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ON NON-STANDARD n-NORM ON SOME SEQUENCE SPACES

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Abstract: In this paper, we construct some difference sequence spaces which we call the spaces of $\Delta^s_{(r)}$-convergent, $\Delta^s_{(r)}$-null and $\Delta^s_{(r)}$-bounded sequences with respect to n-norm on a real linear space X. We study these spaces by defining non-standard n-norm and $(n-r)$-norm for every $r = 1, 2 \ldots, n-1$. We show that under certain cases, convergence and completeness in the n-norm is equivalent to those in the $(n-r)$-norm. We also prove the fixed point theorem for these spaces, which are n-Banach spaces.

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1. Introduction

Let w, ℓ_∞, c and c_0 denote the spaces of all, bounded, convergent and null sequences $x = (x_k)$ with complex terms respectively, normed by $\|x\|_\infty = \sup\limits_k |x_k|$, where $k \in N = \{1, 2, 3, \ldots\}$ – the set of positive integers.

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Kizmaz [12] defined the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ by $Z(\Delta) = \{x = (x_k) : \Delta x \in Z\}$, for $Z = \ell_\infty$, c and c_0, where Δx=(Δx_k) = $(x_k - x_{k+1})$ for all $k \in N$ and these spaces are Banach spaces with norm $\|x\|= |x_1|+\|\Delta x\|_\infty$.

Mikail Et and Rifat Colak [3] defined further generalization of the difference sequence spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ by $Z(\Delta^m)$=$\{x = (x_k) : \Delta^m x \in Z\}$, for $Z = \ell_\infty$, c and c_0, where $m \in N$, $\Delta^0 x = (x_k)$, Δx=$(\Delta x_k) = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ for all $k \in N$ and which is equivalent to the binomial representation $\Delta^m x_k$=$\sum\limits_{v=0}^{m}(-1)^v \binom{m}{v} x_{k+v}$ and these spaces are Banach spaces with norm $\|x\|_{\Delta^m} = \sum\limits_{i=1}^{m}|x_i|+\|\Delta^m x\|_\infty$=$\sum\limits_{i=1}^{m}|x_i|+\sup\limits_k|\Delta^m x_k|$.

B.C. Tripathy and Ayhan Esi [14] defined another type of generalization of the difference sequence spaces $\ell_\infty(\Delta_r)$, $c(\Delta_r)$ and $c_0(\Delta_r)$ by $Z(\Delta_r) = \{x = (x_k) : \Delta_r x \in Z\}$, for $Z = \ell_\infty, c$ and c_0, where $r \in N$, $\Delta_r x = (\Delta_r x_k) = (x_k - x_{k+r})$ for all $k \in N$ and these spaces are Banach spaces with norm $\|x\|_{\Delta_r}$ $= \sum\limits_{i=1}^{r}|x_i| + \|\Delta_r x\|_\infty = \sum\limits_{i=1}^{r}|x_i| + \sup\limits_k|\Delta_r x_k|$. For $r = 1$, we get the spaces $\ell_\infty(\Delta_r) = \ell_\infty(\Delta)$, $c(\Delta_r) = c(\Delta)$ and $c_0(\Delta_r) = c_0(\Delta)$.

Let r, s be non-negative integers then Hemen Dutta [1] defined the difference sequence spaces $\ell_\infty(\Delta^s_{(r)})$, $c(\Delta^s_{(r)})$ and $c_0(\Delta^s_{(r)})$ by $Z(\Delta^s_{(r)})$=$\{x = (x_k) : \Delta^s_{(r)} x \in Z\}$, for $Z = \ell_\infty$, c and c_0, where $\Delta(r)^s x = (\Delta^s_{(r)} x_k) = (\Delta^{s-1}_{(r)} x_k - \Delta^{s-1}_{(r)} x_{k-r})$ and $\Delta^0_{(r)} x = (x_k)$ for all $k \in N$ and which is equivalent to the binomial representation $\Delta^s_{(r)} x_k = \sum\limits_{v=0}^{s}(-1)^v \binom{s}{v} x_{k-rv}$. For $s = 1$, we get the spaces $\ell_\infty(\Delta^s_{(r)})$ $= \ell_\infty(\Delta_{(r)})$, $c(\Delta^s_{(r)}) = c(\Delta_{(r)})$ and $c_0(\Delta^s_{(r)}) = c_0(\Delta_{(r)})$ (see [2]). For $r = s = 1$, we get the spaces $\ell_\infty(\Delta^s_{(r)}) = \ell_\infty(\Delta)$, $c(\Delta^s_{(r)}) = c(\Delta)$ and $c_0(\Delta^s_{(r)}) = c_0(\Delta)$.

The concept of 2-normed spaces was first developed by Gähler [4], [5], [6], [7] in the mid of 1960's, while that of n-normed space can be found in Misiak [13]. A systematic development of n-normed spaces has been extensively made by many authors including Gunawan [8], [9], Gunawan and Mashadi [11].

Let $n \in N$ and X be a real linear space of dimension $d \geq n \geq 2$. A real valued function $\|\bullet, \bullet, \ldots, \bullet\|$: $X^n \to R$ satisfying the following four properties:

(nN_1) $\|x_1, x_2, \ldots, x_n\|= 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent vectors.

(nN_2) $\|x_1, x_2, \ldots, x_n\|= \|x_{j_1}, x_{j_2}, \ldots, x_{j_n}\|$ for every permutation $(j_1, j_2, \ldots, j_n)$ of $(1, 2, \ldots, n)$, i.e. $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$.

(nN_3) $\|\alpha x_1, x_2, \ldots, x_n\|= |\alpha|\ \|x_1, x_2, \ldots, x_n\|$ for all $\alpha \in R$.

(nN_4) $\|x+x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$ for all $x, x', x_2, \ldots, x_n \in X$, is called an n-norm on X and the pair $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is called linear n-normed space.

A trivial example of an n-normed space is $X = R^n$ equipped with the following Euclidean n-norm:

$$\|x_1, x_2, \ldots, x_n\|_E = |\det(x_{ij})| = \text{abs}\left(\begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{vmatrix} \right),$$

where $x_i = (x_{i1}, \ldots, x_{in}) \in R^n$ for each $i = 1, 2, \ldots, n$.

Example 1. Consider the linear space P_m of real polynomials of degree $\leq m$ on the interval $[0,1]$. Let $\{x_i\}_{i=0}^{nm}$ be $nm+1$ arbitrary but distinct fixed points in $[0,1]$. For $f_1, f_2, \ldots, f_n$ in P_m, let us define

$$\|f_1, f_2, \ldots, f_n\| = \begin{cases} 0, & \text{if } f_1, f_2, \ldots, f_n \text{ are linearly dependent,} \\ \sum_{i=0}^{nm} |f_1(x_i) f_2(x_i) \ldots f_n(x_i)|, & \text{if } f_1, f_2, \ldots, f_n \text{ are linearly independent.} \end{cases}$$

Then $\|\bullet, \bullet, \ldots, \bullet\|$ is an n-norm on P_m.

Proof. If $f_1, f_2, \ldots, f_n$ are linearly dependent, then $\|f_1, f_2, \ldots, f_n\| = 0$. Conversely assume

$$\sum_{i=0}^{nm} |f_1(x_i) f_2(x_i) \ldots f_n(x_i)| = 0.$$

This implies that

$$f_1(x_i) f_2(x_i) \ldots f_n(x_i) = 0 \text{ at } nm+1 \text{ distinct points.}$$

Since the degree of each $f_i \leq m$, we must have at least one $f_i = 0$. Thus

$$\|f_1, f_2, \ldots, f_n\| = 0 \text{ if and only if } f_1, f_2, \ldots, f_n \text{ are linearly dependent.}$$

Other properties for n-norm can be easily verified. □

If $(X, \|\bullet, \bullet, \dots, \bullet\|)$ is an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \dots, a_n\}$ be a linearly independent set in X, then the following function $\|\bullet, \bullet, \dots, \bullet\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \dots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \dots, x_{n-1}, a_i\| : i = 1, 2, \dots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \dots, a_n\}$ and this is known as derived $(n-1)$-norm on X (see [11]).

The standard n-norm on X, a real inner product space of dimension $d \geq n$ is as follows:

$$\|x_1, x_2, \dots, x_n\|_S = \begin{vmatrix} \langle x_1, x_1\rangle & \dots & \langle x_1, x_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1\rangle & \dots & \langle x_n, x_n\rangle \end{vmatrix}^{\frac{1}{2}},$$

where $\langle \bullet, \bullet\rangle$ denotes the inner product on X. If $X = R^n$, then this n-norm is exactly the same as the Euclidean n-norm $\|\bullet, \bullet \dots, \bullet\|_E$ mentioned earlier. For $n = 1$, this n-norm is the usual norm $\|x_1\|_S = \langle x_1, x_1\rangle^{\frac{1}{2}}$.

A sequence (x_k) in an n-normed space $(X, \|\bullet, \bullet, \dots, \bullet\|)$ is said to *converge* to some $L \in X$ in the n-norm if $\lim_{k\to\infty} \|x_k - L, w_2, w_3 \dots, w_n\| = 0$, for every $w_2, w_3 \dots, w_n \in X$.

A sequence (x_k) in an n-normed space $(X, \|\bullet, \bullet, \dots, \bullet\|)$ is said to be *Cauchy* with respect to the n-norm if $\lim_{k,l\to\infty} \|x_k - x_l, w_2, w_3 \dots, w_n\| = 0$, for every $w_2, w_3 \dots, w_n \in X$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Now we state the following three usefull results as Lemmas which can be found in [11].

Lemma 1. *Every n-normed space is an $(n-r)$-normed space for all $r = 1, 2, \dots, n-1$. In particular, every n-normed space is a normed space.*

Lemma 2. *A standard n-normed space is complete if and only if it is complete with respect to the usual norm $\|\bullet\|_S = <\bullet, \bullet>^{\frac{1}{2}}$.*

Lemma 3. *On a standard n-normed space X, the derived $(n-1)$-norm $\|., \dots, .\|_\infty$, defined with respect to orthonormal set $\{e_1, e_2, \dots, e_n\}$, is equivalent to the standard $(n-1)$-norm $\|\bullet, \bullet, \dots, \bullet\|_S$. Precisely, we have*

$$\|x_1, x_2, \dots, x_{n-1}\|_\infty \leq \|x_1, x_2, \dots, x_{n-1}\|_S \leq \sqrt{n}\|x_1, x_2, \dots, x_{n-1}\|_\infty$$

for all $x_1, x_2, \ldots, x_{n-1}$, *where* $\|x_1, x_2, \ldots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \ldots, x_{n-1}, e_i\|_S : i = 1, 2, \ldots, n\}$

Let $(X, \|\bullet, \bullet, \ldots, \bullet\|_X)$ be an n-normed space and $w(n-X)$ denotes X-valued sequence space. Let r and s be non-negative integers, then we define the following sequence spaces:

$$c_0(X, \Delta^s_{(r)}) = \{(x_k) \in w(n-X) : \lim_{k\to\infty} \|\Delta^s_{(r)} x_k, z_1, \ldots, z_{n-1}\|_X = 0, \quad \text{for every } z_1, \ldots, z_{n-1} \in X\},$$

$$c(X, \Delta^s_{(r)}) = \{(x_k) \in w(n-X) : \lim_{k\to\infty} \|\Delta^s_{(r)} x_k - L, z_1, \ldots, z_{n-1}\|_X = 0, \quad \text{for every } z_1, \ldots, z_{n-1} \in X \text{ and some } L\},$$

$$\ell_\infty(X, \Delta^s_{(r)}) = \{(x_k) \in w(n-X) : \sup_k \|\Delta^s_{(r)} x_k, z_1, \ldots, z_{n-1}\|_X < \infty, \quad \text{for every } z_1, \ldots, z_{n-1} \in X\}.$$

We call these spaces $c(X, \Delta^s_{(r)})$, $c_0(X, \Delta^s_{(r)})$ and $\ell_\infty(X, \Delta^s_{(r)})$ as the spaces of $\Delta^s_{(r)}$-convergent, $\Delta^s_{(r)}$-null and $\Delta^s_{(r)}$-bounded sequences with respect to n-norm $\|\bullet, \bullet, \ldots, \bullet\|_X$ on the space X.

In the above definition of spaces, n-norm $\|\bullet, \bullet, \ldots, \bullet\|_X$ on X is either a standard n-norm or a non-standard n-norm. In general, we write $\|\bullet, \bullet, \ldots, \bullet\|_X$ and for standard case we write $\|\bullet, \bullet, \ldots, \bullet\|_S$. For derived norms we use $\|\bullet, \bullet, \ldots, \bullet\|_\infty$.

It is obvious that $c_0(X, \Delta^s_{(r)}) \subset c(X, \Delta^s_{(r)}) \subset \ell_\infty(X, \Delta^s_{(r)})$.

2. Main Results

In this section we investigate the main results of this article involving the sequence spaces $c_0(X, \Delta^s_{(r)})$, $c(X, \Delta^s_{(r)})$ and $\ell_\infty(X, \Delta^s_{(r)})$.

Theorem 1. *The spaces* $c_0(X, \Delta^s_{(r)})$, $c(X, \Delta^s_{(r)})$ *and* $\ell_\infty(X, \Delta^s_{(r)})$ *are linear spaces.*

Proof. The proof of this theorem can be proved very easily. □

Theorem 2. *Let* Y *be any one of the spaces* $c_0(X, \Delta^s_{(r)})$, $c(X, \Delta^s_{(r)})$ *and* $\ell_\infty(X, \Delta^s_{(r)})$. *We define the following function* $\|\bullet, \bullet, \ldots, \bullet\|_Y$ *on* $Y \times \ldots \times Y$ *(n-factors) by*

$$\|x^1, x^2, \ldots, x^n\|_Y = 0, \quad \text{if } x^1, x^2, \ldots, x^n \text{ are linearly dependent and}$$

$$\|x^1, x^2, \ldots, x^n\|_Y = \sup_{k\geq 1} \|\Delta^s_{(r)} x^1_k, z_1, \ldots, z_{n-1}\|_X,$$

for every $z_1, \ldots, z_{n-1} \in X$, *if* $x^1, x^2, \ldots, x^n$ *are linearly independent.*

Then

$$\|\bullet, \bullet, \ldots, \bullet\|_Y \text{ is a non-standard } n\text{-norm on } Y. \tag{2.1}$$

Proof. If $x^1, x^2, \ldots, x^n$ are linearly dependent, then $\|x^1, x^2, \ldots, x^n\|_Y = 0$. Conversely assume that $\|x^1, x^2, \ldots, x^n\|_Y = 0$. Then using (2.1), we have

$$\sup_{k\geq 1} \|\Delta^s_{(r)} x^1_k, z_1, \ldots, z_{n-1}\|_X = 0, \text{ for every } z_1, \ldots, z_{n-1} \in X.$$

This implies that

$$\|\Delta^s_{(r)} x^1_k, z_1, \ldots, z_{n-1}\|_X = 0, \text{ for every } z_1, \ldots, z_{n-1} \in X \text{ and } k \in N.$$

Hence we must have $\Delta^s_{(r)} x^1_k = 0$ for all $k \in N$. Let $k = 1$, then $\Delta^s_{(r)} x^1_k = \sum_{v=0}^{s} (-1)^v \binom{s}{v} x^1_{1-rv} = 0$ and so $x^1_1 = 0$, by putting $x^1_{1-rv} = 0$ for $v = 1, 2, \ldots, s$. Similarly taking $k = 2$, we have $x^1_2 = 0$ and so on. Proceeding in this way inductively, we get $x^1 = \theta$ and hence $x^1, x^2, \ldots, x^n$ are linearly dependent.

Since $\|\bullet, \bullet, \ldots, \bullet\|_X$ is an n-norm on X and

$$\|x^2, x^1, \ldots, x^n\|_Y = \sup_{k\geq 1} \|z_1, \Delta^s_{(r)} x^1_k, \ldots, z_{n-1}\|_X,$$

etc., it is clear that $\|x^1, x^2, \ldots, x^n\|_Y$ is invariant under permutation.

Let $\alpha \in R$ be any element. If $\alpha x^1, x^2, \ldots, x^n$ are linearly dependent, then it is obvious that

$$\|\alpha x^1, x^2, \ldots, x^n\|_Y = |\alpha| \|x^1, x^2, \ldots, x^n\|_Y.$$

Otherwise,

$$\|\alpha x^1, x^2, \ldots, x^n\|_Y = \sup_{k\geq 1} \|\Delta^s_{(r)} \alpha x^1_k, z_1, \ldots, z_{n-1}\|_X = |\alpha| \|x^1, x^2, \ldots, x^n\|_Y.$$

Finally, let $x^1 = (x^1_k)$ and $y^1 = (y^1_k) \in Y$, then clearly

$$\|x^1 + y^1, x^2, \ldots, x^n\|_Y \leq \|x^1, x^2, \ldots, x^n\|_Y + \|y^1, x^2, \ldots, x^n\|_Y.$$

Thus we can conclude that $\|\bullet, \bullet, \ldots, \bullet\|_Y$ is an n-norm on Y. □

The following corollary is a consequence of Lemma 1.

Corollary 3. *The spaces $c_0(X,\Delta^s_{(r)})$, $c(X,\Delta^s_{(r)})$ and $\ell_\infty(X,\Delta^s_{(r)})$ are $(n-r)$-normed spaces for all $r=1,2,\dots,n-1$. In particular these spaces are normed spaces.*

In fact we can define $(n-r)$-norm on the spaces $c_0(X,\Delta^s_{(r)})$, $c(X,\Delta^s_{(r)})$ and $\ell_\infty(X,\Delta^s_{(r)})$ for all $r=1,2,\dots,n-1$. This is the aim of the next theorem.

Theorem 4. *Let Y be any one of the spaces $c_0(X,\Delta^s_{(r)})$, $c(X,\Delta^s_{(r)})$ and $\ell_\infty(X,\Delta^s_{(r)})$. We define the following function $\|\bullet,\bullet,\dots,\bullet\|_Y$ on $Y\times\dots\times Y$ ($(n-r)$-factors) by*
$\|x^1,x^2,\dots,x^{n-r}\|_Y=0$ if $x^1,x^2,\dots,x^{n-r}$ are linearly dependent and
$\|x^1,x^2,\dots,x^{n-r}\|_Y=\sup\limits_{k\geq 1}\|\Delta^s_{(r)}x^1_k,z_1,\dots,z_{n-r-1}\|_X$, for every $z_1,\dots,$ $z_{n-r-1}\in X$, if $x^1,x^2,\dots,x^{n-r}$ are linearly independent.
Then

$$\|\bullet,\bullet,\dots,\bullet\|_Y \quad \text{is a non-standard } (n-r)\text{-norm on } Y \text{ for all } r=1,2,\dots,n-1. \tag{2.2}$$

Proof. The proof is similar to that of Theorem 2. □

Remark 1. In (2.2) above, for any linearly independent set $\{a_1,a_2,\dots,a_n\}$,

$$\|\Delta^s_{(r)}x^1_k,z_1,\dots,z_{n-r-1}\|_\infty=\max\{\|\Delta^s_{(r)}x^1_k,z_1,\dots,z_{n-r-1},a_{i_1},a_{i_2},\dots,a_{i_r}\|_X\},$$

$\{i_1,i_2,\dots,i_r\}\subseteq\{1,2,\dots,n\}$ is an derived $(n-r)$-norm on X for each $k\in N$ and for all $r=1,2,\dots,n-1$.

Theorem 5. *If X is an n-Banach space, then the spaces $c_0(X,\Delta^s_{(r)})$, $c(X,\Delta^s_{(r)})$ and $\ell_\infty(X,\Delta^s_{(r)})$ are n-Banach spaces under the norm (2.1).*

Proof. Let Y be any one of the spaces $c_0(X,\Delta^s_{(r)})$, $c(X,\Delta^s_{(r)})$ and $\ell_\infty(X,\Delta^s_{(r)})$.
Let (x^i) be any Cauchy sequence in Y. Let $\varepsilon>0$ be given. Then there exists a positive integer n_0 such that $\|x^i-x^j,u^2,\dots,u^n\|_Y<\varepsilon$, for every $u^2,\dots,u^n\in Y$ and $i,j\geq n_0$. Using the definition of n-norm, we get

$$\sup_k\|\Delta^s_{(r)}(x^i_k-x^j_k),z_1,\dots,z_{n-1}\|_X<\varepsilon, \text{ for all } i,j\geq n_0.$$

It follows that

$$\|\Delta^s_{(r)}(x^i_k-x^j_k),z_1,\dots,z_{n-1}\|_X<\varepsilon, \text{ for all } i,j\geq n_0 \text{ and } k\in N.$$

Hence $\Delta^s_{(r)}(x^i_k)$ is a Cauchy sequence in X for all $k \in N$ and so it is convergent in X for all $k \in N$, since X is an n-Banach space. For simplicity, let $\lim_{i\to\infty} \Delta^s_{(r)} x^i_k = y_k$, for each $k \in N$. Putting $k = 1, 2, 3, \dots$, we can easily argue that $\lim_{i\to\infty} x^i_k = x_k$, exists for each $k \in N$. Since n-norm is a continuous function, we can have

$$\sup_k \|\Delta^s_{(r)}(x^i_k - x_k), z_1, \dots, z_{n-1}\|_X < \varepsilon, \text{ for all } i \geq n_0 \text{ and } j \to \infty.$$

It follows that $(x^i - x) \in Y$. Since $(x^i) \in Y$ and Y is a linear space, so we have $x = x^i - (x^i - x) \in Y$. This completes the proof of the theorem. □

Using Lemma 2, we get the next corollary.

Corollary 6. *If X is a Banach space under the standard n-norm, then the spaces $c_0(X, \Delta^s_{(r)})$, $c(X, \Delta^s_{(r)})$ and $\ell_\infty(X, \Delta^s_{(r)})$ are n-Banach spaces.*

Theorem 7. *Let Y be any one of the spaces $c_0(X, \Delta^s_{(r)})$, $c(X, \Delta^s_{(r)})$ and $\ell_\infty(X, \Delta^s_{(r)})$. If (x^i) converges to x in Y in the n-norm (2.1), then (x^i) also converges to x in the $(n-1)$-norm (2.2).*

Proof. Let (x^i) converges to x in Y in the n-norm (2.1), then

$$\|x^i - x, u^1, \dots, u^{n-1}\|_Y \to 0 \text{ as } i \to \infty \text{ for every } u^1, \dots, u^{n-1} \in Y.$$

Using definition of n-norm (2.1), we get

$$\sup_k \|\Delta^s_{(r)}(x^i_k - x_k), z_1, \dots, z_{n-1}\|_X \to 0 \text{ as } i \to \infty.$$

Then for any linearly independent set $\{a_1, a_2, \dots, a_n\}$ we have

$$\sup_k \|\Delta^s_{(r)}(x^i_k - x_k), z_1, \dots, z_{n-2}, a_j\|_X \to 0 \text{ as } i \to \infty$$

and for each $j = 1, 2, \dots, n$. This implies that

$$\sup_k \|\Delta^s_{(r)}(x^i_k - x_k), z_1, \dots, z_{n-2}\|_\infty \to 0 \text{ as } i \to \infty.$$

Thus $\|x^i - x, u^1, \dots, u^{n-2}\|_Y \to 0$ as $i \to \infty$ for every $u^1, \dots, u^{n-2} \in Y$.
Hence (x^i) converges to x in the $(n-1)$-norm. □

If X is equipped with standard n-norm and derived norm is with respect to an orthonormal set, then the converse of the above theorem is valid.

Theorem 8. *Let X be a standard n-normed space and the derived $(n-1)$-norm on X is with respect to an orthonormal set. Let Y be any one of the spaces $c_0(X, \Delta^s_{(r)})$, $c(X, \Delta^s_{(r)})$ and $\ell_\infty(X, \Delta^s_{(r)})$. If (x^i) convergent in Y in the n-norm* (2.1) *if and only if (x^i) is convergent in Y in the $(n-1)$-norm* (2.2).

Proof. In view of the above theorem, it is enough to prove that (x^i) is convergent in the $(n-1)$-norm which implies (x^i) is convergent in the n-norm.

Let (x^i) converge to x in Y in the $(n-1)$-norm. Then

$$\|x^i - x, u^1, \ldots, u^{n-2}\|_Y \to 0 \text{ as } i \to \infty \text{ for every } u^1, \ldots, u^{n-2} \in Y.$$

Using the definition of $(n-1)$-norm (2.2), we get

$$\sup_k \|\Delta^s_{(r)}(x^i_k - x_k), z_1, \ldots, z_{n-2}\|_\infty \to 0 \text{ as } i \to \infty.$$

Now we have

$$\|\Delta^s_{(r)}(x^i_k - x_k), z_1, \ldots, z_{n-1}\|_S \leq \|\Delta^s_{(r)}(x^i_k - x_k), z_1, \ldots, z_{n-2}\|_S \ \|z_{n-1}\|_S,$$

where $\|\bullet, \bullet, \ldots, \bullet\|_S$ and $\|\bullet\|_S$ on the right hand side denote the standard $(n-1)$-norm and the usual norm on X respectively. Since derived $(n-1)$-norm on X is with respect to an orthonormal set, using Lemma 3, we have $\|\Delta^s_{(r)}(x^i_k - x_k), z_1, \ldots, z_{n-1}\|_S \leq \sqrt{n}\|\Delta^s_{(r)}(x^i_k - x_k), z_1, \ldots, z_{n-2}\|_\infty \ \|z_{n-1}\|_S$ and in this case $\|\bullet, \bullet, \ldots, \bullet\|_\infty$ on the right hand side is the derived $(n-1)$-norm which we use to define the $(n-1)$-norm (2.2). Hence

$$\begin{aligned}\sup_k \|\Delta^s_{(r)}(x^i_k - x_k), z_1, \ldots, z_{n-1}\|_S \\ \leq \sup_k (\sqrt{n}\|\Delta^s_{(r)}(x^i_k - x_k), z_1, \ldots, z_{n-2}\|_\infty \ \|z_{n-1}\|_S).\end{aligned}$$

Thus we get $\|x^i - x, u^1, \ldots, u^{n-1}\|_Y \to 0$ as $i \to \infty$ for every $u^1, \ldots, u^{n-1} \in Y$. Therefore (x^i) converges to x in Y in the n-norm. □

Corollary 9. *Let X be a standard n-normed space and derived $(n-r)$-norms on X for all $r = 1, 2, \ldots, n-1$ are with respect to an orthonormal set. Let Y be any one of the spaces $c_0(X, \Delta^s_{(r)})$, $c(X, \Delta^s_{(r)})$ and $\ell_\infty(X, \Delta^s_{(r)})$. Then a sequence in Y is convergent in the n-norm if and only if it is convergent in the $(n-1)$-norm and by induction in the $(n-r)$-norm for all $r = 1, 2, \ldots, n-1$. In particular, a sequence in Y is convergent in the n-norm* (2.1) *if and only if it is convergent in the norm* (2.2) *for $r = n-1$.*

Theorem 10. *Let X be a standard n-normed space and derived $(n-r)$-norms on X for all $r = 1, 2, \ldots, n-1$ are with respect to an orthonormal set. Let Y be any one of the spaces $c_0(X, \Delta^s_{(r)})$, $c(X, \Delta^s_{(r)})$ and $\ell_\infty(X, \Delta^s_{(r)})$. Then Y is complete with respect to the n-norm if and only if it is complete with respect to the $(n-1)$-norm. By induction, Y is complete with respect to the n-norm* (2.1) *if and only if it is complete with respect to the norm* (2.2) *for $r = n-1$.*

Proof. By replacing the phrases "(x^i) converges to x" with "(x^i) is Cauchy" and "$x^i - x$" with "$x^i - x^j$" we see that the analogues of Theorem 7, Theorem 8 and Corollary 9 hold for Cauchy sequences. This completes the proof. □

For the next theorem, we shall consider that X is a standard n-normed space and derived $(n-r)$-norms on X for all $r = 1, 2, \ldots, n-1$ are with respect to an orthonormal set. Also we assume Y to be any one of the spaces $c_0(X, \Delta^s_{(r)})$, $c(X, \Delta^s_{(r)})$ and $\ell_\infty(X, \Delta^s_{(r)})$.

Theorem 11. (Fixed Point Theorem) *Let Y be an n-Banach space under the n-norm* (2.1) *and T be a contractive mapping of Y into itself, that is, there exists a constant $C \in (0,1)$ such that $\|Ty^1 - Tz^1, x^2, \ldots, x^n\|_Y \leq C\|y^1 - z^1, x^2, \ldots, x^n\|_Y$, for all $y^1, z^1, x^2, \ldots, x^n$ in Y. Then T has a unique fixed point in Y.*

Proof. If we can show that T is also contractive with respect to norm (2.2) for $r = n$-1, then we are done by Theorem 10 and the fixed point theorem for Banach spaces. Now by hypothesis $\|Ty^1 - Tz^1, x^2, \ldots, x^n\|_Y \leq C\|y^1 - z^1, x^2, \ldots, x^n\|_Y$, for all $y^1, z^1, x^2, \ldots, x^n$ in Y. Using (2.1), we get

$$\sup_k \|\Delta^s_{(r)}(Ty^1_k - Tz^1_k), z_1, \ldots, z_{n-1}\|_S \leq C \sup_k \|\Delta^s_{(r)}(y^1_k - z^1_k), z_1, \ldots, z_{n-1}\|_S.$$

Then for an orthonormal set $\{e_1, e_2, \ldots, e_n\}$ in X, we get

$$\sup_k \|\Delta^s_{(r)}(Ty^1_k - Tz^1_k), e_{i_1}, \ldots, e_{i_{n-1}}\|_S \leq C \sup_k \|\Delta^s_{(r)}(y^1_k - z^1_k), e_{i_1}, \ldots, e_{i_{n-1}}\|_S,$$

for all y^1, z^1 in Y and $\{i_1, i_2, \ldots, i_{n-1}\} \subseteq \{1, 2, \ldots, n-1\}$. Thus we have $\sup_k \|\Delta^s_{(r)}(Ty^1_k - Tz^1_k)\|_\infty \leq C \sup_k \|\Delta^s_{(r)}(y^1_k - z^1_k)\|_\infty$, where derived norms are defined as in Remark 1. Hence $\|Ty^1 - Tz^1\|_Y \leq C\|y^1 - z^1\|_Y$, for all y^1, z^1 in Y. This completes the proof. □

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