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DERIVATION AND NUMERICAL STUDY OF A NEW VISCOUS SHALLOW WATER BIFLUID MODEL

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Abstract: We consider the flow of two Newtonian, incompressible and non-miscible fluids in a 2D thin domain. Starting from the Navier-Stokes equations and using the method of asymptotic expansions we derive a 1D viscous shalow water bifluid model. Numerical simulations are performed using a finite volume method.

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1. Introduction

The study of a two-phase flow in a thin domain is introduced in [9]: in this paper, using the method of asymptotic expansion with respect to the thickness of the domain, A. Mikelić and L. Paoli derived an effective 1D equation. Supposing that the fluids are strictly separated at $t = 0$, they show that the derived equations equal the Buckley-Leverett equation for the saturation, in conjunction

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with the Reynolds law for each of the flowing phases. Then in [11] starting from the Stokes equations, L. Paoli derives a generalized Buckley-Leverett equation for the first fluid saturation.

In the two papers quoted above the authors start from the Stokes equations. We consider here the complete Naviers-Stokes equations. Note that we perform here a formal asymptotic analysis.

In this paper, we obtain a viscous shallow water bifluid model extending the Gerbeau-Perthame model (see [6]) without using the mean velocity $\bar{u} = \frac{1}{h}\int_0^h u\,dy$, where u is the velocity of the fluid, h the depth of the domain and y the vertical space variable.

This paper is orgonised as follows: In Section 2 we present th initial problem. Next, in Section 3 we just present our model. In Section 4 we perform a formal asymptotic analysis. In Section 5 we show how to recover our model by an averaging procedure. Lastly we present in Section 6 some numerical experiments.

2. The Initial Problem

We consider a thin domain which has a small characteristic depth noted H_0, compared to its characteristic length denoted L (for example a channel). This domain is defined as:

$$\Omega = \{(x,y),\ \ x \in [0,L]\ \ y \in [0,h(t,x)]\}\ \text{(see Figure 1).}$$

The domain Ω is occupied by two immiscible fluids. We denote a the height of the lower fluid and h the total height. We assume that the velocity is continuous at the interface between the two fluids. Then the flow is governed by the following Navier-Stokes equations

$$(NS1)\ \left\{\begin{array}{ll} \rho\,\partial_t u + \rho\,(u.\nabla)u - 2\mathrm{div}(\eta D(u)) + \nabla p = \rho\, g\, e_2 & \text{in}\ \ \Omega\times]0,T[= Q_T, \\ \mathrm{div}(u) = 0 & \text{in}\ \ Q_T, \end{array}\right.$$

with the initial data $u(t=0) = u_0$ in Ω. We use here and in the sequel the notations:

$$\partial_x = \frac{\partial}{\partial x},\ \ \partial_y = \frac{\partial}{\partial y},\ \ \partial_t = \frac{\partial}{\partial t}.$$

We note $u = (v,w)$ the velocity, $D(u) = \dfrac{1}{2}(\nabla u + {}^t\nabla u)$ the viscosity tensor, $e_2 = {}^t(0,1)$, p the pressure and g the gravitational constant.

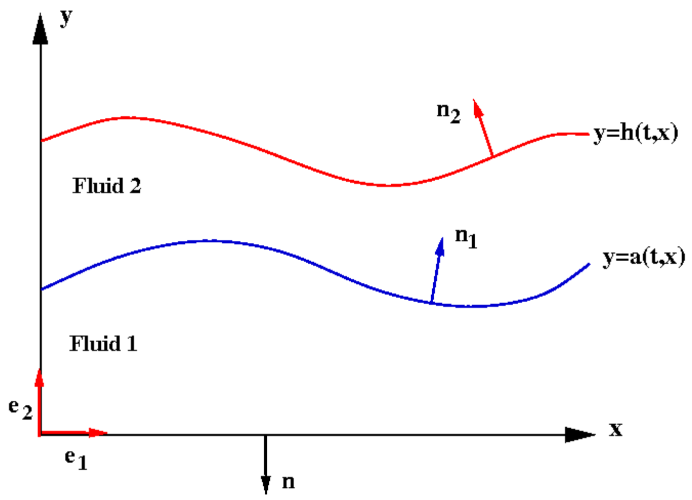

Figure 1: Bifluid flow

The density ρ and the viscosity η are constant for each fluid. They satisfy the following advection equations

$$(adv) \qquad \begin{cases} \partial_t \eta + \operatorname{div}(\eta\, u) = 0 & \text{in } Q_T, \\ \partial_t \rho + \operatorname{div}(\rho\, u) = 0 & \text{in } Q_T, \\ \eta(t=0) = \eta_0 \in \{\eta_1, \eta_2\} & \text{in } \Omega, \\ \rho(t=0) = \rho_0 \in \{\rho_1, \rho_2\} & \text{in } \Omega, \end{cases}$$

where $\eta_1 > 0$, $\rho_1 > 0$ are respectively the viscosity and the density of the lower fluid and $\eta_2 > 0$, $\rho_2 > 0$ are respectively the viscosity and the density of the upper fluid.

Let us define the unit vectors $n_1 = {}^t(0, -1)$, $n_3 = \frac{1}{\sqrt{1+(\partial_x a)^2}}\, {}^t(-\partial_x a, 1)$ and $n_2 = \frac{1}{\sqrt{1+(\partial_x h)^2}}\, {}^t(-\partial_x h, 1)$ normal to the bottom, the interface and the free boundary, respectively (see Figure 1), $\tau = \frac{1}{\sqrt{1+(\partial_x a)^2}}\, {}^t(1, \partial_x a)$ the tangent vector at the interface and let $\sigma_i = -p_i\, I + 2\eta_i\, D(U)$ be the stress tensor for the fluid i, $i = 1, 2$.

One adds the following boundary conditions:

- *At the free surface*, i.e. for $y = h(t, x)$.

$$\begin{cases} \partial_t h - \sqrt{1 + (\partial_x h)^2}\, u.n_2 = 0: \text{ classical free boundary condition,} \\ \sigma_2.n_2 = 0: \text{ normal-stress continuity.} \end{cases}$$

These conditions write also:

$$(fs)\qquad \begin{cases} \partial_t h + v\,\partial_x h - w = 0, \\ p\,\partial_x h + \eta_2\,(-2\partial_x h \partial_x v + \partial_y v + \partial_x w) = 0 \quad \text{at } \; y = h(t,x), \\ -p + \eta_2\,(-\partial_x h\,\partial_y v - \partial_x w\,\partial_x h + 2\partial_y w) = 0. \end{cases}$$

• *At the interface*, i.e. for $y = a$.

With the continuity of the velocity, the friction vanishes at the interface. Thus we have

$$(\sigma_i(u).n_3)_\tau = 0, \quad i = 1,\, 2.$$

From this equation we obtain

$$-2\eta_i\,\partial_x v\,\partial_x a + \eta_i\,(\partial_x w + \partial_y v)(1 - (\partial_x a)^2) + 2\eta_i\,\partial_y w\,\partial_x a = 0, \quad i = 1,\, 2. \tag{1}$$

• *At the bottom*, i.e. $y = 0$.

We consider a Navier condition with a friction coefficient K and a no-penetration condition

$$\begin{cases} w = 0, \\ \sigma_1.n_1 = -Ku. \end{cases}$$

That is also:

$$(b)\qquad \begin{cases} w = 0, & \text{at } \; y = 0 \\ Kv - \eta_1\,(\partial_y v + \partial_x w) = 0. \end{cases}$$

System $(NS1) - (adv)$ can be written as follows:

$$(NS)\qquad \begin{cases} \rho\,\partial_t v + \rho\,v\,\partial_x v + \rho\,w\,\partial_y v - 2\partial_x(\eta\,\partial_x v) - \partial_y(\eta\,\partial_y v) - \partial_y(\eta\,\partial_x w) \\ \qquad\qquad + \partial_x p = 0, \\ \rho\,\partial_t w + \rho\,v\,\partial_x w + \rho\,w\,\partial_y w - \partial_x(\eta\,\partial_x w) - 2\partial_y(\eta\,\partial_y w) - \partial_x(\eta\,\partial_y v) \\ \qquad\qquad + \partial_y p = -\rho\,g, \\ \partial_x v + \partial_y w = 0, \\ \partial_t \eta + \partial_x(\eta\,v) + \partial_y(\eta\,w) = 0, \\ \partial_t \rho + \partial_x(\rho\,v) + \partial_y(\rho\,w) = 0, \\ \eta(t=0) = \eta_0 \in \{\eta_1, \eta_2\}, \;\; (\eta_1, \eta_2) \in \mathbb{R}^2, \\ \rho(t=0) = \rho_0 \in \{\rho_1, \eta_2\}, \;\; (\rho_1, \rho_2) \in \mathbb{R}^2. \end{cases}$$

We will start from these equations joined to (fs) and (b) to derive our bifluid model presented in the next section.

3. A Viscous Shalow Water Bifluid Model

Using a suitable scaling, then averaging according to the vertical direction, we will obtain the model in the following form:
(S)

$$\begin{cases} \partial_t a + \partial_x (a\, v) = 0, \\ \partial_t h + \partial_x (h\, v) = 0, \\ (\rho_2\, h + (\rho_1 - \rho_2)\, a)\partial_t v + (\rho_2\, h + (\rho_1 - \rho_2)\, a)\, v \partial_x v - 4(\eta_1 - \eta_2)\Big(\partial_x (a\, \partial_x v) \\ \qquad + \partial_x a \partial_x v\Big) - 4\eta_2\, \partial_x (h\, \partial_x v) + \rho_2\, g\, h \partial_x h + g\, (\rho_1 - \rho_2)\, a \partial_x a = -K\, v. \end{cases}$$

Notice that this model uses a common velocity for the two fluids. The heights a and h are advected with this velocity.

We notice that if $\eta_1 = \eta_2 = \mu$ and $\rho_1 = \rho_2 = 1$, then we obtain the Gerbeau-Perthame's system (see [6]):

$$\begin{cases} \partial_t h + \partial_x (h\, v) = 0, \\ \partial_t (h\, v) + \partial_x (h\, v^2) + \dfrac{g}{2}\, \partial_x (h^2) = -K\, v + 4\mu \partial_x (h\, \partial_x v). \end{cases}$$

The continuation of the paper is devoted to the details of this procedure.

4. Asymptotic Analysis

We introduce a small parameter ε equal to the ratio between the characteristic depth, noted H_0, and the characteristic length L of the channel.

4.1. Changes of Variables

As in [1], we use the following changes of variables

$$z = \frac{y}{\varepsilon}, \quad H(t,x) = \frac{h(t,x)}{\varepsilon}, \quad A(t,x) = \frac{a(t,x)}{\varepsilon}, \quad K = \varepsilon k, \quad G = \varepsilon g.$$

We have $0 \le y \le h(t,x)$, thus $0 \le z \le H(t,x)$. With these changes of variable, System (NS) becomes:

$$(NS_\varepsilon)\quad \begin{cases} \partial_t v + v\,\partial_x v + \dfrac{1}{\varepsilon}\, w\,\partial_z v - 2\partial_x(\eta\,\partial_x v) - \dfrac{1}{\varepsilon^2}\,\partial_z(\eta\,\partial_z v) - \dfrac{1}{\varepsilon}\,\partial_z(\eta\,\partial_x w) \\ \qquad\qquad + \partial_x p = 0, \\ \partial_t w + v\,\partial_x w + \dfrac{1}{\varepsilon}\, w\,\partial_z w - \partial_x(\eta\,\partial_x w) - \dfrac{2}{\varepsilon^2}\,\partial_z(\eta\,\partial_z w) - \dfrac{1}{\varepsilon}\,\partial_x(\eta\,\partial_z v) \\ \qquad\qquad + \frac{1}{\varepsilon}\,\partial_z p = -\frac{\rho G}{\varepsilon}, \\ \partial_x v + \frac{1}{\varepsilon}\,\partial_z w = 0, \\ \partial_t \eta + \partial_x(\eta\, v) + \frac{1}{\varepsilon}\,\partial_z(\eta\, w) = 0, \\ \partial_t \rho + \partial_x(\rho\, v) + \frac{1}{\varepsilon}\,\partial_z(\rho w) = 0, \\ \eta(t=0) = \eta_0 \in \{\eta_1, \eta_2\}, \\ \rho(t=0) = \eta_0 \in \{\rho_1, \rho_2\} \end{cases}$$

and the boundary conditions (fs), (1) and (b) become, respectively:

$$(fs_\varepsilon)\quad \begin{cases} \varepsilon\,\partial_t H + \varepsilon\, v\,\partial_x H - w = 0, \\ \varepsilon\, p\,\partial_x H + \eta_2\,(-2\varepsilon\,\partial_x H\,\partial_x v + \dfrac{1}{\varepsilon}\,\partial_z v + \partial_x w) = 0 \quad \text{at} \quad z = H, \\ -p + \eta_2\,(-\partial_x H\,\partial_z v - \varepsilon\,\partial_x v\,\partial_x H + \dfrac{2}{\varepsilon}\,\partial_z w) = 0, \end{cases}$$

$$-2\varepsilon\eta_i\partial_x v\partial_x A + \eta_i(\partial_x w + \frac{1}{\varepsilon}\partial_z v)(1 - \varepsilon^2(\partial_x A)^2) + 2\eta_i\partial_z w\partial_x A = 0, \tag{2}$$

$$(b_\varepsilon)\quad \begin{cases} w = 0, & \text{at } z = 0, \\ K\, v - \eta_1\,(\dfrac{1}{\varepsilon}\,\partial_z v + \partial_x w) = 0. \end{cases}$$

4.2. Formal Asymptotic Expansions

Here we suppose that the unknowns can be formally written as

$$v = \sum_{i\geq 0} \varepsilon^i\, v^i, \quad w = \sum_{i\geq 0} \varepsilon^i\, w^i, \quad \eta = \sum_{i\geq 0} \varepsilon^i\, \eta^i, \quad \rho = \sum_{i\geq 0} \varepsilon^i\, \rho^i,$$

$$p = \sum_{i\geq 0} \varepsilon^i\, p^i, \quad H = \sum_{i\geq 0} \varepsilon^i\, H^i.$$

Then we replace each unknown by its value in the preceding equations and do an identification according to the power of ε.

In what follows the equations given by (fs_ε) are evaluated at the free surface and those given by (b_ε) are evaluated at the bottom.

- At order ε^{-2}, (NS_ε) gives

$$\partial_z(\eta^0\,\partial_z v^0) = 0. \tag{3}$$

- At order ε^{-1}, (NS_ε) gives

$$w^0\,\partial_z v^0 - \partial_z(\eta^0\,\partial_z v^1) - \partial_z(\eta^1\partial_z v^0) - \partial_z(\eta^0\,\partial_x w^0) = 0, \tag{4}$$

$$w^0\,\partial_z w^0 - 2\partial_z(\eta^0\,\partial_z w^1) - 2\partial_z(\eta^1\,\partial_z w^0) - \partial_x(\eta^0\,\partial_z v^0) + \partial_z p^0 = -\rho^0\,G, \tag{5}$$

$$\partial_z w^0 = 0. \tag{6}$$

On the boundary, $(f s_\varepsilon)$ gives

$$\partial_z v^0 = 0 \qquad \text{at} \qquad z = H(t,x), \tag{7}$$

and (b_ε) gives

$$\partial_z v^0 = 0 \qquad \text{at} \qquad z = 0. \tag{8}$$

- At order ε^0, (NS_ε) gives

$$\partial_t v^0 + v^0\,\partial_x v^0 + w^0\partial_z v^1 + w^1\,\partial_z v^0 - 2\partial_x(\eta^0\,\partial_x v^0)$$
$$-\partial_z(\eta^0\,\partial_z v^2 + \eta^1\,\partial_z v^1 + \eta^2\,\partial_z v^0 + \eta^0\,\partial_x w^1 + \eta^1\,\partial_x w^0) + \partial_x p^0 = 0, \tag{9}$$

$$\partial_x v^0 + \partial_z w^1 = 0, \tag{10}$$

$$\partial_t \eta^0 + \partial_x(\eta^0\,v^0) + \partial_z(\eta^0\,w^1) + \partial_z(\eta^1\,w^0) = 0. \tag{11}$$

On the boundary $(f s_\varepsilon)$ gives

$$w^0 = 0 \qquad \text{at} \qquad z = H(t,x), \tag{12}$$

$$\partial_z v^1 + \partial_x w^0 = 0 \qquad \text{at} \qquad z = H(t,x), \tag{13}$$

$$-p^0 + \eta_2\,(-\partial_x H^0\,\partial_z v^0 + 2\partial_z w^1) = 0 \qquad \text{at} \qquad z = H(t,x), \tag{14}$$

and (b_ε) gives

$$w^0 = 0 \qquad \text{at} \qquad z = 0, \tag{15}$$

$$\eta_1\,(\partial_z v^1 + \partial_x w^0) = 0 \qquad \text{at} \qquad z = H(t,x). \tag{16}$$

- At order ε we use only the boundary conditions, so $(f s_\varepsilon)$ gives

$$\partial_t H^0 + v^0\,\partial_x H^0 - w^1 = 0 \qquad \text{at} \qquad z = H(t,x), \tag{17}$$

$$p^0\,\partial_x H^0 + \eta_2\,(-2\partial_x H^0\,\partial_x v^0 + \partial_z v^2 + \partial_x w^1) = 0 \qquad \text{at} \qquad z = H(t,x). \tag{18}$$

From (i_ε), we have

$$-2\eta_i^0\,\partial_x v^0\partial_x A^0 + \eta_i^0\,(\partial_x w^1 + \partial_z v^2) + 2\eta_i^0\partial_z w^1\,\partial_x A^0 = 0 \quad \text{at} \quad z = A(t,x), \tag{19}$$

and (b_ε) gives

$$w^1 = 0 \qquad \text{at} \qquad z = 0, \tag{20}$$

$$k\,v^0 - \eta_1\,(\partial_z v^2 + \partial_x w^1) = 0 \qquad \text{at} \qquad z = 0. \tag{21}$$

5. Derivation of the Model

In this section we do the preliminary computations allowing us to establish the basic equations. Next, we average these equations with respect to the vertical space variable to obtain our model.

5.1. Preliminary Computations

With (6), (15) and (12) we get $w^0 = 0$.

By using (3), (7) and (8) one shows that $\partial_z v^0(t,x,z) = 0$ in each subdomain. In the same way (4), (13) and (16) imply $\partial_z v^1 = 0$, thus v^0 and v^1 are independent of z in each subdomain. So (9) becomes

$$\partial_t v^0 + v^0 \partial_x v^0 - 2\partial_x(\eta^0 \partial_x v^0) - \partial_z\left(\eta^0\, \partial_z v^2 + \eta^0\, \partial_x w^1\right) + \partial_x p^0 = 0. \tag{22}$$

Equation (5) becomes

$$-2\partial_z\left(\eta^0\, \partial_z w^1\right) + \partial_z p^0 = -\rho^0\, G \tag{23}$$

and (11) becomes

$$\partial_t \eta^0 + v^0\, \partial_x \eta^0 + \eta^0\, \partial_x v^0 + \partial_z(\eta^0\, w^1) = 0. \tag{24}$$

Using (10) we can write (17) as

$$\partial_t H^0 + v^0\, \partial_x H^0 + H^0\, \partial_x v^0 = 0. \tag{25}$$

The main difficulty is related to the discontinuity of η^0 through the interface:

$$\eta^0 = \eta_1\, \chi_{[0,a(t,x)[}(z) + \eta_2\, \chi_{]a(t,x),H(t,x)]}(z).$$

As in [10], we get

$$\int_0^H \partial_t \eta^0\, dz = (\eta_1 - \eta_2)\, \partial_t A \tag{26}$$

and

$$\int_0^H \partial_x \eta^0\, dz = (\eta_1 - \eta_2)\, \partial_x A. \tag{27}$$

Here we derive the basic equations at the boundary. Taking in account the independence of v^0 with respect to z and using (10), (14) gives

$$p^0 = -2\eta_2\, \partial_x v^0 \ \text{ at } \ z = H(t,x).$$

By replacing p^0 by its value in (18) we obtain

$$\eta_2\,(\partial_z v^2 + \partial_x w^1) = 4\eta_2\,\partial_x H^0\,\partial_x v^0 \qquad \text{at} \qquad z = H^0(t,x). \tag{28}$$

Equation (21) gives too

$$\eta_1\,(\partial_z v^2 + \partial_x w^1) = k\,v^0 \qquad \text{at} \qquad z = 0. \tag{29}$$

From (19) we get

$$(\partial_z v^2 + \partial_x w^1) = 4\partial_x v^0\,\partial_x A^0 \qquad \text{at} \qquad z = A(t,x). \tag{30}$$

5.2. Averaging Step

The objective of this step is to pass from 2D to 1D by integrating according to the vertical direction. So by integrating (24) with respect to z on $[0, H(t,x)]$ and using (20) we get

$$\int_0^{H^0} \partial_t \eta^0\,dz + v^0 \int_0^{H^0} \partial_x \eta^0\,dz + \partial_x v^0 \int_0^{H^0} \eta^0\,dz + \eta_2 w^1(H^0) = 0.$$

Replacing $w^1(H^0)$ by its value in (17) and using (26) and (27) we obtain

$$\begin{aligned}(\eta_1-\eta_2)\,\partial_t A + (\eta_1-\eta_2)\,v^0\,\partial_x A + (\eta_1-\eta_2)\,A\,\partial_x v^0 \\ + \eta_2\,(H^0\,\partial_x v^0 + \partial_t H^0 + v^0\,\partial_x H^0) = 0.\end{aligned}$$

Using (27), we get

$$(\eta_1-\eta_2)\,\partial_t A^0 + (\eta_1-\eta_2)\,\partial_x(A^0\,\partial_x v^0) = 0. \tag{31}$$

To obtain the expression of the pressure, we replace $\partial_z w^1$ by $-\partial_x v^0$ in (23) (see (10)) and we get

$$\partial_z(2\eta^0\,\partial_x v^0 + p^0) = -\rho^0\,G.$$

This equation can be written as

$$2\eta^0\,\partial_x v^0 + p^0(t,x,z) = \int_z^{H^0} \rho\,g.$$

On the other hand we have

$$\int_z^{H^0} \rho\,g = \begin{cases} \rho_2\,g\,(H^0 - z) & \text{if} \quad z > A^0, \\ \rho_2\,g\,(H^0 - A^0) + \rho_1\,(A^0 - z) & \text{if} \quad z < A^0, \end{cases}$$

so

$$p^0 = -2\eta^0\,\partial_x v^0 + \rho_2\, g\,(H^0 - z) + g\,(\rho_1 - \rho_2)\,(A^0 - z)\,\chi_{z<A^0}(z).$$

By replacing $p^0(t,x,z)$ by its value in (22) we obtain

$$\begin{aligned} &\rho\,\partial_t v^0 + \rho\, v^0\,\partial_x v^0 - 4\partial_x(\eta^0\,\partial_x v^0) - \partial_z(\eta^0\,\partial_z v^2 + \eta^0\,\partial_x w^1) + \rho_2\, g\,\partial_x H^0 \\ &\quad + g\,(\rho_1-\rho_2)\,\partial_x(A^0)\,\chi_{z<A^0}(z) + g\,(\rho_1-\rho_2)\,(A^0 - z)\,\partial_x(\chi_{z<A^0}(z)) = 0. \quad (32) \end{aligned}$$

By integrating (32) on $[0, H^0(t,x)]$ and using (28), (29) and (30) we have

$$\begin{aligned} &(\rho_2\, H^0 + (\rho_1-\rho_2)\,A^0)\,\partial_t v^0 + (\rho_2\, H^0 + (\rho_1-\rho_2)\,A^0)\, v^0\,\partial_x v^0 \\ &\qquad - 4(\eta_1-\eta_2)\left(\partial_x(A^0\,\partial_x v^0) + \partial_x A^0\,\partial_x v^0\right) - 4\eta_2\,\partial_x(H^0\partial_x v^0) \\ &\qquad\qquad + \rho_2\, g\, H^0\,\partial_x H^0 + g\,(\rho_1-\rho_2)\,A^0\,\partial_x A^0 + k\, v^0 = 0. \quad (33) \end{aligned}$$

Finally, with (31), (25) and (33) we get the following system

$$\begin{cases} \partial_t A^0 + \partial_x(A^0\,\partial_x v^0) = 0, \\ \partial_t H^0 + v^0\,\partial_x H^0 + H^0\,\partial_x v^0 = 0, \\ (\rho_2\, H^0 + (\rho_1-\rho_2)\,A^0)\,\partial_t v^0 + (\rho_2 H^0 + (\rho_1-\rho_2)\,A^0)\, v^0\,\partial_x v^0 \\ \quad -4(\eta_1-\eta_2\,)(\partial_x(A^0\,\partial_x v^0) + \partial_x A^0\,\partial_x v^0) - 4\eta_2\,\partial_x(H^0\,\partial_x v^0) \\ \qquad + \rho_2\, g\, H^0\,\partial_x H^0 + g\,(\rho_1-\rho_2)\,A^0\,\partial_x A^0 + k\, v^0 = 0. \end{cases}$$

Returning to the initial variables, we get the viscous shallow water bifluid model (S).

6. Numerical Results

In this section we assume $\rho_1 = \rho_2 = 1$ and we discretize system (S) using an operator splitting. The hyperbolic part of the system is solved using a finite volume method and the diffusive part by an implicit method. We present some numerical simulations implementing the ratio $\frac{\eta_2}{\eta_1}$. The considered channel is closed downstream $(v = 0)$ and the values of the heigts a and h are prescribed at the upstream boundary.

6.1. Discretization by Finite Volume Method

We use a regular mesh $\Delta x = \frac{L}{N}$ on the spacial domain $[0, L]$ and the time step Δt must be defined at each step.

At the initial time $t = 0$ we are given $(a_i)_{1\le i\le N}$, $(h_i)_{1\le i\le N}$, $(v_i)_{1\le i\le N}$ (values in the internal meshes) as well as the upstream and downstream boundary conditions: h_0, a_0 and v_0 at $x = 0$, a_{N+1}, h_{N+1} and v_{N+1} at $x = L$.

Starting at time $t = t_n$ with the known states $U_i^n = \begin{pmatrix} h_i^n \\ a_i^n \\ h_i^n v_i^n \end{pmatrix}$, $0 \le i \le N+1$, we compute U_i^{n+1} using an operator splitting. First we update a and h and we get a first estimate of v by solving the hyperbolic part of the system, i.e

$$\begin{cases} \partial_t a + \partial_x (a\, v) = 0, \\ \partial_t h + \partial_x (h\, v) = 0, \\ \partial_t (h\, v) + \partial_x (h\, v^2 + g\, \frac{h^2}{2}) = -K\, v. \end{cases}$$

Second, we update v by solving the equation:

$$\partial_t (h\, v) = 4(\eta_1 - \eta_2)\, \partial_x (a\, \partial_x v) + 4\eta_2\, \partial_x (h\, \partial_x v), \tag{34}$$

where a and h are the values issued from the previous step.

More precisely we compute h_i^{n+1}, a_i^{n+1} and $v_i^{n+1/2}$ the provisional velocity, by solving:

$$\begin{cases} \partial_t U + \partial_x (F(U)) = G(U), \\ U(t_n, x) = U^n, \end{cases}$$

where U^n is the piecewise constant function equal to U_i^n on the mesh m_i,

$$U = \begin{pmatrix} h \\ a \\ q \end{pmatrix}, \quad F(U) = \begin{pmatrix} q \\ \frac{a\, q}{h} \\ \frac{q^2}{h} + g\frac{h^2}{2} \end{pmatrix} \text{ and } G(U) = \begin{pmatrix} 0 \\ 0 \\ -Kv \end{pmatrix},$$

where $q = hv$. Setting $s = \frac{a}{h}$ and $c = \sqrt{g\, h}$, we get the Jacobian matrix of F under the form

$$DF(U) = \begin{pmatrix} 0 & 0 & 1 \\ -vs & v & s \\ c^2 - v^2 & 0 & 2v \end{pmatrix}.$$

The eigenvalues of this matrix are $v - c$, v and $v + c$ and the corresponding eigenvectors are:

$$r_1 = \begin{pmatrix} 1 \\ s \\ v-c \end{pmatrix}, \quad r_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad r_3 = \begin{pmatrix} 1 \\ s \\ v+c \end{pmatrix}.$$

We use an explicit first order VFRoe scheme (see [3, 4, 5] for instance) which writes

$$U_i^{n+1/2} = U_i^n - \frac{\Delta t}{\Delta x}(F_{i+\frac{1}{2}}^{n,-} - F_{i-\frac{1}{2}}^{n,+}) + \Delta t\, G(U_i^n), \quad 1 \le i \le N,$$

with

$$U_i^{n+1/2} = \begin{pmatrix} h_i^{n+1} \\ a_i^{n+1} \\ q_i^{n+1/2} = h_i^{n+1} v_i^{n+1/2} \end{pmatrix}$$

where $q_i^{n+1/2}$ is the provisional discharge at time $n+1$ at the point x_i.

We set $F_{i+\frac{1}{2}}^{n,\pm} = F(U^{PRL}(\zeta = 0^\pm, U_i^n, U_{i+1}^n))$ where $U^{PRL}(\zeta = 0, U_i^n, U_{i+1}^n)$ is the solution of the following linearized Riemann's problem

$$PRL : \begin{cases} \partial_t U + M(U_i^n, U_{i+1}^n)\partial_x U = 0, \\ U(t=0,x) = \begin{cases} U_i^n & \text{if} \quad x < 0, \\ U_{i+1}^n & \text{if} \quad x > 0, \end{cases} \end{cases}$$

with

$$M(U_i^n, U_{i+1}^n) = \begin{pmatrix} 0 & 0 & 1 \\ -\tilde{v}\tilde{s} & \tilde{v} & \tilde{s} \\ \tilde{c}^2 - \tilde{v}^2 & 0 & 2\tilde{v} \end{pmatrix},$$

where $\tilde{h} = \frac{h_i^n + h_{i+1}^n}{2}$, $\tilde{a} = \frac{a_i^n + a_{i+1}^n}{2}$, $\tilde{q} = \frac{q_i^n + q_{i+1}^n}{2}$, $\tilde{v} = \frac{\tilde{q}}{\tilde{h}}$, $\tilde{s} = \frac{\tilde{a}}{\tilde{h}}$, $\tilde{c} = \sqrt{g\,\tilde{h}}$.

Of course we have to take carefully in account the signs of the eigenvalues. We ship here the details of the rather classical computations. At this stage, we have the new states $U_i^{n+1/2}$, $1 \le i \le N$ in the internal meshes. Next we update the boundary states $U_0^{n+1/2}$ and $U_{N+1}^{n+1/2}$ (in the external fictious meshes) using the prescribed boundary conditions and a "ghost waves method" in the spirit of Dubois [2] or Kumbaro [7] for instance. This method makes use of the known values at the same level time and thus is naturally implicit.

In a second step we compute the final values of the discharge q^{n+1}, $0 \le n \le N+1$ by a classical implicit scheme applied to equation (34) with the values of a and h issued from the first step and the values $q^{n+1/2}$, $0 \le n \le N+1$ as initial data.

6.2. Some Numerical Experiments

In this subsection we investigate qualitatively, using a numerical experiment, the behavior of the interface between the two fluids as well as the free surface in two situations of the type of dam failure, depending on the position of the denser fluid. We therefore compare, in each situation, the cases $\eta_1 > \eta_2$ (with $\eta_1/\eta_2 = 100$) and $\eta_1 < \eta_2$ (with $\eta_1/\eta_2 = 0.01$).

6.2.1. First Test: Initial Discontinuity of the Free Surface

We start from an initial state at zero speed, with a horizontal interface and a discontinuity of the free surface by passing a constant height H_1 to a constant height H_2 with $H_1 > H_2$, see Figure 2. The upstream heights are kept constant while the downstream speed is zero (reflection). We then observe the evolution of the liquid-liquid and liquid-air interfaces: the figures on the left are relative to $\eta_1 < \eta_2$ (the more viscous fluid is above) and the right figures to $\eta_1 > \eta_2$ (the more viscous fluid is below).

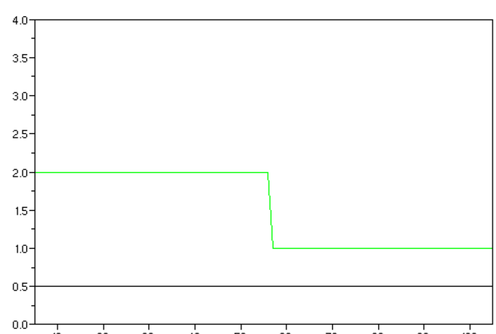

Figure 2: Initial common conditions for the cases $\eta_1 < \eta_2$ and $\eta_1 > \eta_2$

There is a ripple effect of the lower layer by the upper layer, similar regardless of the viscosity ratio (Figures 3, 4). The top layer, more viscous, stabilizes first (Figure 5), followed by the corresponding lower layer (ripple effect) but in a longer time (Figure 6).

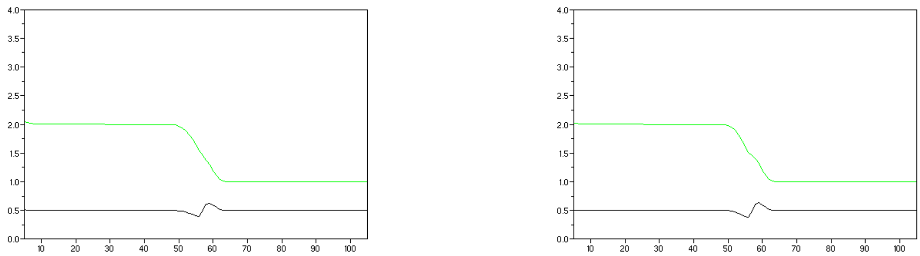

Figure 3: $t = 0.01$, $\eta_1 < \eta_2$ (left) and $\eta_1 > \eta_2$ (right)

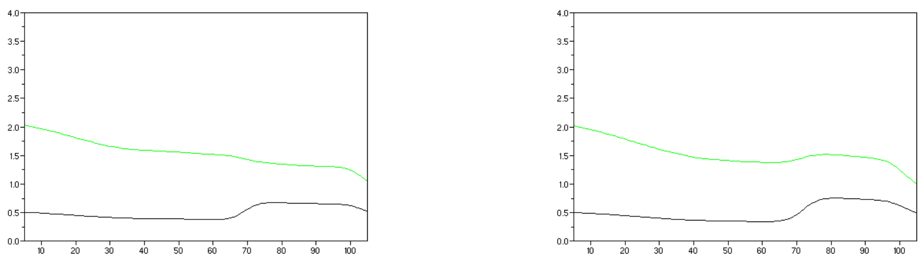

Figure 4: $t = 0.1$, $\eta_1 < \eta_2$ (left) and $\eta_1 > \eta_2$ (right)

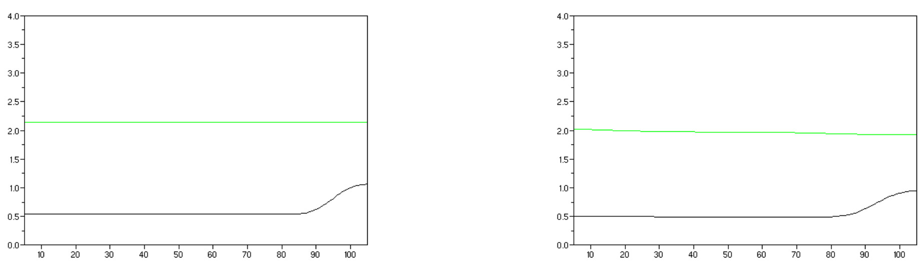

Figure 5: $t = 1.057$, $\eta_1 < \eta_2$ (left) and $\eta_1 > \eta_2$ (right)

6.2.2. Second Test: Initial Discontinuity of the Interface

We go back to initial state at zero speed, with a horizontal free surface and a discontinuity of the liquid-liquid interface of the same type as that imposed on the free surface in the previous case, see Figure 7. The boundary conditions are

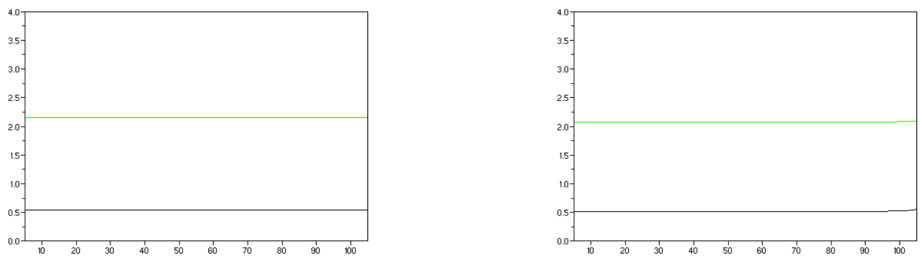

Figure 6: $t = 3.850$, complete stabilization for $\eta_1 < \eta_2$ (left)

the same as before. Similarly, the figures on the left are relative to $\eta_1 < \eta_2$ (the more viscous fluid is above) and the right figures to $\eta_1 > \eta_2$ (the more viscous fluid is below).

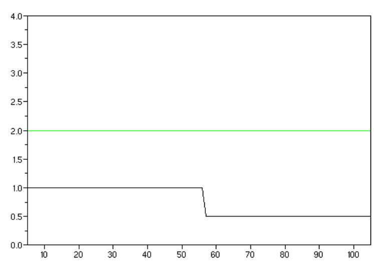

Figure 7: Common initial conditions for the cases $\eta_1 < \eta_2$ and $\eta_1 > \eta_2$

It is observed that the deformation of the interface betweenthe two fluids has little effect on the free surface (Figure 8). The stabilization arises first for the configuration where the more viscous fluid is above (Figure 9).

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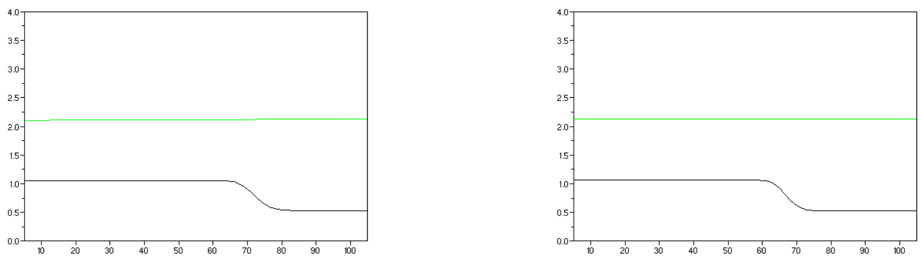

Figure 8: $t = 1.263$, $\eta_1 < \eta_2$ (left) and $\eta_1 > \eta_2$ (right)

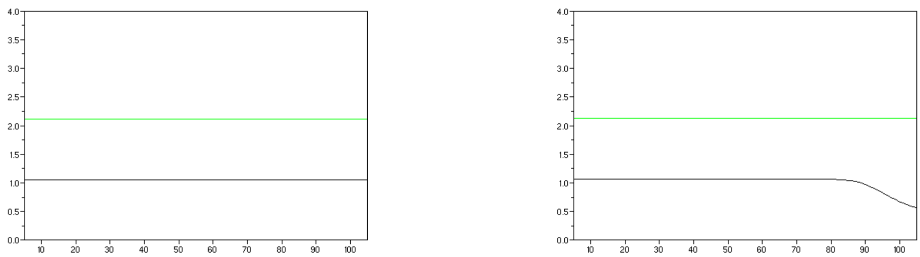

Figure 9: $t = 6.370$, complete stabilization for $\eta_1 < \eta_2$ (left)

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