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QUANTUM ENTANGLEMENT AND INFORMATION: THE EFFECT OF ENTROPY REVISITED

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Abstract: In non-extensive statistical mechanics [14], it is a nonsense statement to say that the entropy of a system is extensive (or not), without mentioning a law of composition of its elements. In this theory quantum correlations might be perceived through quantum information process. This article, that is an extension of recent work [4], is a comparative study between the entropies of Von Neumann and of Tsallis, with some implementations of the effect of entropy in quantum entanglement, important as a process for transmission of quantum information. We consider two factorized (Fock number) states, which interact through a beam splitter bilinear Hamiltonian with two entries. This comparison showed us that the entropies of Tsallis and Von Neumann behave differently depending on the reflectance of the beam splitter.

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1. Introduction

Entanglement is a physical phenomenon of quantum systems that allows two or more objects, even separated spatially, to be somehow linked so that an object cannot be correctly described without mentioning his counterpart, see [5]. This

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concept is defined as a somewhat quality of any physical condition that cannot be represented as a simple tensor product of elements in Hilbert spaces, see [6].

Entanglement is critical in the development of quantum information processing [2], permitting increased transmission capacity of information and better efficiency [3], [1]; may be also the key to security in communication, via quantum cryptography [8], [13] and is one of the most remarkable features of quantum mechanics [7]. It has been under intense study, both in systems involving light and matter [10], [12], and in solids [17], [9]. In this article, in order to quantify the degree of entanglement for input fields that are in "Fock states", it is required the use of a measure of purity, and this measure will be made through the entropy. We shall proceed a comparison between the entropies of Von Neumann [11] and Tsallis [16], in such a way that it extends a previous recent work [4]. The classical Boltzmann entropy has its equivalent quantum known as the Von Neumann entropy [11] associated with the quantum state of a system described by the density operator ρ:

$$S(\rho) \equiv -\mathrm{Tr}\,[\rho \, \ln \, \rho]. \tag{1}$$

In 1988 a generalization was proposed by Tsallis [16]

$$S_q(\rho) \equiv -\frac{1 - \mathrm{Tr}\,[\rho^q]}{1-q} \tag{2}$$

and has been successfully applied to various problems as in the analysis of blackbody radiation, see [15].

With a purpose to make a sound comparison between Von Neumann and Tsallis entropies, we use a beam splitter, where the input beam of light, leads to an outflow of entangled states. A linear transformation is performed with the vector amplitudes into output spans, so when a single photon approaches a beam splitter, the wave splits into two: one part is reflected and the other is transmitted. The photon is then in a superposition of two different paths.

Looking at Figure 1, consider the operators belonging to the Hilbert space, the input field described by the operator a, the operator whose output is c, is superimposed with the input field b, the operator whose output is d, where the reflection coefficients (R) and transmission (T) with $R^2 + T^2 = 1$. The operators of the output field are:

$$c = BaB^{\dagger}, \quad \text{and} \quad d = BbB^{\dagger}, \tag{3}$$

where

$$B = e^{i\phi_0} \begin{pmatrix} \cos\theta e^{i\phi_T} & \sin\theta e^{i\phi_R} \\ -\sin\theta e^{-i\phi_R} & \cos\theta e^{-i\phi_T} \end{pmatrix}, \tag{4}$$

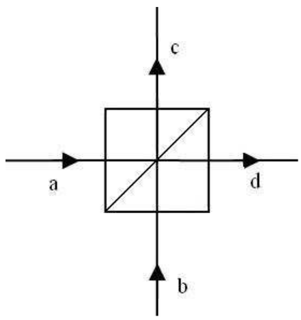


Figure 1: Scheme of a beam splitter

with $T=\cos\frac{\theta}{2}$, $R=\sin\frac{\theta}{2}$ and ϕ the phase difference between the fields reflected and transmitted. As

$$c = BaB^{\dagger} = e^{\theta/2\,(a^{\dagger}be^{i\phi}-ab^{\dagger}e^{i\phi})}a\;e^{-\theta/2\,(a^{\dagger}be^{i\phi}-ab^{\dagger}e^{-i\phi})}, \tag{5}$$

then $c = a\cos(\frac{\theta}{2}) - be^{i\theta}\sin(\frac{\theta}{2}) = Ta - Re^{i\theta}b$, and similarly $d = Tb + Re^{-i\theta}a$.

For two input states *(Fock states)* independent $|n_1n_2> = |n_1>_a|n_2>_b$ the superposition of these will be called $|\psi>$, given by

$$\begin{aligned}|\psi> = B|n_1n_2> &= \sum_{N_1N_2} < N_1N_2|B|n_1n_2> |N_1N_2> \\ &= \sum_{N_1N_2} B_{n_1n_2}^{N_1N_2}|N_1N_2>, \end{aligned}\tag{6}$$

where

$$\begin{aligned}B_{n_1n_2}^{N_1N_2} = e^{-i\theta(n_1-N_1)}\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}(-1)^{n_1-k}R^{n_1+n_2-k+l}T^{k+l}\frac{\sqrt{n_1!n_2!N_1!N_2!}}{k!(n_1-k)!l!(n_2-l)!} \\ \times\,\delta_{N_1,n_2+k-l}\delta_{N_2,n_1-k+l}, \end{aligned}\tag{7}$$

knowing that δ is the Kronecker function, thus

$$\begin{aligned}&B|n_1>_a|n_2>_b \\ &= \sum_{k=0}^{n_1}\sum_{l=0}^{n_2}\frac{(-1)^{n_1-k}e^{-i\theta(n_1N_1)}R^{n_1+n_2-k+l}T^{k+l}\sqrt{n_1!n_2!(n_2+k-l)(n_1-k+l)}}{k!(n_1-k)!l!(n_2-l)!} \\ &\qquad\times|n_2+k-l,n_1-k+l>. \end{aligned}\tag{8}$$

Finally it is obtained the entropies of: Von Neumann ($S(\rho_c)$) and Tsallis ($S_q(\rho_c)$), given by:

$$S(\rho_c) = -\sum_{N_1 N_2} |B_{n_1 n_2}^{N_1 N_2}|^2 \ \ln \ |B_{n_1 n_2}^{N_1 N_2}|^2, \tag{9}$$

$$S_q(\rho_c) = \frac{1}{q-1}\left(1 - \sum_{N_1 N_2} |B_{n_1 n_2}^{N_1 N_2}|^{2q}\right). \tag{10}$$

2. The Behavior of Entropies

By analyzing the behavior of the entropy of Von Neumann ($S(\rho_c)$) and Tsallis ($S_q(\rho_c)$) depending on the entropic index q such that with $q \in [0,1]$, variations in the correlation of variables can be detected. For the tests or measurements, it is used both that the limit $q \to 1$ of the Tsallis entropy which corresponds to the Von Neumann entropy, and q varying in $0,3$, $0,5$ and $0,8$ (choice of the entropic index was random for the entropy of Tsallis).

Note. All illustrations and/or graphs from now refer to as the entanglement of a number of input photons, depending on the extent of reflection. Taking $r = \sqrt{R}$ and $t = \sqrt{1-r^2}$ and the photon input as $n_1 = a$ and $n_2 = b$ and output $N_1 = n_2 + k - l = c$ and $N_2 = n_1 - k + l = d$.

Developing the equation (9), of Von Neumann, we have that:

$$S = -\sum_{c=0}^{a+b}\sum_{d=0}^{a+b}(|\sum_{k=0}^{a}\sum_{l=0}^{b}((-1)^{a-k}\ r^{a+b-k-l}\ t^{k+l}\frac{\sqrt{a!b!c!d!}}{k!(a-k)!l!(b-l)!}$$
$$\delta_{c,b+k-l}\ \delta_{d,a-k+l})|^2) * \ \log_2\ [|\sum_{k=0}^{a}\sum_{l=0}^{b}((-1)^{a-k}\ \ r^{a+b-k-l}\ t^{k+l}$$
$$\frac{\sqrt{a!b!c!d!}}{k!(a-k)!l!(b-l)!}\delta_{c,b+k-l}\ \delta_{d,a-k+l})|^2 + 1\times 10^{-29}]. \tag{11}$$

And for the equation (10), of Tsallis, we have that:

$$S_q = \frac{1}{q-1}(1 - \sum_{c=0}^{a+b}\sum_{d=0}^{a+b}(|\sum_{k=0}^{a}\sum_{l=0}^{b}((-1)^{a-k}\ r^{a+b-k-l}\ t^{k+l}$$
$$\frac{\sqrt{a!b!c!d!}}{k!(a-k)!l!(b-l)!}\delta_{c,b+k-l}\ \delta_{d,a-k+l})|^{2q})). \tag{12}$$

The equations (11) and (12) were implemented in tests of entanglement, using programs written in *Matlab*® and *Mathematica*®.

2.1. Single Entry

Photons injected into one of the entrances to the beam splitter, produce the graphs in Figure 2 for $n_1 = a = 10$ and $n_2 = b = 0$.

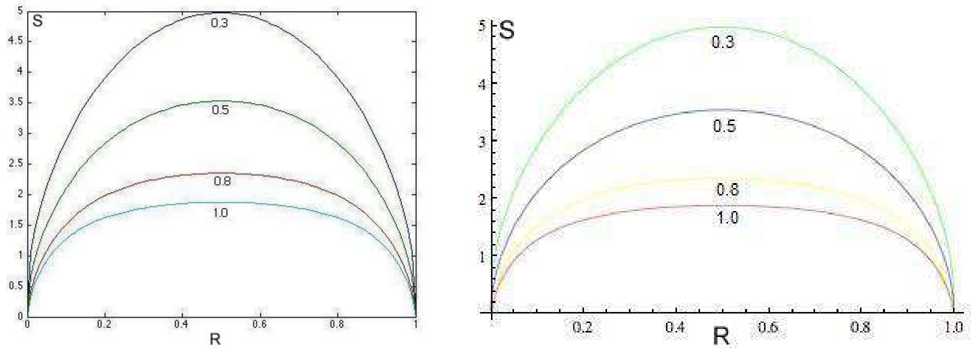


Figure 2: Entropy for inputs $a = 10$ and $b = 0$ photons, using *Matlab* and *Mathematica*, respectively

Comparing the plots in Figure 2 with figures obtained by [4], no difference is found. We continue to inject photons at a single entry, but with larger input values, $a = 500$ and $b = 0$.

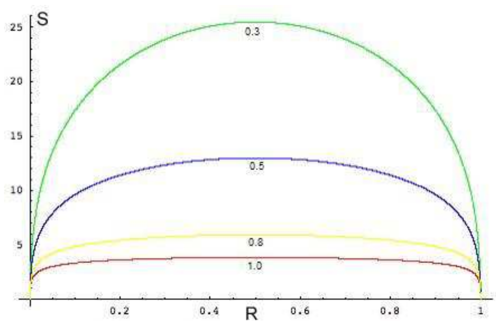


Figure 3: Entropy for inputs $a = 500$ and $b = 0$ photons

It can be seen in Figures 2 and 3, in both entropies, even for a number of input photons much larger, show us that the degree of entanglement is a convex function with maximum at $r = 0,5$.

2.2. Identical Entries

In this section we are testing with the injection of the same number of photons at the entrances of the beam splitter. Injecting the same number of photons in both entries of the beam splitter, it is obtained the following charts to $n_1 = a = 5$ and $n_2 = b = 5$.

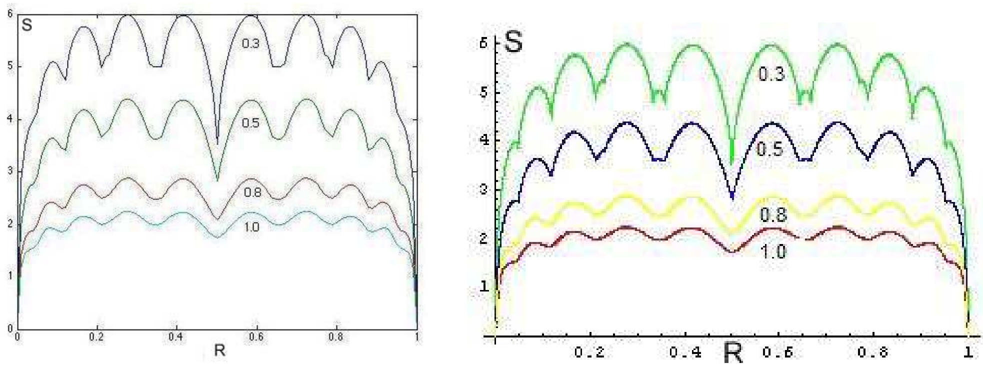


Figure 4: Entropy for inputs $a = b = 5$ photons, using *Matlab* and *Mathematica*, respectively

Comparing the plots in Figure 4 with figures obtained by [4], no difference is found.

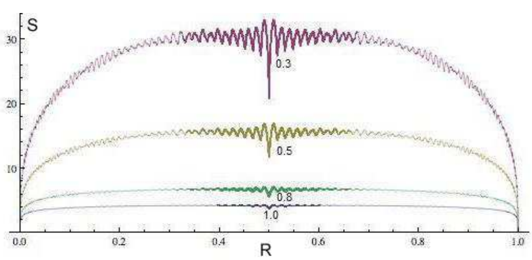


Figure 5: Entropy for entries of $a = b = 50$ photons

And it can be seen in the graphs of Figures 4 and 5 in which the number of incoming photons are equal, a fall of entanglement depending on the extent of reflection, so we have that the degree of entanglement is minimum at $r = 0.5$, for both entropies.

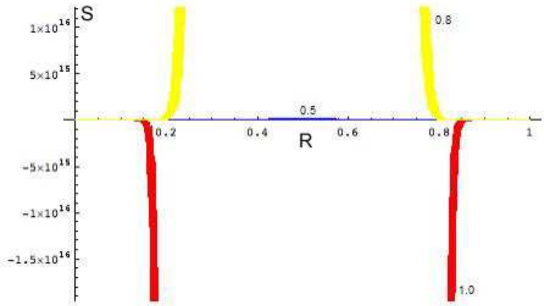
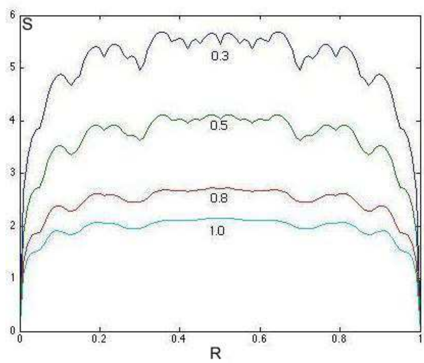


Figure 6: Entropy for entries of $a = b = 100$ photons

Watching now the graph in Figure 6 whose entries were larger, we observe that for the maximum Tsallis entropy entanglement still occurs in $r \sim 0,5$, but for the Von Neumann entropy, at $r = 0,5$, it is obtained the minimum entanglement.

2.3. Entries with Different Numbers of Photons

Injecting different numbers of photons in each input beam splitter, we get the following charts for $n_1 = a = 3$ and $n_2 = b = 7$ are obtained (see Figure 7).



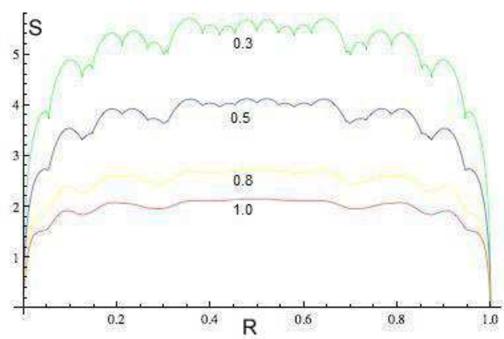


Figure 7: Entropy entries for $a = 3$ and $b = 7$ photons, using *Matlab* and *Mathematica*, respectively

Analyzing the graphs of Figure 7, we have no difference with the figures obtained by [4]; also observing the graphs obtained in Figures 8 and 9 whose entries photons in the beam splitter were different, it is observed that for the

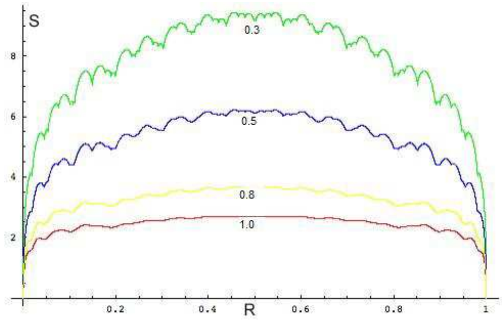


Figure 8: Entropy for entries $a = 4$ and $b = 16$ photons

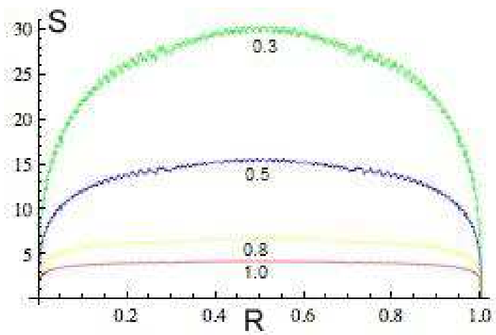


Figure 9: Entropy for entries $a = 30$ and $b = 70$ photons

Von Neumann entropy maximum entanglement recurs at $r = 0,5$, but for the Tsallis entropy the maximum entanglement does not occur in $r = 0,5$, but at $r \sim 0,5$, because at $r = 0,5$ is a slight decrease in amplitude.

Now observing Figure 10 whose entries were higher than different, we observe that there is a cloud of entanglement for the Tsallis entropy when $q = 0,5$, but the maximum entanglement still occurs in $r \sim 0,5$, however for the Von Neumann entropy at $r = 0,5$ again the minimum entanglement is showed.

However, we conclude that when one of the entrances to beam splitter receiving the vacuum state $|0>$, the Tsallis entropy is similar to the Von Neumann entropy, at least for the graphics you see anything new. This result coincides with the work presented by Borges et al. [4]. Even when the same number of photons is injected at the entrances of the beam splitter, the Tsallis entropy still has some resemblance to the Von Neumann entropy, but with greater variation in reflection amplitude. And finally, when a different number of photons is injected at the entrances of the beam splitter is a small difference

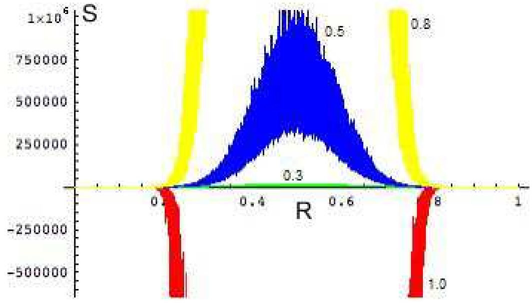


Figure 10: Entropy for entries $a = 40$ and $b = 160$ photons

in the entanglement as a function of amplitude, as seen in recent graphics entanglement is maximum in $r \sim 0,5$ to the Tsallis entropy, but for the Von Neumann entropy, while a much larger number is inserted into the beam splitter that entanglement becomes minimum at $r = 0,5$.

3. Concluding Remarks

The non-extensive statistical mechanics, is a theory originally proposed by Tsallis in 1988. In the Tsallis theory [14] makes no sense to say a priori that the entropy of a system is extensive (extensive or not) without stating the law of composition of its elements.

A non-extensive entropy (S_q) is as a function of the coefficient (amplitude) reflection for different values of $q \in [0, 1]$. For $q \to 1$ the non-extensive entropy is also known as Von Neumann entropy.

In conclusion, to this work, it is evident that the behavior of (S_q) for small values of q is qualitatively different, in comparing both entropies "a la" Von Neumann and "a la" Tsallis. In this comparison between the entropies of Von Neumann and Tsallis, it is noticeable that the latter is able (thanks to the entropic parameter q) to "lift" quantum correlations not perceived by the former. This may represent a connection of the physical system (extensiveness) and the measure of the degree of entanglement.

There are some interesting points to be studied in future, among them an increase in the number of entries in the beam splitter, an analysis by Gaussian states, and a refinement of the software used in implementation.

References

[1] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, W.K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, *Phys. Rev. Lett.*, **70** (1993), 1895-1899.

[2] C.H. Bennett, D.P. Di Vincenzo, Quantum information and computation, *Nature*, **404** (2000), 247-255.

[3] C.H. Bennett, S.J. Wiesner, Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states, *Phys. Rev. Lett.*, **69** (1992), 2881-2884.

[4] M.F. Borges, R. Godoy, G.A. Prataviera, Towards a quantum information process using tsallis entropy, *International Journal of Applied Mathematics*, **21** (2008), 645-656.

[5] M. Cunha, *Emaranhamento: dos gatos de Schrödinger á Álgebra Multilinear*, UFMG (2004).

[6] M. Cunha, *Emaranhamento: Caracterizačão, Manipulačão e Consequências*, UFMG (2005).

[7] A. Einstein, B. Podolsky, N. Rosen, Can quantum-mechanical description of physical reality be considered complete?, *Phys. Rev.*, **47** (1935), 777-780.

[8] A.K. Ekert, Quantum cryptography based on Bell's Theorem, *Phys. Rev. Lett.*, **67** (1991), 661-663.

[9] V.V. França, V.V., K. Capelle, Entanglement of strongly interacting low-dimensional fermions in metallic, superfluid, and antiferromagnetic insulating systems, *Phys. Rev. A*, **74** (2006).

[10] H. Haffner, W. Hänsell, C.F. Roos, J. Benhelm, Scalable multiparticle entanglement of trapped ions, *Nature*, **438** (2005), 643-646.

[11] J. Neumann, *The Mathematical Foundations of Quantum Mechanics*, Princeton University Press (1955).

[12] R.N. Palmer, Moura C. Alves, D. Jaksch, Detection and characterization of multipartite entanglement in optical lattices, *Phys. Rev. A*, **72** (2005).

[13] W. Tittel, J. Brendel, H. Zbinden, N. Gisin, Violation of Bell Inequalities by photons more than 10 km apart, *Phys. Rev. Lett.*, **81** (1998), 3563-3566.

[14] C. Tsallis, Possible generalizations of Boltzmann-Gibbs statistics, *J. Stat. Phys.*, **52** (1988).

[15] C. Tsallis, Nonextensive statical mechanics and thermodynamics, *Braz. J. of Phys.*, **29** (1999).

[16] C. Tsallis, Sá F.C. Barreto, Loh D. Edwin, Generalization of the Planck radiation law and application to the cosmic microwave background radiation, *Phys. Rev. E*, **52** (1995), 1447-1451.

[17] P. Zanardi, Quantum entanglement in fermionic lattices, *Phys. Rev. A*, **65** (2002).