left(1 + o\left(\frac{1}{z}\right)\right)$$

which holds for $z \in \mathbb{C}$ such that $|\arg(z)| < \pi$ and $a > 0$, and where

$$o\Big(\frac{1}{z}\Big) \to 0 \qquad \text{for} \qquad |z| \to +\infty,$$

we can write

$$\begin{aligned}
\Big|\frac{A_k}{A_{k+1}}\Big| &= \frac{\Gamma\Big(\frac{2k+n}{\lambda}\Big)}{\Gamma\Big(\frac{2k+n+2}{\lambda}\Big)} \frac{(k+1)(k+\frac{n}{2})}{\pi^2} \\
&= \Bigg(\frac{\lambda}{2k+n}\Bigg)^{\frac{2}{\lambda}} \frac{(2k)^2}{(2\pi)^2}(1+\frac{1}{k})(1+\frac{n}{2k})(1+o(\frac{1}{k}) \\
&= \Bigg(\frac{\lambda}{1+\frac{n}{2k}}\Bigg)^{\frac{2}{\lambda}} \frac{(2k)^{2-\frac{2}{\lambda}}}{(2\pi)^2}\Big(1+o(\frac{1}{k})\Big).
\end{aligned}$$

Thus the radius of convergence is now obtained from

$$R_\lambda = \sqrt{\lim_{k\to+\infty}\Big|\frac{A_k}{A_{k+1}}\Big|} = \begin{cases} 0 & \text{for} \quad 0 < \lambda < 1, \\ 1/2\pi & \text{for} \quad \lambda = 1, \\ +\infty & \text{for} \quad \lambda > 1. \end{cases}$$

It follows that the Maclaurin series (11) converges for $\lambda > 1$ for any $\omega \in \mathbb{R}^n$ and for $\lambda = 1$ for any $\omega \in \mathbb{R}^n$ such that $\|\omega\| < 1/2\pi$.

Example 1. For $\lambda = 2$ we have

$$\hat{G}_2(\omega) = \pi^{\frac{n}{2}} \sum_{k=0}^{+\infty}(-1)^k \frac{(\pi\,\|\omega\|)^{2k}}{k!} = \pi^{\frac{n}{2}} e^{-(\pi\|\omega\|)^2}.$$

Example 2. For $\lambda = 1$ and $n = 1$ we have

$$\begin{aligned}
\hat{G}_1(\omega) &= 2\pi^{\frac{1}{2}} \sum_{k=0}^{+\infty}(-1)^k \frac{\Gamma(2k+1)}{\Gamma(k+1)\Gamma\Big(\frac{2k+1}{2}\Big)}(\pi\omega)^{2k} \\
&= 2\sum_{k=0}^{+\infty}(-1)^k(2\pi\omega)^{2k} \\
&= \frac{2}{1+(2\pi\omega)^2},
\end{aligned}$$

where the last expression is valid for any $\omega \in \mathbb{R}$.

4. Other Extensions

We consider the following more general form of (6) given by

$$GQ_\lambda(x) = e^{-(x^t\,Q\,x)^{\frac{\lambda}{2}}}, \tag{12}$$

where Q is a real positive-definite symmetric matrix of order n. Then its Fourier transform is given by

$$\widehat{GQ}_\lambda(\omega) = \int e^{-(x^t\,Q\,x)^{\frac{\lambda}{2}}} e^{-2\pi i \omega^t x} dx. \tag{13}$$

We can write $Q = P^t\,D\,P$, where P is an orthogonal matrix and $D = \mathrm{diag}(d_1, d_2, \cdots, d_n)$ is a diagonal matrix with strictly positive real entries $d_i > 0$. We have

$$Q = P^t\,D\,P = P^t\,D^{\frac{1}{2}}\,D^{\frac{1}{2}}\,P, \tag{14}$$

where $D^{\frac{1}{2}} = \mathrm{diag}(d_1^{\frac{1}{2}}, d_2^{\frac{1}{2}}, \cdots, d_n^{\frac{1}{2}})$, then

$$x^t\,Q\,x = x^t\,P^t\,D^{\frac{1}{2}}\,D^{\frac{1}{2}}\,P\,x. \tag{15}$$

Let us set $y = D^{\frac{1}{2}}\,P\,x$ or $x = P^t\,D^{-\frac{1}{2}}\,y$, then

$$\omega^t x = \omega^t P^t D^{-\frac{1}{2}} y = \left(D^{-\frac{1}{2}} P \omega\right)^t y, \tag{16}$$

and (13) becomes

$$\widehat{GQ}_\lambda(\omega) = \frac{1}{\left|\mathrm{Det}(D^{\frac{1}{2}}\,P)\right|} \hat{G}_\lambda(D^{-\frac{1}{2}} P \omega), \tag{17}$$

since $dy = \left|\mathrm{Det}(D^{\frac{1}{2}}\,P)\right| dx$ and where $\mathrm{Det}(D^{\frac{1}{2}}\,P) = \left[\prod_{k=1}^n d_k\right]^{\frac{1}{2}} \mathrm{Det}(P)$.

Finally we could also use the p-norm $\|x\|_p$ in $\mathbb{R}^n$ given by

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$$

and define another generalized Gaussian distribution by the formula

$$G_{\lambda,p}(x) = e^{-\|x\|_p^\lambda}.$$

Formulas like (10) and (11) are not yet established for this form of GG.

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