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HYPERCOMPLEX GEOMETRIC DERIVATIVE FROM A CAUCHY-LIKE INTEGRAL FORMULA

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Abstract: The derivation and integration of hipercomplex functions have been investigated along the years, see [7], [11], [14]. The main purpose of this brief article is to give a geometrical interpretation for quaternionic derivatives, based on a recent determination of a Cauchy-like formula for quaternions, see [3].

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1. Introduction and Motivation

Hipercomplex of octonionic and quaternionic types, and their applications, have been matter of intense research in the last years, both under and geometrical and algebraic point of view, see [15], [4]. In algebra a derivation may be re-

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garded on a algebra which generalizes certain features of the derivative operator. Specially, given an algebra A over a ring or a field K, a *K-derivation* is *K-linear* map $D : A \to A$ that satisfies Leibnitz's law:

$$D(ab) = (Da)b + aD(b).$$

Derivations may occur in many different contexts and in diverse areas of mathematics. In our geometrical context, we will take derivation form the Cauchy-like integral formula, following recent result obtained by two of us, see [5], [10].

The Cauchy integral formula for quaternions expresses a holomorphic function around a point q to the internal field by its value at the boundary of this domain. The purpose of this formula is making the integration a priori knowing the value of q on the boundary of the domain. The following theorem ensures this fact:

Theorem. (Cauchy Integral Formula for Quaternions) *Let Ω be a simply connected domain in the four-dimensional space and $f(q)$ a regular function on Ω. Then the result (see [5], [3], [10]):*

$$\int_{\varphi} \frac{f(q)}{q - q_0} dq = \pi(i + j + 2k) f(q_0), \tag{1}$$

where φ is a closed hypersurface in Ω and q_0 is any point φ.

The formula (1) can be written as follows:

$$f(q_0) = \frac{1}{\pi(i + j + 2k)} \int_{\varphi} \frac{f(q)}{q - q_0} dq. \tag{2}$$

This will facilitate further calculations.

2. The Case of One Complex Variable

For the case of a complex variable, the previous theorem can be written in the formula as follows (see [10]):

$$f(z_0) = \frac{1}{2\pi i} \int_{\varphi} \frac{f(z)}{z - z_0} dz,$$

where now φ is a closed curve, and z_0 is any point in φ. The derivatives are calculated using the formula:

$$f^n(z_0) = \frac{n!}{2\pi i} \int_{\varphi} \frac{f(z')}{(z' - z)^{n+1}} dz'.$$

3. The Quaternionic Case

Considering the following expression:

$$f(q_0) = \frac{1}{\pi(i+j+2k)} \int_{\varphi} \frac{f(q)}{q - q_0} dq.$$

Putting q' as integration variable and q as an internal point to the domain Ω, the expression is rewritten as follows

$$f(q_0) = \frac{1}{\pi(i+j+2k)} \int_{\varphi} \frac{f(q')}{q' - q} dq'.$$

In order to determine the first derivative $f'(q)$, we must take the difference:

$$\begin{aligned}
f(q+\Delta q) - f(q) &= \frac{1}{\pi(i+j+2k)} \int_{\varphi} \frac{f(q')}{q' - q - \Delta q} dq' \\
&\quad - \frac{1}{\pi(i+j+2k)} \int_{\varphi} \frac{f(q')}{q' - q} dq', \\
f(q+\Delta q) - f(q) &= \frac{1}{\pi(i+j+2k)} \int_{\varphi} \frac{f(q')(q'-q) - f(q')(q'-q-\Delta q)}{(q'-q)(q'-q-\Delta q)} dq', \\
f(q+\Delta q) - f(q) &= \frac{1}{\pi(i+j+2k)} \int_{\varphi} \frac{\Delta q f(q')}{(q'-q)(q'-q-\Delta q)} dq', \\
f(q+\Delta q) - f(q) &= \frac{\Delta q}{\pi(i+j+2k)} \int_{\varphi} \frac{f(q')}{(q'-q)(q'-q-\Delta q)} dq'.
\end{aligned}$$

Now it follows that:

$$\frac{f(q+\Delta q) - f(q)}{\Delta q} = \frac{1}{\pi(i+j+2k)} \int_{\varphi} \frac{f(q')}{(q'-q)(q'-q-\Delta q)} dq'.$$

Considering to the limit when $\Delta q \to 0$, we arrive at the expression:

$$f'(q) = \frac{1}{\pi(i+j+2k)} \int_{\varphi} \frac{f(q')}{(q'-q)^2} dq'.$$

In order to ensure that was done, it is necessary to show that the difference:

$$\zeta = \frac{1}{\pi(i+j+2k)} \int_{\varphi} \frac{f(q')}{(q'-q)^2} dq' - \frac{1}{\pi(i+j+2k)} \int_{\varphi} \frac{f(q')}{(q'-q)(q'-q-\Delta q)} dq'$$

tends to zero when $\Delta q \to 0$. Making a difference, it follows that:

$$\zeta = \frac{-\Delta q}{\pi(i+j+2k)} \int_{\varphi} \frac{f(q')}{(q'-q)^2(q'-q-\Delta q)} dq'.$$

The function $f(q')$ is continuous at φ, and its magnitude is limited, i.e. $|f(q')|< A.d$ d indicates the distance from the point q to the boundary φ. So we have that $|q'-q|\geq 2d$ $(q+\Delta q)$ for values of Δq zero near, is near q, so $|q'-(q+\Delta q)|> d$. Therefore

$$|\zeta|< \frac{|\Delta q|}{6\pi} \frac{A}{4d^3}.$$

It follows that $\zeta \to 0$ when $\Delta q \to 0$. Therefore,

$$f''(q) = \frac{2!}{\pi(i+j+2k)} \int_{\varphi} \frac{f(q')}{(q'-q)^3} dq'$$

and proceeding inductively, now it follows that:

$$f^n(q) = \frac{n!}{\pi(i+j+2k)} \int_{\varphi} \frac{f(q')}{(q'-q)^{n+1}} dq'$$

4. Conclusion

The use of Cauchy's integral formula for quaternions, is essential for determining a formula for the derivation of a quaternionic function, a fact that will be relevant in derivatives calculations with quaternion. The result can also be generalized to the octonionic case, which will be subject of future work.

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