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ON THE SOLUTION OF THE CONVECTIVE-DIFFUSION EQUATION

F.I. Arunaye[1], B.S. Bhatt[2 §], P. Nagarani[3]
[1,3]Department of Mathematics
Mona Campus
The University of the West Indies
Mona, Kingston 7, JAMAICA
[1]e-mail: ifarunaye@yahoo.com
[3]e-mail: nagaraniponakala@gmail.com
[2]Department of Mathematics and Computer Science
Faculty of Science and Agriculture
The University of the West Indies
St. Augustine Campus, TRINIDAD AND TOBAGO
e-mail: Bal.Bhatt@sta.uwi.edu

Abstract: Group invariant solutions for an unsteady diffusion equation which describes the local concentration of the solute, namely,

$$\frac{\partial C}{\partial t} + w(r)\frac{\partial C}{\partial z} = D_m \left\{ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right) + \frac{\partial^2 C}{\partial z^2} \right\}$$

have been obtained.

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1. Introduction

The study of convection-diffusion attracted the attention of researchers due to its applications in the fields of chemical engineering, environmental dynamics,

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§Correspondence author

biomedical engineering and physiological fluid dynamics. In chemical engineering, the dispersion of a gaseous tracer injected into a flowing stream of a second gas, is studied under conditions where the tracer gas can also be exchanged by diffusion within a stagnant gaseous zone held in a porous solid. The studies on natural convection mass/heat transfer in an annulus bounded by two horizontal cylinders have applications in nuclear reactors, recovery of geothermal energy, exothermal chemical reaction (Kuehn and Goldstein [5] and Tsui and Trebley [10]). This theory also can be used to analyze the dispersion of waste products in the waterways, for the steady flow in the natural rivers (Fischer [3]). Numerous examples are available in the literature in biological systems where dispersion plays an important role (Truskey et al [9]). The transport processes are critical to understand the transport process of nutrients in blood vessels and various artificial devices (Middleman [7], Lightfoot [6]). Since many intravenous medications are therapeutic at low concentration but toxic at high concentration, it is significant to know the rate of dispersion of drugs in blood flow of the circulatory system. Also, in the indicator dilution technique, it is a common practice to introduce a quantity of solute into the blood stream and to measure its concentration at some down-stream point as it moves along the blood flow.

The well known approach for solving differential equations is the classical Lie symmetric group of transformation for finding the analytical solutions. The group invariant solution technique for solving the partial differential equations is a consequence of the Lie theory (Olver [8], Bluiman and Kumei [2]). With the motivation of the above applications we have analyzed the solution of the convection-diffusion equation by group invariant method as used by Bhatt and Krishnan [1], Krishnan and Bhatt [4].

2. Convection-Diffusion Equation in Cylindrical Coordinate Systems

The unsteady convective diffusion equation which describes the local concentration C of the solute as a function of longitudinal (axial) coordinate z, transverse (radial) coordinate r and time t is given by [9]:

$$\frac{\partial C}{\partial t} + w(r)\frac{\partial C}{\partial z} = D_m \left\{ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right) + \frac{\partial^2 C}{\partial z^2} \right\}, \tag{1}$$

where $w(r)$ is the axial velocity of the fluid in pipe and D_m is coefficient of molecular diffusion (molecular diffusivity) which is assumed to be constant. The velocity of the fluid can be obtained by solving the equation of motion and equation of continuity for the given fluid.

Equation (1) can be rewritten as

$$C_t + w(r)C_z = D_m\left(\frac{1}{r}C_r + C_{rr} + C_{zz}\right). \tag{2}$$

Let the infinitesimal Lie point symmetry generators be defined as

$$\Gamma = \xi(r,t,z,C)\partial_r + \tau(r,t,z,C)\partial_t + \zeta(r,t,z,C)\partial_z + \phi(r,t,z,C)\partial_C. \tag{3}$$

The second prolongation (extension) of (3) is

$$\begin{aligned}\Gamma^{(2)} = \Gamma &+ \phi^r\partial_{C_r} + \phi^z\partial_{C_z} + \phi^t\partial_{C_t} + \phi^{rt}\partial_{C_{rt}} + \phi^{rz}\partial_{C_{rz}} \\ &+ \phi^{zt}\partial_{C_{zt}} + \phi^{rr}\partial_{C_{rr}} + \phi^{zz}\partial_{C_{zz}} + \phi^{tt}\partial_{C_{tt}}.\end{aligned} \tag{4}$$

The invariant property ensures the action of (4) on (2) as follows:

$$\Gamma^{(2)}\{C_t + w(r)C_z - D_m\left(\frac{1}{r}C_r + C_{rr} + C_{zz}\right)\} = 0, \tag{5}$$

i.e.

$$(w^{'}(r)C_z + \frac{D_m}{r^2}C_r)\xi + \phi^t + w(r)\phi^z - \frac{D_m}{r}\phi^r - D_m\phi^{rr} - D_m\phi^{zz} = 0, \tag{6}$$

where $w^{'}(r) = \frac{\partial}{\partial r}w(r)$.

We have:

$$\phi^t = \phi_t - \xi_t C_r + (\phi_C - \tau_t)C_t - \xi_C C_r C_t - \tau_C C_t^2 - \zeta_t C_z - \zeta_C C_z C_t, \tag{7}$$

$$\phi^r = \phi_r + (\phi_C - \xi_r)C_r - \xi_C C_r^2 - \tau_r C_t - \tau_C C_r C_t - \zeta_r C_z - \zeta_C C_z C_r, \tag{8}$$

$$\phi^z = \phi_z + (\phi_C - \zeta_z)C_z - \xi_z C_r - \xi_C C_r C_z - \tau_z C_t - \tau_C C_t C_z - \zeta_C C_z^2, \tag{9}$$

$$\begin{aligned}\phi^{rr} = \phi_{rr} &+ (2\phi_{rC} - \xi_{rr})C_r - \tau_{rr}C_t + (\phi_{CC} - 2\xi_{rC})C_r^2 + (\phi_C - 2\xi_r)C_{rr} - \xi_{CC}C_r^3 \\ &- 3\xi_C C_r C_{rr} - \tau_{CC}C_t C_r^2 - 2\tau_{rC}C_r C_t - \tau_C C_{rr}C_t - 2\tau_r C_{rt} - 2\tau_C C_r C_{rt} \\ &- \zeta_{rr}C_z - 2\zeta_{rC}C_z C_r - \zeta_{CC}C_z C_r - \zeta_C C_z C_{rr} - 2\zeta_r C_{rz} - 2\zeta_r C_r C_{rz},\end{aligned} \tag{10}$$

$$\begin{aligned}\phi^{zz} = \phi_{zz} &+ (2\phi_{zC} - \zeta_{zz})C_z - \tau_{zz}C_t + (\phi_{CC} - 2\zeta_{zC})C_z^2 + (\phi_C - 2\zeta_z)C_{zz} - \zeta_{CC}C_z^3 \\ &- 3\zeta_C C_z C_{zz} - \xi_{zz}C_r - 2\xi_{zC}C_z C_r - \xi_{CC}C_z^2 C_r - \xi_C C_r C_{zz} - \tau_{zz}C_t - 2\tau_{zC}C_z^2 \\ &- \tau_{CC}C_z^2 C_t - \tau_C C_{zz}C_t - 2\xi_z C_{rz} - 2\xi_C C_{rz}C_z - 2\tau_z C_{tz} - \tau_C C_{tz}C_z.\end{aligned} \tag{11}$$

We now substitute equations (7), (8), (9), (10) and (11) into (6) to obtain the Lie point symmetry determining equation below:

$$
\begin{aligned}
&(w^{'}(r)C_z + \frac{D_m}{r^2}C_r)\xi + \phi_t - \xi_t C_r + (\phi_C - \tau_t)C_t \\
&- \xi_C C_r C_t - \tau_C C_t^2 - \zeta_t C_z - \zeta_C C_z C_t \\
&+ w(r)\{\phi_z + (\phi_C - \zeta_z)C_z - \xi_z C_r - \xi_C C_r C_z - \tau_z C_t \\
&- \tau_C C_t C_z - \zeta_C C_z^2\} - \frac{D_m}{r}\{\phi_r + (\phi_C - \xi_r)C_r \\
&- \xi_C C_r^2 - \tau_r C_t - \tau_C C_r C_t - \zeta_r C_z - \zeta_C C_z C_r\} \\
&- D_m\{\phi_{rr} + (2\phi_{rC} - \xi_{rr})C_r - \tau_{rr}C_t + (\phi_{CC} - 2\xi_{rC})C_r^2 \\
&+ (\phi_C - 2\xi_r)C_{rr} - \xi_{CC}C_r^3 - 3\xi_C C_r C_{rr} - \tau_{CC}C_t C_r^2 \\
&- 2\tau_{rC}C_r C_t - \tau_C C_{rr}C_t - 2\tau_r C_{rt} - 2\tau_C C_r C_{rt} - \zeta_{rr}C_z \\
&- 2\zeta_{rC}C_z C_r - \zeta_{CC}C_z C_r - \zeta_C C_z C_{rr} - 2\zeta_r C_{rz} - 2\zeta_r C_r C_{rz}\} \\
&- D_m\{\phi_{zz} + (2\phi_{zC} - \zeta_{zz})C_z - \tau_{zz}C_t + (\phi_{CC} - 2\zeta_{zC})C_z^2 \\
&+ (\phi_C - 2\zeta_z)C_{zz} - \zeta_{CC}C_z^3 - 3\zeta_C C_z C_{zz} - \xi_{zz}C_r - 2\xi_{zC}C_z C_r \\
&- \xi_{CC}C_z^2 C_r - \xi_C C_r C_{zz} - \tau_{zz}C_t - 2\tau_{zC}C_z^2 - \tau_{CC}C_z^2 C_t \\
&- \tau_C C_{zz}C_t - 2\xi_z C_{rz} - 2\xi_C C_{rz}C_z - 2\tau_z C_{tz} - \tau_C C_{tz}C_z\}.
\end{aligned} \tag{12}
$$

Substituting for $C_t = -w(r)C_z + D_m(\frac{1}{r}C_r + C_{rr} + C_{zz})$ in (12) and after collecting the like terms equation (12) becomes

$$
\begin{aligned}
&[\frac{D_m}{r^2}\xi - \xi_t - \frac{D_m}{r}\tau_t + \frac{2wD_m}{r}\tau_C - w\xi_z - \frac{wD_m}{r}\tau_z + \frac{D_m}{r}\xi_r \\
&+ \frac{D_m^2}{r^2}\tau_r - D_m(2\phi_{rC} - \xi_{rr}) + \frac{D_m^2}{r}\tau_{rr} + \frac{D_m^2}{r}\tau_{zz} + 2D_m\xi_{zz}]C_r \\
&+ \left[\frac{2D_m^2}{r}\tau_{rC} - D_m(\phi_{CC} - 2\xi_{rC})\right] C_r^2 \\
&+ [D_m\xi_{CC} + \frac{D_m^2}{r}\tau_{CC}]C_r^3 + \left[\frac{2wD_m}{r}\tau_C - D_m\tau_t\right. \\
&\left. - wD_m\tau_z + \frac{D_m^2}{r}\tau_r + D_m^2\tau_{rr} + 2D_m\xi_r + 2D_m^2\tau_{zz}\right] C_{rr} \\
&+ \left[2D_m\xi_C + 2D_m^2\tau_{rC} + \frac{D_m^2}{r}\tau_C\right] C_r C_{rr} + D_m^2\tau_{CC}C_r^2 C_{rr} \\
&+ [2D_m\zeta_{rC} + D_m\zeta_{CC} + 2D_m\zeta_{zC} - \frac{2wD_m}{r}\tau_C - 2wD_m\tau_{rC}]C_r C_z \\
&+ 2D_m\tau_r C_{rt} - 2D_m\tau_C C_r C_{rt}
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
&+[2D_m\zeta_r + 2D_m\xi_z]C_{rz} + 2D_m\tau_z C_{tz} \\
&+ 2D_m\xi_C C_z C_{rz} + 2D_m\tau_C C_{tz}C_z + 2D_m\zeta_r C_r C_{rz} \\
&+ [D_m\zeta_{CC} - wD_m\tau_{CC}]C_z^3 - 2wD_m\tau_C C_z C_{rr} \\
&+ [w(\phi_C - \zeta_z) - \zeta_t - wD_m(\phi_C - \tau_t) + w^2\tau_z + w^{'}\xi - \frac{wD_m}{r}\tau_r \\
&+ \frac{D_m}{r}\zeta_r - wD_m\tau_{rr} + D_m\zeta_{rr} - D_m(2\phi_{zC} - \zeta_{zz}) - 2wD_m\tau_{zz}]C_z \\
&+ [2D_m\tau_{zC} - D_m(\phi_{CC} - 2\zeta_{zC})]C_z^2 \\
&+ [D_m\xi_{CC} + \frac{D_m^2}{r}\tau_{CC}]C_z^2 C_r + D_m\tau_{CC}C_z^2 C_{rr} \\
&+ [D_m\tau_{zz} + 2D_m\zeta_z + D_m^2\tau_{zz} - D_m\tau_t + D_m^2\tau_{rr} + \frac{D_m^2}{r}\tau_r \\
&- wD_m\tau_z + 2wD_m\tau_C]C_{zz} + D_m\tau_{CC}C_z^2 C_{zz} + 2D_m\tau_{rC}C_{zz}C_r \\
&+ [2D_m\zeta_C - 2wD_m\tau_C]C_{zz}C_z + D_m^2\tau_{CC}C_r^2 C_{zz} + wD_m\tau_{CC}C_r^2 C_z \\
&+ \{w^{'}\xi + \phi_t + w\phi_z - \frac{D_m}{r}\phi_r - D_m\phi_{rr} - D_m\phi_{zz}\} = 0.
\end{aligned}
$$

By equating the coefficients of the monomials of C and its derivatives in (13), we obtain the over-determined systems of partial differential equations (see Table 1)

Now from equations (xvii), (xvi), (xiv), (xiii), (xviii), (ix), (vii), (vi), (v), (iii) and (ii) we observe that τ is a function of t only; i.e., $\tau = \sigma(t)$ and equations (iv), (viii), (x), (xiii) and (xx) imply that ζ, ξ are not functions of C. Thus equation (xi) gives ζ is not a function of r either, hence ζ is a function of t and z alone, i.e., $\zeta(t, z)$. Equation (xv) implies that ξ is not a function of z either, so ξ is a function of t and r alone, i.e., $\xi(t, r)$. From equation (xxiii) we get

$$\xi_r = \frac{1}{2}\tau_t \Rightarrow \xi = \frac{1}{2}\tau_t r + \sigma(t). \tag{14}$$

Similarly equation (xxvii) implies

$$\zeta_z = \frac{1}{2}\tau_t \Rightarrow \zeta = \frac{1}{2}\tau_t z + \sigma(t). \tag{15}$$

Equations (xxv) and (xxii) give

$$\phi_C = \frac{(1 - D_m)w}{2D_m}\left[-\frac{z}{2r}\sigma - \frac{z}{8D_m}\tau_{tt}r^2 - \frac{zr}{4D_m}\sigma_t\right] \tag{16}$$

Monomial	**coefficient equations**	
1	$\phi_t + w\phi_z - \frac{D_m}{r}\phi_r - D_m\phi_{rr} - D_m\phi_{zz} = 0$	(i)
$C_r^2\, C_z$	$wD_m\,\tau_{CC} = 0$	(ii)
$C_r^2\, C_{zz}$	$D_m^2\,\tau_{CC} = 0$	(iii)
$C_{zz}\, C_z$	$2D_m\,\zeta_C\, -\, 2w\,D_m\,\tau_C = 0$	(iv)
$C_{zz}\, C_r$	$2D_m\,\tau_{rC} = 0$	(v)
$C_z^2\, C_{zz}$	$D_m\,\tau_{CC} = 0$	(vi)
$C_z^2\, C_{rr}$	$D_m\,\tau_{CC} = 0$	(vii)
$C_z^2\, C_r$	$D_m\,\xi_{CC} + \frac{D_m^2}{r}\,\tau_{CC} = 0$	(viii)
$C_z\, C_{rr}$	$-2w\,D_m\,\tau_C = 0$	(ix)
C_z^3	$D_m\,\zeta_{CC} - w\,D_m\,\tau_{CC} = 0$	(x)
$C_r\, C_{rz}$	$2D_m\,\zeta_r = 0$	(xi)
$C_{tz}\, C_z$	$2D_m\,\tau_C = 0$	(xii)
$C_z\, C_{rz}$	$2D_m\,\xi_C = 0$	(xiii)
C_{tz}	$2D_m\,\tau_z = 0$	(xiv)
C_{rz}	$2D_m\,\zeta_r + 2D_m\xi_z = 0$	(xv)
$C_r\, C_{rt}$	$-2D_m\,\tau_C = 0$	(xvi)
C_{rt}	$2D_m\,\tau_r = 0$	(xvii)
$C_r^2\, C_{rr}$	$D_m^2\,\tau_{CC} = 0$	(xviii)
$C_r\, C_{rr}$	$2D_m\,\xi_C + 2D_m^2\tau_{rC} + \frac{D_m^2}{r}\tau_C = 0$	(xix)
C_r^3	$D_m\,\xi_{CC} + \frac{D_m^2}{r}\tau_{CC} = 0$	(xx)
C_r^2	$\frac{2D_m^2}{r}\tau_{rC} - D_m(\phi_{CC} - 2\xi_{rC}) = 0$	(xxi)
C_r	$\frac{D_m}{r^2}\xi - \xi_t - \frac{D_m}{r}\tau_t + \frac{2wD_m}{r}\tau_C - w\xi_z - \frac{wD_m}{r}\tau_z$ $+\frac{D_m}{r}\xi_r + \frac{D_m^2}{r^2}\tau_r - D_m(2\phi_{rC} - \xi_{rr}) + \frac{D_m^2}{r}\tau_{rr}$ $+\frac{D_m^2}{r}\tau_{zz} + 2D_m\xi_{zz} = 0$	(xxii)
C_{rr}	$\frac{2wD_m}{r}\tau_C - D_m\tau_t - wD_m\tau_z + \frac{D_m^2}{r}\tau_r + D_m^2\tau_{rr}$ $+2D_m\xi_r + 2D_m^2\tau_{zz} = 0$	(xxiii)
$C_r\, C_z$	$2D_m\zeta_{rC} + D_m\zeta_{CC} + 2D_m\zeta_{zC}\, -\, \frac{2wD_m}{r}\tau_c$ $-2wD_m\tau_{rC} = 0$	(xxiv)
C_z	$w(\phi_C - \zeta_z) - \zeta_t - wD_m(\phi_C - \tau_t)w^2\tau_z$ $-\frac{wD_m}{r}\tau_r + \frac{D_m}{r}\zeta_r - wD_m\tau_{rr} + D_m\zeta_{rr}$ $+w^{'}\xi - D_m(2\phi_{zC} - \zeta_{zz}) - 2wD_m\tau_{zz} = 0$	(xxv)
C_z^2	$2D_m\tau_{zC} - D_m(\phi_{CC} - 2\zeta_{zC}) = 0$	(xxvi)
C_{zz}	$D_m\tau_{zz} + 2D_m\zeta_z + D_m^2\tau_{zz} - D_m\tau_t + D_m^2\tau_{rr}$ $+\frac{D_m^2}{r}\tau_r - wD_m\tau_z + 2wD_m\tau_C = 0$	(xxvii)

Table 1

$$-\frac{1}{8D_m}\tau_{tt}z^2 - \frac{z}{2D_m}\sigma_t - \frac{wz}{4D_m}\tau_t + \frac{zw^{'}}{2D_m}(\frac{1}{2}\tau_t r + \sigma) + \frac{wz}{2}\tau_t + \eta(t).$$

Equations (xxi) and (xxvi) imply

$$\phi_{CC} = 0 \Rightarrow \phi(t, r, z, C) = \beta(t, r, z)C + \alpha(t, r, z). \tag{17}$$

If $\phi(t, r, z, C)$ is a solution of (xii) then $\beta(t, r, z)$ and $\alpha(t, r, z)$ are also solutions satisfying

$$\beta_t + w(r)\beta_z = D_m(\frac{1}{r}\beta_r + \beta_{rr} + \beta_{zz}), \tag{18}$$

$$\alpha_t + w(r)\alpha_z = D_m(\frac{1}{r}\alpha_r + \alpha_{rr} + \alpha_{zz}). \tag{19}$$

Thus taking equation (16) to be identically $\beta(t, r, z)$ we have the following relations:

$$\begin{aligned}\beta_z = &\frac{(1-D_m)w}{2D_m}\left[-\frac{1}{2r}\sigma - \frac{1}{8D_m}\tau_{tt}r^2 - \frac{r}{2D_m}\sigma_t\right] - \frac{1}{4D_m}\tau_{tt}z \\ &+ \frac{w^{'}}{2D_m}(\frac{1}{2}\tau_t r + \sigma) - \frac{1}{2D_m}\sigma_t - \frac{w}{4D_m}\tau_t + \frac{w}{2}\tau_t;\end{aligned} \tag{20}$$

$$\beta_{zz} = -\frac{1}{4D_m}\tau_{tt}\,; \tag{21}$$

$$\begin{aligned}\beta_r = &\frac{(1-D_m)w}{2D_m}\left[\frac{z}{2r^2}\sigma - \frac{z}{4D_m}\tau_{tt}r - \frac{z}{4D_m}\sigma_t\right] - \frac{w^{'}z}{2}\tau_t \\ &+ \frac{w^{''}z}{2D_m}(\frac{1}{2}\tau_t r + \sigma) + \frac{(1-D_m)w^{'}}{2D_m}\left[-\frac{z}{2r}\sigma - \frac{z}{8D_m}\tau_{tt}r^2 - \frac{zr}{2D_m}\sigma_t\right];\end{aligned} \tag{22}$$

$$\begin{aligned}\beta_{rr} = &\frac{(1-D_m)w}{2D_m}\left[-\frac{z}{r^3}\sigma - \frac{z}{4D_m}\tau_{tt}\right] \\ &+ \frac{(1-D_m)w^{'}}{D_m}\left[\frac{z}{2r^2}\sigma - \frac{z}{4D_m}\tau_{tt}r - \frac{z}{2D_m}\sigma_t\right] \\ &+ \frac{(1-D_m)w^{''}}{2D_m}[-\frac{z}{2r}\sigma - \frac{z}{8D_m}\tau_{tt}r^2 - \frac{zr}{2D_m}\sigma_t] \\ &+ \left(\frac{w^{''}z}{2} + \frac{w^{''}zr}{4D_m} + \frac{w^{'}z}{4D_m}\right)\tau_t + \frac{w^{'''}z}{2D_m}\sigma\,;\end{aligned} \tag{23}$$

$$\beta_t = \frac{(1-D_m)w}{2D_m}\left[-\frac{z}{2r}\sigma_t - \frac{z}{8D_m}\tau_{ttt}r^2 - \frac{zr}{2D_m}\sigma_{tt}\right] + \frac{zw'}{2D_m}\left(\frac{1}{2}\tau_{tt}r + \sigma_t\right) \\ - \frac{1}{8D_m}\tau_{ttt}z^2 - \frac{z}{2D_m}\sigma_{tt} - \frac{wz}{4D_m}\tau_{tt} + \frac{wz}{2}\tau_{tt} + \eta_t. \tag{24}$$

Substituting for (20), (21), (22), (23) and (24) in equation (i) we get

$$\begin{aligned} &\frac{(1-D_m)w}{2D_m}\left\{-\frac{z}{2D_m r}\sigma_t - \frac{z}{8D_m}\tau_{ttt}r^2 - \frac{zr}{4D_m}\sigma_{tt}\right\} \\ &- \frac{1}{8D_m}\tau_{ttt}z^2 - \frac{z}{2D_m}\sigma_{tt} - \frac{wz}{4D_m}\tau_{tt} \\ &+ \frac{wz}{2}\tau_{tt} + \frac{zw'}{2D_m}\left(\frac{1}{2}\tau_{tt}r + \sigma_t\right) + \eta_t \\ &+ \frac{(1-D_m)w^2}{2D_m}\left[-\frac{1}{2r}\sigma - \frac{1}{8D_m}\tau_{tt}r^2 - \frac{r}{2D_m}\sigma_t\right] - \frac{wz}{4D_m}\tau_{tt} \\ &- \frac{w}{2D_m}\sigma_t - \frac{w^2}{4D_m}\tau_t + \frac{w^2}{2}\tau_t + \frac{ww'r}{4D_m}\tau_t + \frac{ww'}{2D_m}\sigma \\ &- \frac{(1-D_m)w}{2}\left\{\frac{z}{2r^3}\sigma - \frac{z}{4D_m}\tau_{tt} - \frac{z}{4D_m r}\sigma_t\right\} - \frac{w'z}{2r}\tau_t - \frac{w''z}{4}\tau_t \\ &- \frac{w''z}{2r}\sigma - \frac{(1-D_m)w'}{2}\left\{-\frac{z}{2r^2}\sigma - \frac{z}{8D_m}\tau_{tt}r - \frac{z}{2D_m}\sigma_t\right\} \\ &+ \frac{(1-D_m)w}{2r^3}\sigma + \frac{(1-D_m)wz}{8D_m}\tau_{tt} \\ &- (1-D_m)w'\left\{-\frac{z}{2r^2}\sigma - \frac{z}{4D_m}\tau_{tt}r - \frac{z}{2D_m}\sigma_t\right\} \\ &- \frac{(1-D_m)w''}{2}\left\{-\frac{z}{2r}\sigma - \frac{z}{8D_m}\tau_{tt}r^2 - \frac{zr}{4D_m}\sigma_t\right\} \\ &- \left(\frac{D_m w''z}{2} + \frac{w''z}{4} + \frac{w'''zr}{4}\right)\tau_t - \frac{w'''z}{2}\sigma = -\frac{1}{4}\tau_{tt}. \end{aligned} \tag{25}$$

Setting

$$\sigma_{tt} = 0 \Rightarrow \sigma = c_3 t + c_1, \tag{26}$$

$$\tau_{ttt} = 0 \Rightarrow \tau = 4c_6 t^2 + 2c_5 t + c_2. \tag{27}$$

Then equation (25) becomes

$$
\begin{aligned}
&-\frac{(1-D_m)wz}{4D_m^2 r}\sigma_t-\frac{wz}{4D_m}\tau_{tt}+\frac{wz}{2}\tau_{tt}+\frac{zw^{'}}{4D_m}\tau_{tt}r+\frac{zw^{'}}{2D_m}\sigma_t\\
&+\eta_t+\frac{(1-D_m)w^2}{2D_m}\left[-\frac{1}{2r}\sigma-\frac{1}{8D_m}\tau_{tt}r^2-\frac{r}{2D_m}\sigma_t\right]\\
&-\frac{wz}{4D_m}\tau_{tt}-\frac{w}{2D_m}\sigma_t-\frac{w^2}{4D_m}\tau_t+\frac{w^2}{2}\tau_t+\frac{ww^{'}r}{4D_m}\tau_t+\frac{ww^{'}}{2D_m}\sigma\\
&-\frac{(1-D_m)w}{2}\left\{\frac{z}{2r^3}\sigma-\frac{z}{4D_m}\tau_{tt}-\frac{z}{4D_m r}\sigma_t\right\}-\frac{w^{'}z}{2r}\tau_t-\frac{w^{''}z}{4}\tau_t\\
&-\frac{w^{''}z}{2r}\sigma-\frac{(1-D_m)w^{'}}{2}\left\{-\frac{z}{2r^2}\sigma-\frac{z}{8D_m}\tau_{tt}r-\frac{z}{2D_m}\sigma_t\right\}\\
&+\frac{(1-D_m)w}{2r^3}\sigma+\frac{(1-D_m)wz}{8D_m}\tau_{tt}\\
&-(1-D_m)w^{'}\left\{-\frac{z}{2r^2}\sigma-\frac{z}{4D_m}\tau_{tt}r-\frac{z}{2D_m}\sigma_t\right\}\\
&-\frac{(1-D_m)w''}{2}\left\{-\frac{z}{2r}\sigma-\frac{z}{8D_m}\tau_{tt}r^2-\frac{zr}{4D_m}\sigma_t\right\}\\
&-\left(\frac{D_m w^{''}z}{2}+\frac{w^{''}z}{4}+\frac{w^{'''}zr}{4}\right)\tau_t-\frac{w^{'''}z}{2}\sigma=-\frac{1}{4}\tau_{tt}.
\end{aligned}
\tag{28}
$$

By setting

$$
\begin{aligned}
f(r,z)=&\frac{wz}{2}-\frac{wz}{2D_m}+\frac{zw^{'}}{4D_m}r-\frac{(1-D_m)w^2}{16D_m^2}r^2+\frac{(1-D_m)zw}{4D_m}\\
&+\frac{(1-D_m)zw^{''}}{16D_m}r^2+\frac{5(1-D_m)zw^{'}}{16D_m}r+\frac{1}{4},
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
g(r,z)=&\frac{zw^{'}}{2D_m}-\frac{w}{2D_m}-\frac{(1-D_m)w^2}{4D_m^2}r-\frac{(1-D_m)zw}{8D_m r}\\
&+\frac{3(1-D_m)zw^{'}}{4D_m}+\frac{(1-D_m)zw^{''}}{8D_m}r,
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
h(r,z)=&\frac{ww^{'}}{2D_m}-\frac{zw^{''}}{2r}-\frac{zw^{''}}{2}+\frac{3(1-D_m)zw^{'}}{4r^2}-\frac{(1-D_m)w^2}{4D_m r}\\
&-\frac{(1-D_m)zw}{4r^3}-\frac{(1-D_m)w}{2r^3}+\frac{(1-D_m)zw^{''}}{2r},
\end{aligned}
\tag{31}
$$

$$k(r,z) = \frac{w^2}{2} - \frac{w^2}{4D_m} + \frac{ww'}{4D_m}r - \frac{zw'}{2r} - \frac{zw''}{2} + \frac{D_m zw''}{2} - \frac{zw'''}{4}r; \quad (32)$$

We obtain from (28) after substituting for τ_{tt}, τ_t, σ_t and σ that

$$\begin{aligned} \eta_t = & -8[f(r,z) + k(r,z)t]c_6 - 2k(r,z)c_5 - [g(r,z) + h(r,z)t]c_3 \\ & - h(r,z)c_1, \\ \eta = & -8[f(r,z)t + k(r,z)\frac{t^2}{2}]c_6 - 2k(r,z)tc_5 - [g(r,z)t + h(r,z)\frac{t^2}{2}]c_3 \\ & - h(r,z)tc_1 + c_4. \end{aligned} \quad (33)$$

So the infinitesimals of the Lie point symmetry generators (3) are as follows:

$$\begin{aligned} \tau(t,r,z) &= 4c_6t^2 + 2c_5t + c_2, \\ \xi(t,r,z) &= 4c_6tr + c_5r + c_3t + c_1, \\ \zeta(t,r,z) &= 4c_6tz + c_5z + c_3t + c_1, \\ \phi(t,r,z,C) &= \beta(t,r,z)C + \alpha(t,r,z), \end{aligned}$$

where

$$\begin{aligned} \beta(t,r,z) = \{ & -8f(r,z)t - 4k(r,z)t^2 - \frac{1}{D_m}z^2 + 4wzt - \frac{2zw'}{D_m}t - \frac{(1-D_m)zw}{2D_m^2}r^2 \\ & + \frac{2zw'}{D_m}tr\}c_6 + \left\{-2k(r,z)t + wz - \frac{1}{2D_m}wz + \frac{zw'}{2D_m}r\right\}c_5 \\ & + c_4 + \{-g(r,z)t - h(r,z)\frac{t^2}{2} - \frac{z}{2D_m} + \frac{zw'}{2D_m}t - \frac{(1-D_m)w}{4D_m r}t \\ & - \frac{(1-D_m)zw}{4D_m^2}r\}c_3 + \{-h(r,z)t + \frac{(1-D_m)w}{4D_m r} + \frac{zw'}{2D_m}\}c_1; \end{aligned}$$

and $\alpha(t,r,z)$ is any other solution of convection diffusion equation (1).

2.1. Lie Point Symmetries

The Lie point symmetries of system (1) are:

$$\Gamma_1 = \partial_r + \partial_z + \left(\frac{(1-D)w}{4D_m r} + \frac{zw'}{2D_m} - h(r,z)t\right) C\partial_C,$$

$$\Gamma_2 = \partial_t,$$

$$\Gamma_3 = t\partial_r + t\partial_z + \{-g(r,z)t - h(r,z)\frac{t^2}{2} - \frac{z}{2D_m} + \frac{zw'}{2D_m}t - \frac{(1-D)w}{4D_m r}t - \frac{(1-D)zw}{4D_m^2}r\}C\partial_C,$$

$$\Gamma_4 = C\partial_C,$$

$$\Gamma_5 = 2t\partial_t + r\partial_r + z\partial_z + \left\{-2k(r,z)t + wz - \frac{1}{2D_m}wz + \frac{zw'}{2D_m}r\right\}C\partial_C,$$

$$\Gamma_6 = 4t^2\partial_t + 4tr\partial_r + 4tz\partial_z + \{-8f(r,z)t - 4k(r,z)t^2 - \frac{1}{D_m}z^2 + 4wzt - \frac{2zw'}{D_m}t - \frac{(1-D_m)zw}{2D_m^2}r^2 + \frac{2zw'}{D_m}tr\}C\partial_C,$$

$$\Gamma_\alpha = \alpha(t,r,z)\partial_C.$$

2.2. Lie Groups

The Lie group invariants of system (1) are:

$$G_1 : \left(t, r+\lambda, z+\lambda, C\ \exp\left\{\left[\frac{(1-D)w}{4D_m r} + \frac{zw'}{2D_m} - h(r,z)t\right]\lambda\right\}\right),$$

$$G_2 : (t+\lambda, r, z, C),$$

$$G_3 : (t, r+\lambda t, z+\lambda t, C\ \exp\{[-g(r,z)t - h(r,z)\frac{t^2}{2} - \frac{z}{2D_m} + \frac{zw'}{2D_m}t - \frac{(1-D)w}{4D_m r}t - \frac{(1-D)zw}{4D_m^2}r]\lambda\}),$$

$$G_4 : (t, r, z, C\,e^{\lambda}),$$

$$G_5 : \left(te^{2\lambda}, re^{\lambda}, ze^{\lambda}, C \exp\left\{ \left[-2k(r,z)t + wz - \frac{1}{2D_m} wz + \frac{zw^{'}}{2D_m} r \right] \lambda \right\} \right),$$

$$G_6 : (\frac{t}{1-4\lambda t}, \frac{r}{1-4\lambda t}, \frac{z}{1-4\lambda t}, C \exp\{-\lambda[8f(r,z)t + 4k(r,z)t^2 + \frac{1}{D_m} z^2 - 4wzt$$
$$+ \frac{2zw^{'}}{D_m} t + \frac{(1-D_m)zw}{2D_m^2} r^2 - \frac{2zw^{'}}{D_m} tr]\}),$$

$$G^{\alpha} : (t, r, z, C + \lambda\alpha(t,r,z)).$$

2.3. Group Invariant Solutions

The all possible solutions of system (1) are as follows:

$$C^{(1)} = \beta(t, r-\lambda, z-\lambda) \exp\left\{ -\lambda \left[\frac{(1-D)\overline{w}}{4D_m \overline{r}} + \frac{\overline{z}\,\bar{w}^{'}}{2D_m} - h(\overline{r}, \overline{z})t \right] \right\},$$

where $\overline{r} = r - \lambda$, $\overline{z} = z - \lambda$, $\overline{w} = w(\overline{r})$;

$$C^{(2)} = \beta(t-\lambda, r, z),$$

$$C^{(3)} = \beta(t, r-\lambda t, z-\lambda t) \exp\{\lambda[g(\overline{r}, \overline{z})t + h(\overline{r}, \overline{z})\frac{t^2}{2} + \frac{\overline{z}}{2D_m}$$

$$-\frac{\overline{z}\,\bar{w}^{'}}{2D_m} t + \frac{(1-D)\overline{w}}{4D_m \overline{r}} t + \frac{(1-D)\overline{z}\,\overline{w}}{4D_m^2} \overline{r}]\},$$

where $\overline{r} = r - \lambda t$, $\overline{z} = z - \lambda t$, $\overline{w} = w(\overline{r})$;

$$C^{(4)} = \beta(t, r, z)e^{-\lambda},$$

$$C^{(5)} = \beta(te^{-2\lambda}, re^{-\lambda}, ze^{-\lambda}) \exp\left\{ \lambda \left[2k(\overline{r}, \overline{z})\overline{t} - \overline{w}\,\overline{z} + \frac{1}{2D_m} \overline{w}\,\overline{z} - \frac{\overline{z}\,\bar{w}^{'}}{2D_m} \overline{r} \right] \right\},$$

where $\overline{t} = te^{-2\lambda}$, $\overline{r} = re^{-\lambda}$, $\overline{z} = ze^{-\lambda}, \overline{w} = w(\overline{r})$;

$$C^{(6)} = \beta(\frac{t}{1-4\lambda t}, \frac{r}{1-4\lambda t}, \frac{z}{1-4\lambda t}) \exp\{\lambda[8f(\overline{r},\overline{z})\overline{t} + 4k(\overline{r},\overline{z})\overline{t}^2 + \frac{1}{D_m}\overline{z}^2 - 4\overline{w}\,\overline{z}\,\overline{t} + \frac{2\overline{z}\,\overline{w}^{'}}{D_m}\overline{t} + \frac{(1-D_m)\overline{z}\,\overline{w}}{2D_m^2}\overline{r}^2 - \frac{2\overline{z}\,\overline{w}^{'}}{D_m}\overline{t}\,\overline{r}]\},$$

where $\overline{t} = \frac{t}{1-4\lambda t}$, $\overline{r} = \frac{r}{1-4\lambda t}$, $\overline{z} = \frac{z}{1-4\lambda t}$, $\overline{w} = w(\overline{r})$;

$$C^{(\alpha)} = \beta(t, r, z) + \lambda\alpha(t, r, z).$$

3. The Convection-Diffusion Equation in Rectangular Coordinate Systems

A special case of the convection-diffusion equation in rectangular coordinate systems is given by

$$C_t + W(r)C_z = D_m(C_{rr} + C_{zz}). \tag{34}$$

We note that the Lie point symmetry defining equation do not contain terms in equation (8) and $\frac{D_m}{r^2}C_r\xi$ in equation (6). Thus after deligent substitutions as in the preceding, the corresponding infinitesimals for (34) are as follows:

$$\begin{aligned}
\tau(t, r, z) &= 4c_6t^2 + 2c_5t + c_2,\\
\xi(t, r, z) &= 4c_6tr + c_5r + c_3t + c_1,\\
\zeta(t, r, z) &= 4c_6tz + c_5z + c_3t + c_1,\\
\phi(t, r, z, C) &= \beta(t, r, z)C + \alpha(t, r, z),
\end{aligned}$$

where

$$\begin{aligned}
\beta(t, r, z) = \{&-\frac{z^2}{D_m} - 2t + \frac{2zw^{'}}{D_m}t - \frac{2wr^2z}{D_m} - 2w^{'}t^2r - w^{''}zt^2 + \frac{w^2t^2}{D_m}\\
&- 8t(\frac{wz}{2D_m} + \frac{w^{'}rz}{D_m} + \frac{w^{''}r^2z}{4D_m}) - 4t^2(\frac{w^{''}rz}{2} + \frac{w^{'}z}{D_m} - \frac{ww^{'}}{8D_m} + \frac{w^{'''}}{8})\\
&+ 2t(\frac{w^2r}{D_m^2} + \frac{wz}{D_m})\}c_6 - \{\frac{wz}{2D_m} - \frac{zw^{'}}{2D_m} + (w^{'}r - \frac{ww^{'}}{2D_m} + \frac{w^{'''}}{2})t\\
&- \frac{w^2t}{2D_m} + \frac{w^{''}zt}{2} + 2t(\frac{w^{''}rz}{2} + \frac{w^{'}z}{D_m})\}c_5 + c_4\\
&-\{\frac{wrz}{2D_m^2} - \frac{zw^{'}}{2D_m} + \frac{z}{2D_m} + \frac{w^{'}t^2}{2} + (\frac{ww^{'}}{4D_m} - \frac{w^{'''}}{4})t^2 - (\frac{w^2r}{2D_m^2} + \frac{w}{2D_m})t\}c_3\\
&- (\frac{zw^{'}}{2D_m} - \frac{ww^{'}}{2D_m} + \frac{w^{'''}}{2})tc_1.
\end{aligned}$$

3.1. Lie Point Symmetries

The corresponding Lie point symmetries for (34) are as follows:

$$\Gamma_1 = \partial_r + \partial_z - \left(\frac{zw^{'}}{2D_m} - \frac{ww^{'}}{2D_m} + \frac{w^{'''}}{2}\right) tC\partial_C,$$

$$\Gamma_2 = \partial_t,$$

$$\Gamma_3 = t\partial_r + t\partial_z - \{\frac{wrz}{2D_m^2} - \frac{zw^{'}}{2D_m} + \frac{z}{2D_m} + \frac{w^{'}t^2}{2} + (\frac{ww^{'}}{4D_m} - \frac{w^{'''}}{4})t^2$$
$$-(\frac{w^2r}{2D_m^2} + \frac{w}{2D_m})t\}C\partial_C,$$

$$\Gamma_4 = C\partial_C,$$

$$\Gamma_5 = 2t\partial_t + r\partial_r + z\partial_z - \{\frac{wz}{2D_m} - \frac{zw^{'}}{2D_m} + (w^{'}r - \frac{ww^{'}}{2D_m} + \frac{w^{'''}}{2})t$$
$$-\frac{w^2t}{2D_m} + \frac{w^{''}zt}{2} + 2t(\frac{w^{''}rz}{2} + \frac{w^{'}z}{D_m})\}C\partial_C,$$

$$\Gamma_6 = 4t^2\partial_t + 4tr\partial_r + 4tz\partial_z - \{\frac{z^2}{D_m} + 2t - \frac{2zw^{'}}{D_m}t + \frac{2wr^2z}{D_m}$$
$$+2w^{'}t^2r + w^{''}zt^2 - \frac{w^2t^2}{D_m} + 8t(\frac{wz}{2D_m} + \frac{w^{'}rz}{D_m} + \frac{w^{''}r^2z}{4D_m})$$
$$+4t^2(\frac{w^{''}rz}{2} + \frac{w^{'}z}{D_m} - \frac{ww^{'}}{8D_m} + \frac{w^{'''}}{8}) - 2t(\frac{w^2r}{D_m^2} + \frac{wz}{D_m})\}C\partial_C,$$

$$\Gamma_\alpha = \alpha(t,r,z)\partial_C.$$

3.2. Lie Groups

The Lie groups for (34) are as follows:

$$G_1 : \left(t, r+\lambda, z+\lambda, C\,\exp\left\{-\lambda\left(\frac{zw^{'}}{2D_m} - \frac{ww^{'}}{2D_m} + \frac{w^{'''}}{2}\right)t\right\}\right),$$

$$G_2 : (t+\lambda, r, z, C),$$

$$G_3 : (t, r+\lambda t, z+\lambda t, C\,\exp\{[-\lambda\{\frac{wrz}{2D_m^2} - \frac{zw^{'}}{2D_m} + \frac{z}{2D_m}$$
$$+\frac{w^{'}t^2}{2} + \left(\frac{ww^{'}}{4D_m} - \frac{w^{'''}}{4}\right)t^2 - \left(\frac{w^2r}{2D_m^2} + \frac{w}{2D_m}\right)t\}]\}),$$

$$G_4 : (t, r, z, Ce^{\lambda}),$$

$$G_5 : (te^{2\lambda}, re^{\lambda}, ze^{\lambda}, C\ \exp\{-\lambda\{\frac{wz}{2D_m} - \frac{zw^{'}}{2D_m} + (w^{'}r - \frac{ww^{'}}{2D_m} + \frac{w^{'''}}{2})t$$
$$-\frac{w^2t}{2D_m} + \frac{w^{''}zt}{2} + 2t(\frac{w^{''}rz}{2} + \frac{w^{'}z}{D_m})\}\}),$$

$$G_6 : (\frac{t}{1-4\lambda t}, \frac{r}{1-4\lambda t}, \frac{z}{1-4\lambda t}, C\ \exp[-\lambda\{\frac{z^2}{D_m} + 2t - \frac{2zw^{'}}{D_m}t$$
$$+\frac{2wr^2z}{D_m} + 2w^{'}t^2r + w^{''}zt^2 - \frac{w^2t^2}{D_m} + 8t(\frac{wz}{2D_m} + \frac{w^{'}rz}{D_m} + \frac{w^{''}r^2z}{4D_m})$$
$$+4t^2(\frac{w^{''}rz}{2} + \frac{w^{'}z}{D_m} - \frac{ww^{'}}{8D_m} + \frac{w^{'''}}{8}) - 2t(\frac{w^2r}{D_m^2} + \frac{wz}{D_m})\}]),$$

$$G^{\alpha} : (t, r, z, C + \lambda\alpha(t, r, z)).$$

3.3. Group Invariant Solutions

The group invariant solutions for (34) are as follows:

$$C^{(1)} = \beta(t, r-\lambda, z-\lambda)\ \exp\left\{\lambda\left(\frac{\overline{z}\,\overline{w}^{'}}{2D_m} - \frac{\overline{w}\,\overline{w}^{'}}{2D_m} + \frac{\overline{w}^{'''}}{2}\right)t\right\},$$

where $\overline{w}^{'} = w^{'}(\overline{r}); \overline{z} = z - \lambda$

$$C^{(2)} = \beta(t-\lambda, r, z),$$

$$C^{(3)} = \beta(t, r-\lambda t, z-\lambda t)\ \exp\{\lambda\{\frac{\overline{w}\,\overline{r}\,\overline{z}}{2D_m^2} - \frac{\overline{z}\,\overline{w}^{'}}{2D_m} + \frac{\overline{z}}{2D_m} + \frac{\overline{w}^{'}t^2}{2}$$
$$+\left(\frac{\overline{w}\,\overline{w}^{'}}{4D_m} - \frac{\overline{w}^{'''}}{4}\right)t^2 - \left(\frac{\overline{w}^2\,\overline{r}}{2D_m^2} + \frac{\overline{w}}{2D_m}\right)t\}\},$$

where $\overline{r} = r - \lambda t$, $\overline{z} = z - \lambda t$, $\overline{w} = w(\overline{r})$;

$$C^{(4)} = \beta(t, r, z)e^{-\lambda},$$

$$C^{(5)} = \beta(te^{-2\lambda}, re^{-\lambda}, ze^{-\lambda})\ \exp\{\lambda\{\frac{\overline{w}\,\overline{z}}{2D_m} - \frac{\overline{z}\,\overline{w}^{'}}{2D_m} + (\overline{w}^{'}\overline{r} - \frac{\overline{w}\,\overline{w}^{'}}{2D_m}$$
$$+\frac{\overline{w}^{'''}}{2})\,\overline{t} - \frac{\overline{w}^2\,\overline{t}}{2D_m} + \frac{\overline{w}^{''}\,\overline{z}\,\overline{t}}{2} + 2\overline{t}\left(\frac{\overline{w}^{''}\,\overline{r}\,\overline{z}}{2} + \frac{\overline{w}^{'}\,\overline{z}}{D_m}\right)\}\},$$

where $\bar{t} = te^{-2\lambda}$, $\bar{r} = re^{-\lambda}$, $\bar{z} = ze^{-\lambda}$, $\overline{w} = w(\overline{r})$;

$$C^{(6)} = \beta\left(\frac{t}{1-4\lambda t}, \frac{r}{1-4\lambda t}, \frac{z}{1-4\lambda t}\right) \exp\{\lambda[\frac{\bar{z}^2}{D_m} + 2\bar{t} - \frac{2\bar{z}\,\bar{w}'}{D_m}\bar{t}$$

$$+\frac{2\bar{w}\,\bar{r}^2\,\bar{z}}{D_m} + 2\bar{w}'\,\bar{t}^2\,\bar{r} + \bar{w}''\,\bar{z}\,\bar{t}^2 - \frac{\bar{w}^2\,\bar{t}^2}{D_m} + 8\bar{t}\left(\frac{\bar{w}\,\bar{z}}{2D_m} + \frac{\bar{w}'\,\bar{r}\,\bar{z}}{D_m} + \frac{\bar{w}''\,\bar{r}^2\,\bar{z}}{4D_m}\right)$$

$$+4\bar{t}^2\left(\frac{\bar{w}''\,\bar{r}\,\bar{z}}{2} + \frac{\bar{w}'\,\bar{z}}{D_m} - \frac{\bar{w}\,\bar{w}'}{8D_m} + \frac{\bar{w}'''}{8}\right) - 2\bar{t}\left(\frac{\bar{w}^2\,\bar{r}}{D_m^2} + \frac{\overline{w}\,\overline{z}}{D_m}\right)]\},$$

where $\bar{t} = \frac{t}{1-4\lambda t}$, $\bar{r} = \frac{r}{1-4\lambda t}$, $\bar{z} = \frac{z}{1-4\lambda t}$, $\overline{w} = w(\overline{r})$;

$$C^{(\alpha)} = \beta(t, r, z) + \lambda\alpha(t, r, z).$$

4. Conclusions

We have been able to use the method of group invariant solution as given in [8] to the unsteady diffusion equation. It may be remarked that the group invariant method is very effective and gives number of solutions for a given partial differential equation.

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