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ON THE FOURIER TRANSFORM OF THE DIAMOND KLEIN-GORDON KERNEL

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Abstract: In this article, the operator $(\Diamond + m^2)^k$ is introduced and named as the diamond Klein-Gordon operator iterated k-times and is defined by

$$(\Diamond + m^2)^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 + m^2\right]^k,$$

where $p+q=n$ is the dimension of the space $\mathbb{R}^n$, for $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, m is a nonnegative real number and k is a nonnegative integer. In this work, we study the fundamental solution of operator $(\Diamond + m^2)^k$ and this fundamental solution is called the diamond Klein-Gordon kernel. Then, we study the Fourier transform of the diamond Klein-Gordon kernel and also the Fourier transform of their convolution.

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1. Introduction

The operator $\diamondsuit^k$ has been first introduced by A. Kananthai [7], is named as the diamond operator iterated k-times, and is defined by

$$\diamondsuit^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k, \quad p+q=n, \tag{1.1}$$

where n is the dimension of the space $\mathbb{R}^n$, for $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ and k is a nonnegative integer. The operator $\diamondsuit^k$ can be expressed in the form $\diamondsuit^k = \Box^k \triangle^k = \triangle^k \Box^k$, where the operator $\triangle^k$ is Laplacian iterated k-times, and is defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k, \tag{1.2}$$

and the operator $\Box^k$ is the ultra-hyperbolic operator iterated k-times, and is defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k. \tag{1.3}$$

In 1997, A. Kananthai [7] has shown that the convolution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is the fundamental solution of the operator $\diamondsuit^k$, that is

$$\diamondsuit^k \left((-1)^k R^e_{2k}(x) * R^H_{2k}(x)\right) = \delta, \tag{1.4}$$

where the function $R^H_{2k}(x)$ is defined by (2.1) and $R^e_{2k}(x)$ is defined by (2.6), with $\alpha = 2k$. The fundamental solution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is called the diamond kernel of Marcel Riesz. Moreover, A. Kananthai [2] has proved the convolution equation related to the diamond kernel of Marcel Riesz.

Next, A. Kananthai [3] has proved the convolutions of the diamond kernel of Marcel Riesz and he has also studied the linear equation (see [4])

$$\diamondsuit^k u(x) = f(x). \tag{1.5}$$

This equation is the generalization of the ultra-hyperbolic equation and it can be applied to the wave equation. The solution of an equation (1.5) is $u(x) = (-1)^k M_{2k,2k}(x) * f(x)$, where

$$M_{2k,2k}(x) = R^e_{2k}(x) * R^H_{2k}(x). \tag{1.6}$$

Later, A. Kananthai [6] has proved the nonlinear diamond operator are related to the n-dimensional wave equation. M.A. Tellez and A. Kananthai [12] have proved the convolution product of the distributional families related to the diamond operator. Moreover, A. Kananthai [5] has studied Fourier transform and convolutions of the diamond kernel of Marcel Riesz and also the Fourier transform of their convolution.

In 2004, H. Yıldırım et al [17] have first introduced the operator $\diamondsuit_B^k$ that is name as diamond Bessel operator iterated k-times, and is defined by

$$\diamondsuit_B^k = \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right]^k, \tag{1.7}$$

$B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$. In addition, they have studied the fundamental solution of the equation $\diamondsuit_B^k u(x) = \delta$, and this solution is called the Bessel diamond kernel of Riesz. Moreover, they have studied Fourier-Bessel transform and convolutions of the Bessel diamond kernel of Riesz and also the Fourier-Bessel transform of their convolution. Later, M.Z. Sarikaya and H. Yıldırım [10, 16] have studied the Bessel diamond and the nonlinear Bessel diamond operator related to the Bessel wave equation and B-convolution of the Bessel diamond kernel of Riesz.

Furthermore, S. E. Trione [14] has studied the fundamental solution of the ultra-hyperbolic Klein-Gordon operator iterated k-times, and is defined by

$$(\Box + m^2)^k = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} + m^2\right)^k. \tag{1.8}$$

The fundamental solution of the operator $(\Box+m^2)^k$ is $W_{2k}(x,m)$, and is defined by

$$W_{2k}(x,m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r!\Gamma(k)} (m^2)^r R_{2k+2r}^H(x), \tag{1.9}$$

where $R_{2k+2r}^H(x)$ is defined by (2.1) with $\alpha = 2k + 2r$. Next, Tellez [11] has studied the convolution product of $W_\alpha(x,m) * W_\beta(x,m)$ where α and β are any complex parameters. In addition, S.E. Trione [15] has studied the fundamental $(P \pm i0)^\lambda$-ultrahyperbolic solution of the Klein-Gordon operator iterated k-times and studied the convolution of such fundamental solution.

Later, K. Nonlaopon et al [8] have introduced the operator $(\diamondsuit+m^2)^k$ that is named as the diamond Klein-Gordon operator iterated k-times, and is defined by

$$(\diamondsuit + m^2)^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 + m^2 \right]^k, \tag{1.10}$$

where $p+q=n$ is the dimension of the space $\mathbb{R}^n$, for $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$, m is a nonnegative real number and k is a nonnegative integer. In this work, we study the fundamental solution of operator $(\diamondsuit+m^2)^k$ and this fundamental solution is called the diamond Klein-Gordon kernel. Then, we study the Fourier transform of the diamond Klein-Gordon kernel and also the Fourier transform of their convolution.

2. Preliminaries

Definition 2.1. Let $x=(x_1,x_2,\dots,x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$, denoted by

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 + \cdots + x_{p+q}^2,$$

the nondegenerated quadratic form, $p+q=n$ is the dimension of the space $\mathbb{R}^n$. Let $\Gamma_+ = \{x\in\mathbb{R}^n : x_1>0 \text{ and } u>0\}$ be the interior of forward cone and let $\overline{\Gamma}_+$ denote its closure. For any complex number α, define the function

$$R_\alpha^H(x) = \begin{cases} \frac{u^{(\alpha-n)/2}}{K_n(\alpha)}, & \text{for } x\in\Gamma_+, \\ 0, & \text{for } x\notin\Gamma_+, \end{cases} \tag{2.1}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma((2+\alpha-n)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((2+\alpha-p)/2)\Gamma((p-\alpha)/2)}. \tag{2.2}$$

The function $R_\alpha^H(x)$ is called the ultra-hyperbolic kernel of Marcel Riesz and was introduced by Y. Nozaki [9]. It is well known that $R_\alpha^H(x)$ is an ordinary function if $\mathrm{Re}(\alpha)\geq n$ and is a distribution of α if $\mathrm{Re}(\alpha)<n$. Let supp $R_\alpha^H(x)$ denote the support of $R_\alpha^H(x)$ and suppose that supp $R_\alpha^H(x)\subset\overline{\Gamma}_+$, that is, supp $R_\alpha^H(x)$ is compact.

By putting $p=1$ in $R_{2k}^H(x)$ and taking into account Legendre's duplication formula for $\Gamma(z)$, that is

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{2.3}$$

we obtain

$$I_\alpha^H(x) = \frac{v^{(\alpha-n)/2}}{H_n(\alpha)}, \tag{2.4}$$

and $v = x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2$, where

$$H_n(\alpha) = \pi^{(n-2)/2} 2^{\alpha-1} \Gamma\left(\frac{\alpha+2-n}{2}\right) \Gamma\left(\frac{\alpha}{2}\right). \tag{2.5}$$

$I_\alpha^H(x)$ is called the hyperbolic kernel of Marcel Riesz.

Definition 2.2. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$ and $\omega = x_1^2 + x_2^2 + \cdots + x_n^2$, then the function $R_\alpha^e(x)$ denote the elliptic kernel of Marcel Riesz, and is defined by

$$R_\alpha^e(x) = \frac{\omega^{(\alpha-n)/2}}{W_n(\alpha)}, \tag{2.6}$$

where

$$W_n(\alpha) = \frac{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}, \tag{2.7}$$

α is a complex parameter and n is the dimension of $\mathbb{R}^n$.

By (2.1) and (2.2) with $q = 0$, then $u^{(\alpha-n)/2}$ reduces to $\omega_p^{(\alpha-p)/2}$, where $\omega_p = x_1^2 + x_2^2 + \cdots + x_p^2$ and $K_n(\alpha)$ reduces to

$$K_p(\alpha) = \frac{\pi^{(p-1)/2} \Gamma((1-\alpha)/2) \Gamma(\alpha)}{\Gamma((p-\alpha)/2)}.$$

By using the Legendre's duplication formula

$$\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right), \tag{2.8}$$

and

$$\Gamma\left(\frac{1}{2} + z\right) \Gamma\left(\frac{1}{2} - z\right) = \pi \sec(\pi z). \tag{2.9}$$

We obtain

$$K_p(\alpha) = \frac{1}{2} \sec\left(\frac{\pi\alpha}{2}\right) W_p(\alpha), \tag{2.10}$$

where $W_p(\alpha)$ is defined by (2.7) with $n = p$. Thus, for $q = 0$,

$$R_\alpha^H(x) = \frac{u^{(\alpha-p)/2}}{K_p(\alpha)} = 2\cos\left(\frac{\pi\alpha}{2}\right) \frac{u^{(\alpha-p)/2}}{W_p(\alpha)} = 2\cos\left(\frac{\pi\alpha}{2}\right) R_\alpha^e(x), \tag{2.11}$$

Thus, in case of $\alpha = 2k$,

$$R^H_{2k}(x) = 2(-1)^k R^e_{2k}(x), \tag{2.12}$$

for $q = 0$ and $\omega_p = x_1^2 + x_2^2 + \cdots + x_p^2$.

The proof of the following lemma is given in [7] and [3].

Lemma 2.1. *$R^e_\alpha(x)$ and $R^H_\alpha(x)$ are the tempered distributions.*

From S.E. Trione [13], $R^H_{2k}(x)$ is the fundamental solution of the operator $\Box^k$, that is

$$\Box^k \left(R^H_{2k}(x)\right) = \delta. \tag{2.13}$$

Moreover, we obtain $(-1)^k R^e_{2k}(x)$ is the fundamental solution of the operator $\triangle^k$ (see [1]). That is,

$$\triangle^k((-1)^k R^e_{2k}(x)) = \delta. \tag{2.14}$$

It can be shown that $R^H_{-2k}(x) = \Box^k\delta$ and $R^e_{-2k}(x) = (-1)^k\triangle^k\delta$ for k is a nonnegative integer (see [13, 12]).

Let K be a compact set and $K \subset \overline{\Gamma}_+$ where $\overline{\Gamma}_+$ is defined as in the beginning. Choose the support of $R^H_{2k}(x)$ such that it is equal to K, then supp $R^H_{2k}(x)$ is compact (closed and bounded). So the convolution

$$(-1)^k R^e_{2k}(x) * R^H_{2k}(x) \tag{2.15}$$

exists and is a tempered distribution from lemma 2.1.

Lemma 2.2. *The convolution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is the fundamental solution of the diamond operator iterated k-times, that is*

$$\diamondsuit^k \left((-1)^k R^e_{2k}(x) * R^H_{2k}(x)\right) = \delta. \tag{2.16}$$

The proof of this Lemma is given in [7] and [12].

It can be shown that $R^e_{-2k}(x) * R^H_{-2k}(x) = (-1)^k \diamondsuit^k \delta(x)$, for k is a nonnegative integer.

Definition 2.3. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$, the function $T_\alpha(x, m)$ is defined by

$$T_\alpha(x, m) = \sum_{r=0}^{\infty} \binom{-\alpha/2}{r} (m^2)^r (-1)^{\alpha/2+r} R^e_{\alpha+2r}(x) * R^H_{\alpha+2r}(x), \tag{2.17}$$

where α is a complex parameter, m is a nonnegative real number, $R^H_{\alpha+2r}(x)$ and $R^e_{\alpha+2r}(x)$ are defined by (2.1) and (2.6), respectively.

From the definition of $T_\alpha(x,m)$ and by putting $\alpha = -2k$, we have

$$T_{-2k}(x,m) = \sum_{r=0}^{\infty} \binom{k}{r} (m^2)^r (-1)^{-k+r} R^e_{2(-k+r)}(x) * R^H_{2(-k+r)}(x).$$

Since the operator $(\diamondsuit + m^2)^k$ defined by (1.10) is linearly continuous and has 1-1 mapping of this possess its own inverses. From Lemma 2.2, we obtain

$$T_{-2k}(x,m) = \sum_{r=0}^{\infty} \binom{k}{r} (m^2)^r \diamondsuit^{k-r}\delta = (\diamondsuit + m^2)^k \delta. \tag{2.18}$$

By putting $k = 0$ in (2.18), we have $T_0(x,m) = \delta$. By putting $\alpha = 2k$ into (2.22), we have

$$\begin{aligned} T_{2k}(x,m) =& \binom{-k}{0} (m^2)^0 (-1)^{k+0} R^e_{2k+0}(x) * R^H_{2k+0}(x) \\ &+ \sum_{r=1}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R^e_{2k+2r}(x) * R^H_{2k+2r}(x). \end{aligned} \tag{2.19}$$

The second summand of the right-hand member of (2.19) vanishes for $m = 0$ and then, we have

$$T_{2k}(x, m = 0) = (-1)^k R^e_{2k}(x) * R^H_{2k}(x),$$

which is the fundamental solution of the diamond operator.

Definition 2.4. Let $f(x) \in L^1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\widehat{f}(\xi) = \mathcal{F}f(x)) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx, \tag{2.20}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n), x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \xi \cdot x = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the usual inner product in $\mathbb{R}^n$ and $dx = dx_1\, dx_2 \cdots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi\cdot x} \widehat{f}(\xi)\, d\xi. \tag{2.21}$$

If $f(x)$ is distribution with compact supports by [18], the equation (2.20) can be written as

$$\widehat{f}(\xi) = \mathcal{F}f(x) = \frac{1}{(2\pi)^{n/2}} \langle f(x), e^{-i\xi\cdot x}\rangle. \tag{2.22}$$

The proof of the following Lemmas 2.3 and 2.4 are given in [8].

Lemma 2.3. *Given the equation*

$$(\diamondsuit + m^2)^k u(x) = \delta, \tag{2.23}$$

where $(\diamondsuit + m^2)^k$ *is the diamond Klein-Gordon operator, and is defined by*

$$(\diamondsuit + m^2)^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 + m^2 \right]^k, \tag{2.24}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, k *is a nonnegative integer,* m *is a nonnegative real number and* δ *is the Dirac-delta distribution. Then we obtain*

$$T_{2k}(x,m) = \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R^e_{2k+2r}(x) * R^H_{2k+2r}(x) \tag{2.25}$$

is the fundamental solution of the operator $(\diamondsuit + m^2)^k$*, defined by (3.1), where* $R^H_{2k}(x)$ *and* $R^e_{2k}(x)$ *are defined by (2.1) and (2.6), respectively. Moreover,* $u(x) = T_{2k}(x,m)$ *is tempered distribution.*

Lemma 2.4. *Let* $T_{2k}(x,m)$ *be the diamond Klein-Gordon kernel is defined by (3.2), then* $T_{2k}(x,m)$ *is a tempered distribution and can be expressed by*

$$T_{2k}(x,m) = T_{2k-2v}(x,m) * T_{2v}(x,m)$$

where v *is nonnegative integer and* $v < k$*. Moreover, if we put* $l = k - v$ *and* $h = v$*, then we obtain*

$$T_{2l}(x,m) * T_{2h}(x,m) = T_{2l+2h}(x,m) \qquad \text{for } l + h = k.$$

Lemma 2.5. (The Fourier Transform of $(\diamondsuit^k + m^2)^k \delta$) *Let* $\|\xi\| = (\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^{1/2}$ *for* $\xi \in \mathbb{R}^n$*. Then*

$$\left| \mathcal{F}(\diamondsuit^k + m^2)^k \delta \right| \leq \frac{1}{(2\pi)^{n/2}} (\|\xi\|^2 + m^2)^k. \tag{2.26}$$

That is, $\mathcal{F}(\diamondsuit^k + m^2)^k \delta$ *is bounded and continuous on the space* $\mathcal{S}'$ *of the tempered distribution. Moreover, by the inverse Fourier transformation*

$$(\diamondsuit^k + m^2)^k \delta = \mathcal{F}^{-1} \frac{1}{(2\pi)^{n/2}} \left[\left(\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2 \right)^2 - \left(\xi_{p+1}^2 + \cdots + \xi_{p+q}^2 \right)^2 + m^2 \right]^k.$$

Proof. From the Fourier transform (2.22), we have

$$\mathcal{F}(\diamondsuit^k + m^2)^k\delta = \frac{1}{(2\pi)^{n/2}}\left\langle(\diamondsuit^k + m^2)^k\delta, e^{-i\xi\cdot x}\right\rangle$$
$$=\frac{1}{(2\pi)^{n/2}}\left\langle\delta, (\diamondsuit^k + m^2)^k e^{-i\xi\cdot x}\right\rangle$$
$$=\frac{1}{(2\pi)^{n/2}}\left\langle\delta, \left[\left(\xi_1^2+\cdots+\xi_p^2\right)^2 - \left(\xi_{p+1}^2+\cdots+\xi_{p+q}^2\right)^2 + m^2\right]^k e^{-i\xi\cdot x}\right\rangle$$
$$=\frac{1}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^2 - \left(\xi_{p+1}^2+\cdots+\xi_{p+q}^2\right)^2 + m^2\right]^k.$$

Now

$$\left|\mathcal{F}(\diamondsuit^k + m^2)^k\delta\right|$$
$$=\frac{1}{(2\pi)^{n/2}}\left[\left|\xi_1^2+\cdots+\xi_n^2\right|\left|\xi_1^2+\cdots+\xi_p^2-\xi_{p+1}^2\cdots-\xi_{p+q}^2\right|+m^2\right]^k$$
$$\leq\frac{1}{(2\pi)^{n/2}}\left[\left|\xi_1^2+\xi_2^2+\cdots+\xi_n^2\right|^2+m^2\right]^k$$
$$=\frac{1}{(2\pi)^{n/2}}(\|\xi\|^2+m^2)^k,$$

where $\|\xi\| = \left(\xi_1^2+\xi_2^2+\cdots+\xi_n^2\right)^{1/2}$, $\xi_i(i=1,2,\ldots,n)\in\mathbb{R}$. Hence we obtain (2.26) and $\mathcal{F}(\diamondsuit^k + m^2)^k\delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution.

Since $\mathcal{F}$ is 1-1 transformation from the space $\mathcal{S}'$ of the tempered distribution to the real space $\mathbb{R}$, then by (2.21) we have

$$(\diamondsuit^k + m^2)^k\delta$$
$$=\frac{1}{(2\pi)^{n/2}}\mathcal{F}^{-1}\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^2 - \left(\xi_{p+1}^2+\cdots+\xi_{p+q}^2\right)^2 + m^2\right]^k.$$

That completes the proof. □

3. Main Results

Theorem 3.1.

$$\mathcal{F}T_{2k}(x,m) = \frac{1}{(2\pi)^{n/2}\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^2 - \left(\xi_{p+1}^2+\cdots+\xi_{p+q}^2\right)^2 + m^2\right]^k}$$

and
$$|\mathcal{F}T_{2k}(x,m)| \leq \frac{1}{(2\pi)^{n/2}} M \quad \text{for a large } \xi_i \in \mathbb{R}, \tag{3.1}$$
where M is a constant. That is, $\mathcal{F}$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distributions.

Proof. By Lemma 2.3,
$$(\Diamond^k + m^2)^k T_{2k}(x,m) = \delta,$$
or
$$(\Diamond^k + m^2)^k \delta * T_{2k}(x,m) = \delta. \tag{3.2}$$
If we applied the Fourier transform on both sides of (3.2), then we obtain
$$\mathcal{F}\left((\Diamond^k + m^2)^k \delta * T_{2k}(x,m)\right) = \mathcal{F}\delta = \frac{1}{(2\pi)^{n/2}}.$$
By (2.22), we have
$$\frac{1}{(2\pi)^{n/2}} \left\langle (\Diamond^k + m^2)^k \delta * T_{2k}(x,m), e^{-i\xi\cdot x} \right\rangle = \frac{1}{(2\pi)^{n/2}}.$$
By the definition of convolution
$$\frac{1}{(2\pi)^{n/2}} \left\langle (\Diamond^k + m^2)^k \delta, \left\langle T_{2k}(y,m), e^{-i\xi\cdot(x+y)} \right\rangle \right\rangle = \frac{1}{(2\pi)^{n/2}},$$
$$\frac{1}{(2\pi)^{n/2}} \left\langle T_{2k}(y,m), e^{-i\xi\cdot y} \right\rangle \left\langle (\Diamond^k + m^2)^k \delta, e^{-i\xi\cdot x} \right\rangle = \frac{1}{(2\pi)^{n/2}},$$
$$\mathcal{F}T_{2k}(x,m)(2\pi)^{n/2}\mathcal{F}\left((\Diamond^k + m^2)^k \delta\right) = \frac{1}{(2\pi)^{n/2}}.$$
By Lemma 2.5, we obtain
$$\mathcal{F}T_{2k}(x,m)\left[\left(\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2\right)^2 - \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2\right)^2 + m^2\right]^k = \frac{1}{(2\pi)^{n/2}}.$$
It follows that
$$\mathcal{F}T_{2k}(x,m) = \frac{1}{(2\pi)^{n/2}\left[\left(\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2\right)^2 - \left(\xi_{p+1}^2 + \cdots + \xi_{p+q}^2\right)^2 + m^2\right]^k}.$$

Now,

$$\mathcal{F}T_{2k}(x,m) = \frac{1}{\left(\left|\xi_1^2+\cdots+\xi_n^2\right|\left|\xi_1^2+\cdots+\xi_p^2-\xi_{p+1}^2-\cdots-\xi_{p+q}^2\right|+m^2\right)^k},$$

where $\xi \in (\xi_1, \xi_2, \ldots, \xi_n) \in \Gamma_+$ with Γ_+ defined by Definition 2.1. Then $(\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots \xi_{p+q}^2) > 0$ and for a large ξ_i and a large k, the right-hand side of (3.2) tends to zero. It follows that it is bounded by positive constant say M, that is, we obtain (3.1) as required and also by (3.1), $\mathcal{F}$ is continuous on the space $\mathcal{S}'$ of the tempered distribution. □

Theorem 3.2.

$$\begin{aligned}&\mathcal{F}\left(T_{2k}(x,m) * T_{2l}(x,m)\right) = 2\pi^{n/2}\mathcal{F}[T_{2k}(x,m)]\mathcal{F}[T_{2l}(x,m)]\\ &= \frac{1}{(2\pi)^{n/2}}\frac{1}{\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^2-\left(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2\right)^2+m^2\right]^{k+l}},\end{aligned}$$

where k and l are nonnegative integers and $\mathcal{F}$ is bounded and continuous on the spaces $\mathcal{S}'$ of tempered distribution.

Proof. From Lemma 2.4, we have

$$T_{2k}(x,m) * T_{2l}(x,m) = T_{2k+2l}(x,m), \tag{3.3}$$

where k and l are nonnegative integers. Taking Fourier transform on both sides of (3.3) and using Theorem 3.1, we obtain

$$\begin{aligned}&\mathcal{F}\left(T_{2k}(x,m) * T_{2l}(x,m)\right) = \mathcal{F}\left(T_{2k+2l}(x,m)\right)\\ &=\frac{1}{(2\pi)^{n/2}\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^2-\left(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2\right)^2+m^2\right]^{k+l}},\\ &=\frac{1}{(2\pi)^{n/2}\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^2-\left(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2\right)^2+m^2\right]^{k}}\\ &\quad\times\frac{(2\pi)^{n/2}}{(2\pi)^{n/2}\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^2-\left(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2\right)^2+m^2\right]^{l}},\\ &=2\pi^{n/2}\mathcal{F}\left[T_{2k}(x,m)\right]\mathcal{F}\left[T_{2l}(x,m)\right].\end{aligned}$$

Since $T_{2k+2l}(x,m) \in \mathcal{S}'$, the space of tempered distribution, and by Theorem 3.1 we obtain that $\mathcal{F}$ is bounded and continuous on $\mathcal{S}'$. □

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