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MOCK HECKE EIGENFUNCTIONS AND A CONJECTURE OF HEJHAL

C.J. Mozzochi
P.O. Box 1424, Princeton, NJ 08542, USA
e-mail: cjm@ix.netcom.com

Abstract: Let $\Gamma = \mathrm{SL}(2, Z)$. Hejhal has formulated a conjecture for Maass waveforms $\Psi(z, R)$ on $\Gamma\backslash H$, namely, that the $\Psi(z, R)$ "go Gaussian" as $R \to \infty$, where $R = \sqrt{\lambda - \frac{1}{4}}$.

Let $f(z, k)$ be a Hecke eigenfunction of even weight $k \geq 2$ on $\Gamma\backslash H$.

$$f(z, k) = \sum_{n=1}^{\infty} \lambda_n n^{\frac{k-1}{2}} e(nz).$$

Let $\Psi(z, k) = y^{k/2}|f(z, k)|$.

In this paper we first replace the λ_n with an arbitrary family of independent random variables $\lambda_n(\omega)$ on an arbitrary probability space and show that within our probability model, if the $\Psi(z, k)$ "go Gaussian" as $k \to \infty$, they do not do so uniformly.

We then briefly discuss a specific model based on the Sato-Tate probability measure.

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1. Introduction

Let $\mathbb{H}$ be the upper half plane and let $\Gamma = \mathrm{SL}(2, Z)$.

Let $\mathscr{F} \subset \Gamma\backslash H$ be the usually chosen fundamental domain.

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In a series of papers (cf. [10], [12], [13]) Hejhal and his collaborators have carefully investigated the topography and the statistical behavior of Maass waveforms on $\Gamma\backslash H$.

A Maass waveform $\Psi(z,R)$ on $\Gamma\backslash H$ is simply a square-integrable, nonconstant, Γ invariant eigenfunction of the hyperbolic Laplacian

$$\Delta = y^2\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right] .$$

That is

$$\Delta\Psi(z,R) = -\lambda\Psi(z,R) = -(1/4+R^2)\Psi(z,R)$$

on H.

Hejhal shows (cf. [8]) the $\Psi(z,R)$ can be taken to be real valued and can be expanded, since $\lambda_1 > \frac{1}{4}$,

$$\Psi(z,R) = \sum_{n=1}^{\infty} d_n y^{1/2} K_{iR}(2\pi ny)\begin{Bmatrix}\cos(2\pi nx)\\ \sin(2\pi nx)\end{Bmatrix}$$

depending on whether $\Psi(z,R)$ is even or odd.

Technically, of course, Ψ shold be normalized so that

$$\int\limits_{\Gamma\backslash H} |\Psi(z,R)|^2 d\mu_z = 1.$$

Hejhal then conjectures (cf. [10], [12], [13]) based on computation and theoretical considerations the limit as $R\to\infty$ of the relative frequency measures

$$\frac{\mu\{x\in B \mid \Psi(z,R)\in[\alpha,\beta]\}}{\mu\{B\}}$$

over arbitrary Jordan regions $B\subseteq\mathscr{F}$ is the Gaussian distribution with mean 0 and standard deviation $\|\Psi\|_2/\sqrt{\mu(\mathscr{F})}$.

For each even $k\geq 2$ let $f(z,k)$ be a Hecke eigenfunction of weight k on $\Gamma\backslash H$.

$$f(z,k) = \sum_{n=1}^{\infty}\lambda_n n^{\frac{k-2}{2}} e(nz) .$$

We know from Hecke theory and the work of Deligne that

$$\lambda_n\lambda_m = \sum_{d|(m,n)} \lambda_{nm/d^2} \tag{1.1}$$

$$|\lambda_\rho| \le 2 \quad \text{for each prime } p \ . \tag{1.2}$$

Let $\phi(z,k) = y^{\frac{k}{2}} f(z,k)$. Then

$$\phi(z,k) = \sum_{n=1}^{\infty} \lambda_n n^{\frac{k-2}{2}} y^{\frac{k}{2}} e^{-2\pi n y} e(nx)$$

$$\phi(z,k) = \sum_{n=1}^{\infty} \lambda_n y^{1/2} K(ny) e(nx),$$

where $K(y) = y^{\frac{k-1}{2}} e^{-2\pi y}$. Note the analogy with (1.1).

Let $\Psi(z,k) = |\phi(z,k)| = y^{k/2}|f(z,k)|$. Ψ should be normalized so that

$$\int_{\Gamma\backslash H} |\Psi(z)|^2 d\mu_z = 1.$$

One would like to know the limit as $k \to \infty$ of the relative frequency measures

$$\frac{\mu\left\{x \in B \;\middle|\; \Psi(z,k) \in [\alpha,\beta]\right\}}{\mu(B)}$$

over arbitrary Jordan regions $B \subseteq \mathscr{F}$.

Let (Ω,μ) be an arbitrary probability space. Let $\{\lambda_p(\omega) \mid p \text{ is a prime}\}$ be an arbitrary family of independent random variables on Ω, whose range is contained in $[-2,2]$.

Let $n = p_1^{k_1} \dots p_\ell^{k_\ell}$.

By the Hecke relations (1.1) we have

$$\lambda_n(\omega) = \lambda_{p_1^{k_1}}(\omega) \dots \lambda_{p_\ell^{k_\ell}}(\omega),$$

where the $\lambda_{p^k}(\omega)$ are evaluated from the $\lambda_p(\omega)$. Also

$$\text{Exp}(\lambda_n(\omega)) = \text{Exp}(\lambda_{P_1^{k_1}}(\omega)) \dots \text{Exp}(\lambda_{P_\ell^{k_\ell}}(\omega)).$$

Instead of $\Gamma\backslash H$ we restrict our attention to

$$R = \{(x,y) | -1/2 \le x \le 1/2;\; a \le y \le b\} \subseteq \mathscr{F}.$$

We rewrite $\phi(z,k)$ as

$$\phi(z,k) = \tfrac{k}{2} \operatorname*{Limit}_{T\to\infty} \phi(z;k,T),$$

where

$$\phi(z,k,T)=\sum_{n=1}^{T}\lambda_n(\omega)g(n,k)h(n,y)e(nx)$$

with

$$g(n,k)=n^{\frac{k-1}{2}},$$
$$h(n,y)=e^{-2\pi ny}\ .$$

Then

$$\Psi(z,k)=cy^{k/2}|\operatorname*{Limit}_{T\to\infty}\phi(z,k,T)|\ ,$$

where c is a normalization factor.

To simplify the notation we let $\lambda_n=\lambda_n(\omega)$ throughout.

Define $I(n,k)\mathrel{\hat=}\int_a^b y^{k-2}e^{-4\pi ny}dy$

Define $I(n,m,k)\mathrel{\hat=}\int_a^b y^{mk-2}e^{-4\pi nmy}dy$

Define $I(\theta_1,\dots,\theta_T,m,k)\mathrel{\hat=}\int_a^b y^{mk-2}e^{-2\pi(\theta_1+2\theta_2+\cdots+T\theta_T)y}dy$, where $\theta_1+\theta_2+\cdots+\theta_T=2m$.

The normalization factor is determined from the constraint

$$\operatorname{Exp}\int_R \Psi^2(z,k)d\mu_z=1\ ;$$

so that

$$1=c^2\operatorname*{Limit}_{T\to\infty}\operatorname{Exp}\int_a^b y^{k-2}\left(\int_0^1\left|\sum_{n=1}^{T}\lambda_n g(n,k)h(n,y)e(nx)\right|^2dx\right)dy,$$

$$1=c^2\operatorname*{Limit}_{T\to\infty}\operatorname{Exp}\int_a^b y^{k-2}\left(\sum_{n=1}^{T}\lambda_n^2g^2(n,k)h^2(n,y)\right)dy,$$

$$1=c^2\operatorname*{Limit}_{T\to\infty}\operatorname{Exp}\sum_{n=1}^{T}\lambda_n^2g^2(n,k)\int_a^b y^{k-2}h^2(n,y)dy.$$

Hence

$$c = \frac{1}{\operatorname*{Limit}_{T\to\infty}\left(\sum_{n=1}^{T} g^2(n,k)I(n,k)\mathrm{Exp}(\lambda_n^2)\right)^{1/2}}.$$

The following theorem is well known in the literature (cf. [1] p. 231).

Theorem 1.1. *Let $\gamma = \gamma(k,T)$ be a double sequence. For each $T = 1, 2, \ldots$ define a function g_T on Z^+*

$$g_T(k) = \gamma(k,T)\ ,$$
$$\text{Let } g(k) = \operatorname*{Limit}_{T\to\infty}\gamma(k,T).$$

Suppose $g_T \to g$ uniformly in Z^+. Then if

$$\operatorname*{Limit}_{k\to\infty}\left(\operatorname*{Limit}_{T\to\infty}\gamma(k,T)\right) \quad \textit{exists},$$
$$\operatorname*{Limit}_{k,T\to\infty}\gamma(k,T) \quad \textit{exists},$$

the two limits are equal, and we say the convergence is uniform.

For each $k \geq 2$, $T \geq 1$, $m \geq 1$ let

$$\gamma(k,T,m) = \frac{\int_R y^{km}\left|\sum_{k=1}^{T}\lambda_n g(n,k)h(n,y)e(nx)\right|^{2m} d\mu_z}{\left(\sum_{n=1}^{T} g^2(n,k)I(n,k)\mathrm{Exp}(\lambda_n^2)\right)^m}\ .$$

By Theorem 30.2 on p. 408 in [3] we have

Theorem 1.2. *The $\Psi(z,k)$ go Gaussian as $k \to \infty$ if and only if for each $m \geq 1$*

$$\operatorname*{Limit}_{k\to\infty}\operatorname*{Limit}_{T\to\infty}\mathrm{Exp}\,\gamma(k,T,m) = G(m),$$

where

$$G(m) = 1\cdot 3\cdot 5\ldots(2m-1).$$

Theorem 1.3. *If the $\Psi(z,k)$ go Gaussian as $k \to \infty$, then the convergence is not uniform.*

By Theorem 1.1 to prove Theorem 1.3 it is sufficient to establish

Theorem 1.4. *For each $m \geq 1$*

$$\operatorname*{Limit}_{k,T\to\infty}\mathrm{Exp}\,\gamma(k,T,m) \neq G(m)\ .$$

This theorem is an immediate consequence of

Theorem 1.5. *Let T be fixed but arbitrary large. Then for each $m \geq 1$*

$$\operatorname*{Limit}_{k\to\infty} \operatorname{Exp} \gamma(k,T,m) = \infty \ .$$

2. Proof of Theorem 1.5

We first establish

Lemma 2.1. For each n, m, k

$$k^{m-1} c_1(n,m) \leq \frac{I(n,m,k)}{I^m(n,k)} \leq c_2(n,m) k^{m-1}.$$

Proof.

$$I(n,m,k) = \int_a^b y^{mk-2} e^{-4\pi nmy} dy,$$

$$I(n,m,k) \leq e^{-4\pi nma} \int_a^b y^{mk-2} dy,$$

$$I(n,m,k) \leq \frac{\epsilon^{-4\pi nma}}{mk-1} b^{mk-1},$$

$$\frac{c_1 e^{-4\pi nmb} b^{mk-1}}{mk-1} \leq I(n,m,k),$$

$$\frac{c_1 e^{-4\pi nmb} b^{mk-1}}{(mk-1)} \leq I(n,m,k) \leq \frac{e^{-4\pi nma}}{(mk-1)} b^{mk-1},$$

$$I(n,k) = \int_a^b y^{k-2} e^{-4\pi ny} dy,$$

$$I(n,k) \leq e^{-4\pi na} \int_a^b y^{k-2} dy,$$

$$I(n,k) \leq \frac{e^{-4\pi na}}{(k-1)} b^{k-1},$$

$$c_2 \frac{e^{-4\pi nb} b^{k-1}}{(k-1)} \leq I(n,k),$$
$$c_2 \frac{e^{-4\pi nb} b^{k-1}}{(k-1)} \leq I(n,k) \leq \frac{\epsilon^{-4\pi na} b^{k-1}}{(k-1)},$$
$$\frac{c_2^m e^{-4\pi nmb} b^{m(k-1)}}{(k-1)^m} \leq I^m(n,k) \leq \frac{\epsilon^{-4\pi nma} b^{m(k-1)}}{(k-1)^m},$$
$$\frac{(k-1)^m}{\epsilon^{-4\pi nma} b^{m(k-1)}} \leq \frac{1}{I^m(n,k)} \leq \frac{(k-1)^m}{c_2^m e^{-4\pi nmb} b^{m(k-1)}},$$
$$\frac{c_1 b^{m-1}(k-1)^m}{\epsilon^{4\pi nm(b-a)}(mk-1)} \leq \frac{I(n,m,k)}{I^m(n,k)} \leq \frac{\epsilon^{4\pi nm(b-a)} b^{m-1}(k-1)^m}{c_2^m(mk-1)}$$

$$\operatorname{Exp}\gamma(k,T,m) = \frac{\operatorname{Exp} U(k,T,m)}{V(k,T,m)},$$

where

$$U(k,T,m) = \int_R y^{km} \Big|\sum_{n=1}^{T} \lambda_n g(n,k) h(n,y) e(nx)\Big|^{2m} d\mu_z,$$

and

$$V(k,T,m) = \left(\sum_{n=1}^{T} g^2(n,k) I(n,k) \operatorname{Exp}(\lambda_n^2)\right)^m.$$

We have

$$U(k,T,m) = \int_a^b y^{mk-2}\left(\int_0^1 \Big|\sum_{n=1}^{T} \lambda_n g(n,k) h(n,y)\epsilon(nx)\Big|^{2m} dx\right) dy.$$

Let $\Delta_n = \Delta_n(k,y) = \lambda_n g(n,k) h(ny)$. Then

$$\int_0^1 \Big|\sum_{n=1}^{T} \Delta_n e(nx)\Big|^{2m} dx = \int_0^1 \Big|\Big(\sum_{n=1}^{T} \Delta_n e(nx)\Big)^m\Big|^2 dx$$

$$= \int_0^1 \Big|\Big(\sum_{\substack{k_1+\cdots+k_T=m \\ k_1+2k_2+\cdots+Tk_T=m}} P(k_1,\ldots,k_T)[\Delta_1 \epsilon(x)]^{k_1} \cdots [\Delta_T e(Tx)]^{k_T}\Big)$$

$$+\Bigg(\sum_{\substack{k_1+\cdots+k_T=m\\ k_1+2k_2+\cdots Tk_T=(m+1)}} P(k_1,\ldots,k_T)[\Delta_1\epsilon(x)]^{k_1}\cdots[\Delta_T e(Tx)]^{k_T}\Bigg)+\cdots$$

$$+\Bigg(\sum_{\substack{k_1+\cdots+k_T\\ k_1+2k_2+\cdots Tk_T=mT}} P(k_1,\ldots,k_T)[\Delta_1\epsilon(x)]^{k_1}\cdots[\Delta_T e(Tx)]^{k_T}\Bigg)\Bigg|^2 dx,$$

where $P(k_1,\ldots,k_T)=\frac{(k_1+\cdots+k_T)!}{(k_1!)\cdots(k_T!)}$. □

There are T^m terms in this sum. Hence

$$\int_0^1 \left|\sum_{n=1}^{T}\Delta_n e(nx)\right|^{2m} dx$$

$$=\Bigg(\sum_{\substack{k_1+\cdots+k_T=m\\ k_1+2k_2+\cdots+Tk_T=m}} P(k_1,\ldots,k_T)\Delta_1^{k_1}\cdots\Delta_T^{k_1}\Bigg)^2$$

$$+\Bigg(\sum_{\substack{k_1+\cdots+k_T=m\\ k_1+2k_2+\cdots+Tk_T=(m+1)}} P(k_1,\ldots,k_T)\Delta_1^{k_1}\cdots\Delta_T^{k_T}\Bigg)^2+\cdots$$

$$+\Bigg(\sum_{\substack{k_1+\cdots+k_T=m\\ k_1+2k_2+\cdots+Tk_T=mT}} P(k_1,\ldots,k_T)\Delta_1^{k_1}\cdots\Delta_T^{k_1}\Bigg)^2$$

$$=\Delta_1^{2m}+\Delta_2^{2m}+\cdots+\Delta_T^{2m}+\sum_{\substack{\theta_1+\cdots\theta_T=2m\\ \theta_1+2\theta_2+\cdots+T\theta_T=2m\\ 0\le\theta_i<2m}}\Delta_1^{\theta_1}\cdots\Delta_T^{\theta_T}$$

$$+\sum_{\substack{\theta_1+\cdots+\theta_T=2m\\ \theta_1+2\theta_2+\cdots T\theta_T=(2m+2)\\ 0\le\theta_i<2m}}\Delta_1^{\theta_1}\cdots\Delta_T^{\theta_T}+\cdots+\sum_{\substack{\theta_1+\cdots+\theta_T=2m\\ \theta_1+2\theta_2+\cdots T\theta_T=(2mT)\\ 0\le\theta_i<2m}}\Delta_1^{\theta_1}\cdots\Delta_T^{\theta_T}\ .$$

There are less than T^{2m} terms in this sum.
Hence

$$U(k,T,m)=\sum_{n=1}^{T}\int_a^b y^{mk-2}\Delta_n^{2m}dy$$

$$+ \sum_{\substack{\theta_1+\cdots+\theta_T=2m \\ \theta_1+2\theta_2+\cdots+T\theta_T=s \\ 2m\le s\le 2mT \\ 0\le\theta_i<2m}} \int_a^b y^{mk-2}\Delta_1^{\theta_1}\cdots\Delta_T^{\theta_T}\,dy,$$

where there are less than T^{2m} terms in the second sum.

In what follows we abbreviate the notation in the obvious way.

$$\Delta_n^{2n} = \lambda_n^{2m} g^{2m}(n,k) e^{-4\pi mny} \int_a^b y^{mk-2}\Delta_n^{2m}\,dy$$
$$= \lambda_n^{2m} g^{2m}(n,k) \int_a^b y^{mk-2} e^{-4\pi mny}\,dy = \lambda_m^{2m} g^{2m}(n,k) I(n,m,k)\ ,$$

so that

$$\operatorname{Exp} \int_a^b y^{mk-2}\Delta_n^{2m}\,dy = g^{2m}(n,k) I(n,m,k) \operatorname{Exp}(\lambda_n^{2m}).$$

$$\Delta_n^{\theta_n} = \lambda_n^{\theta_n} g^{\theta_n}(n,k) e^{-2\pi\theta_n ny}$$
$$\Delta_1^{\theta_1}\cdots\Delta_T^{\theta_T} = g_1^{\theta_1}\cdots g_T^{\theta_T}\lambda_1^{\theta_1}\cdots\lambda_T^{\theta_T} e^{-2\pi(\theta_1+2\theta_2+\cdots T\theta_T)}$$
$$\int_a^b y^{mk-2}\Delta_1^{\theta_1}\cdots\Delta_T^{\theta_T}\,dy = g_1^{\theta_1}\cdots g_T^{\theta_T}\lambda_1^{\theta_1}\cdots\lambda_T^{\theta_T} \int_a^b y^{mk-2} e^{-2\pi(\theta_1+2\theta_2+\cdots+T\theta_T)}\,dy$$
$$= g_1^{\theta_1}\cdots g_T^{\theta_T} I(\theta_1,\ldots,\theta_T,m,k)\lambda_1^{\theta_1}\cdots\lambda_T^{\theta_T},$$

so that

$$\operatorname{Exp} \int_a^b y^{mk-2}\Delta_1^{\theta_1}\cdots\Delta_T^{\theta_T}\,dy$$
$$= g_1^{\theta_1}\cdots g_T^{\theta_T} I(\theta_1,\ldots,\theta_T,m,k)\operatorname{Exp}(\lambda_1^{\theta_1}\cdots\lambda_T^{\theta_T}).$$

$$\operatorname{Exp}\gamma(k,T,m) = S_1(k,T,m)\times S_2(k,T,m)$$

where

$$S_1(k,T,m) = \frac{S_A(k,T,m)}{S_B(k,T,m)(1+E_1(k,T,m))}$$

and

$$S_2(k,T,m) = (1 + E_2(k,T,m))$$

where

$$S_A(k,T,m) = \sum_{n=1}^{T} g^{2m}(n,k) I(n,k,m) \mathrm{Exp}(\lambda_n^{2m}),$$

$$S_B(k,T,m) = \sum_{n=1}^{T} g^{2m}(n,k) I^m(n,k) [\mathrm{Exp}(\lambda_n^2)]^m,$$

$$E_1(k,T,m) = \frac{\sum\limits_{\substack{k_1+\cdots+k_T=m \\ 0 \le k_i < m}} A_1^{k_1} \cdots A_T^{k_T}}{\sum\limits_{n=1}^{T} g^{2m}(n,k) I^m(n,k) [\mathrm{Exp}(\lambda_n^2)]^m},$$

where

$$A_n = g^2(n,k) I(n,k) \mathrm{Exp}(\lambda_n^2),$$

$$E_2(k,T,m) = \frac{\sum\limits_{\substack{\theta_1+\cdots+\theta_T=2m \\ 0 \le \theta_i < 2m}} g_1^{\theta_1} \cdots g_T^{\theta_T} I(\theta_1,\ldots,\theta_T,m,k) \mathrm{Exp}(\lambda_1^{\theta_1} \cdots \lambda_T^{\theta_T})}{\sum\limits_{n=1}^{T} g^{2m}(n,k) I(n,k,m) \mathrm{Exp}(\lambda_n^{2m})}.$$

Lemma 2.2. $\operatorname*{Limit}_{k\to\infty} E_1(k,T,m) = 0$, *for each* $m > 0$.

Proof. Let A be any term in $E_1(k,T,m)$.

$$A \le \frac{[g^{2k_1}(1,k) \cdots g^{2k_T}(T,k)][I^{k_1}(1,k) \cdots I^{k_T}(T,k)P][\mathrm{Exp}(\lambda_1^2)]^{k_1} \cdots [\mathrm{Exp}(\lambda_T^2)]^{k_T}}{g^{2m}(T,k) I^m(T,k) [\mathrm{Exp}(\lambda_T^2)]^m}.$$

$$\frac{g^{2k_1}(1,k) \cdots g^{2k_T}(T,k)}{g^{2m}(T,k)} = \frac{\left(1^{\frac{k-1}{2}}\right)^{2k_1} \cdot \left(2^{\frac{k-1}{2}}\right)^{2k_2} \cdots \left(T^{\frac{k-1}{2}}\right)^{2k_T}}{\left(T^{\frac{k-1}{2}}\right)^{2m}}$$

$$= \left(\frac{1^{k_1} \cdot 2^{k_2} \cdots T^{k_T}}{T^m}\right)^{k-1} \le \left(\frac{(T-1)^1 (T)^{m-1}}{T^m}\right)^{k-1} = \left(\frac{T-1}{T}\right)^{k-1},$$

since at least two $k_i \neq 0$. Also,

$$I^{k_1}(1,k) \cdot I^{k_2}(2,k) \cdots I^{k_T}(T,k) \leq \left(\int_a^b y^{k-2} dy \right)^m$$

and

$$e^{-4\pi T b} \left(\int_a^b y^{k-2} ky \right) \leq I(T,k),$$

so

$$\frac{1}{I^m(T,k)} \leq e^{4\pi m T b} \left(\int_a^b y^{k-2} dy \right)^{-m}.$$

Hence

$$A \leq \left(\frac{T-1}{T} \right)^{k-1} e^{4\pi m T b} \frac{[\mathrm{Exp}(\lambda_1^2)]^{k_1} \cdots [\mathrm{Exp}(\lambda_T^2)]^{k_T}}{[\mathrm{Exp}(\lambda_T^2)]^m},$$

since there are only a finite number of A. The proof is complete. □

Lemma 2.3. $\underset{k\to\infty}{\mathrm{Limit}}\, E_2(k,T,m) = 0.$

Proof. Let B be any term in $E_2(k,T,m)$.

$$|B| \leq \frac{[g^{\theta_1}(1,k) \cdots g^{\theta_k}(T,k)] I(\theta_1, \ldots, \theta_T, m, k) [\mathrm{Exp}(|\lambda_1|^{\theta_1} \cdots |\lambda_T|^{\theta_T})]}{g^{2m}(T,k) I(T,m,k) [\mathrm{Exp}(\lambda_T^{2m})]}.$$

$$\frac{g^{\theta_1}(1,k) \cdots g^{\theta_T}(T,k)}{g^{2m}(T,k)} = \frac{\left(1^{\frac{k-1}{2}}\right)^{\theta_1} \cdots \left(T^{\frac{k-1}{2}}\right)^{\theta_T}}{\left(T^{\frac{k-1}{2}}\right)^{2m}}$$

$$= \left(\frac{1^{\theta_1} \cdots T^{\theta_T}}{T^{2m}} \right)^{\frac{k-1}{2}} \leq \left(\frac{(T-1)^1 T^{2m-1}}{T^{2m}} \right)^{\frac{k-1}{2}} = \left(\frac{T-1}{T} \right)^{\frac{k-1}{2}},$$

since at least two $\theta_i \neq 0$.

$$I(\theta_1 \cdots \theta_T, m, k) \leq \left(\int_a^b y^{mk-2} dy \right)$$

$$e^{-r\pi mbT}\int\limits_a^b y^{mk-2}dy \le I(T,m,k),$$

since $s \le 2mT$. So

$$\frac{1}{I(T,m,k)} \le e^{4\pi mbT}\left(\int\limits_a^b y^{mk-2}dy\right)^{-1}.$$

Hence

$$|B| \le \left(\frac{T-1}{T}\right)^{k-1} e^{4\pi mTb}\frac{[\mathrm{Exp}(|\lambda_1|^{\theta_1}\cdots|\lambda_T|^{\theta_T})]}{\mathrm{Exp}(\lambda_T^{2m})}.$$

Since there are only a finite number of B, the proof is complete. □

Remark. The proof of Theorem 1.5 is now an immediate consequence of Lemma 2.1, Lemma 2.2 and Lemma 2.3.

3. The Sato-Tate Measure

It is possible to construct from the Sato-Tate measure (or from any distribution of mean zero) by the infinite product of measure spaces a probability space (Ω, μ) and a set of independent random variables $\{\lambda_p(\omega) \mid p \text{ is a prime}\}$ on Ω, whose range is contained in $[-2, 2]$ such that

$$\mathrm{Exp}(\lambda_n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{3.1}$$

$$\mathrm{Exp}(\lambda_n\lambda_m) = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{otherwise}. \end{cases} \tag{3.2}$$

Let

$$M(k,T,m) = \frac{\sum\limits_{n=1}^{T} g^{2m}(n,k)I(n,k,m)[\mathrm{Exp}(\lambda_n^{2m})]}{\left(\sum\limits_{n=1}^{T} g^2(n,k)I(n,k)[\mathrm{Exp}(\lambda_n^2)]\right)^m}, \tag{3.3}$$

$$E(k,T,m) = \frac{\sum\limits_{\substack{\theta_1+\cdots+\theta_T=2m \\ 0\le\theta_i<2m}} g_1^{\theta_1}\cdots g_T^{\theta_T} I(\theta_1,\ldots,\theta_T,m,k)\mathrm{Exp}(\lambda_{\theta_1},\ldots,\lambda_{\theta_T})}{\left(\sum\limits_{n=1}^{T} g^2(n,k)I(n,k)[\mathrm{Exp}(\lambda_n^2)]\right)^m} \tag{3.4}$$

By the calculations in Section 2 we have

$$\operatorname{Exp}\gamma(k,T,m) = M(k,T,m) + E(k,T,m) \tag{3.5}$$

Lemma 3.1. *For each* $n \geq 1$ $\operatorname{Exp}(\lambda_n^2) = 1$.

Proof. Immediate by (3.2). □

Lemma 3.2. *For each* $n \geq 1$ *and* $m \geq 1$ $\operatorname{Exp}(\lambda_n^{2m}) \geq 1$.

Proof. Immediate by (1.1), (3.1) and (3.2). □

Lemma 3.3. *For each* $k \geq 2$, $T \geq 1$, $m \geq 1$ $M(k,T,m) \geq 0$ *and* $E(k,T,m) \geq 0$.

Proof. Immediate from (1.1), (3.1) and (3.2). □

Let

$$M_U(k,T,m) = \frac{\sum_{n=1}^{T} n^{m(k-1)} e^{-4\pi nma} \operatorname{Exp}(\lambda_n^{2m})}{\left(\sum_{n=1}^{T} n^{(k-1)} e^{-4\pi nb} \operatorname{Exp}(\lambda_n^2)\right)^2}, \tag{3.5}$$

$$M_L(k,T,m) = \frac{\sum_{n=1}^{T} n^{m(k-1)} e^{-4\pi nmb} \operatorname{Exp}(\lambda_n^{2m})}{\left(\sum_{n=1}^{T} n^{(k-1)} e^{-4\pi na} \operatorname{Exp}(\lambda_n^2)\right)^2}. \tag{3.6}$$

Lemma 3.4. *For each* $k \geq 2$, $T \geq 1$, $m \geq 1$

$$C_m k^{m-1} M_L(k,T,m) \leq M(k,T,m) \leq C_m^* k^{m-1} M_U(k,T,m)$$

Proof. This follows straightfowardly from the relevant definitions. □

Lemma 3.5. *For each* $k \geq 2$, $T \geq 1$, $m \geq 1$

$$C_m k^{m-1} M_L(k,T,m) \leq \operatorname{Exp}\gamma(k,T,m).$$

Proof. This follows from Lemma 3.3 and Lemma 3.4. □

We have the following well known result.

Lemma 3.6.

$$\int_0^\infty x^n e^{-\mu x} dx = \frac{n!}{\mu^{n+1}}.$$

Let

$$M_U^*(k,T,m) = \frac{\sum_{n=1}^{T} n^{m(k-1)} e^{-4\pi n m a}}{\left(\sum_{n=1}^{T} n^{(k-1)} e^{-4\pi n b}\right)^m}, \tag{3.7}$$

$$M_L^*(k,T,m) = \frac{\sum_{n=1}^{T} n^{m(k-1)} e^{-4\pi n m b}}{\left(\sum_{n=1}^{T} n^{(k-1)} e^{-4\pi n a}\right)^m}. \tag{3.8}$$

Lemma 3.7. *For each* $k \geq 2$, $T \geq 1$, $m \geq 1$

$$M_U^*(k,T,m) \leq M_U(k,T,m).$$

Proof. This follows from Lemma 3.1 and Lemma 3.2. □

Lemma 3.8. *For each* $k \geq 2$, $T \geq 1$, $m \geq 1$

$$M_L^*(k,T,m) \leq M_L(k,T,m).$$

Proof. This follows from Lemma 3.1 and Lemma 3.2. □

Lemma 3.9. $\underset{k\to\infty}{\text{Limit}}\ \underset{T\to\infty}{\text{Limit}}\ M_U^*(k,T,m) = +\infty.$

Proof. This follows from Lemma 3.6 and Sterling's formula. □

Lemma 3.10. $\underset{k\to\infty}{\text{Limit}}\ \underset{T\to\infty}{\text{Limit}}\ C_m k^{m-1} M_L^*(k,T,m) = 0.$

Proof. This follows from Lemma 3.6 and Sterling's formula. □

Let

$$M^*(k,T,m) = \frac{\sum_{n=1}^{T} g^{2m}(n,k)I(n,k,m)}{\left(\sum_{n=1}^{T} g^2(n,k)I(n,k)\right)^m} \tag{3.9}$$

Lemma 3.11. *For each* $k \geq 2$, $T \geq 1$, $m \geq 1$

$$M^*(k,T,m) \leq M(k,T,m)$$

and

$$M^*(k,T,m) \leq \mathrm{Exp}\gamma(k,T,m).$$

Proof. Immediate by Lemma 3.1, Lemma 3.2 and Lemma 3.3. □

Lemma 3.12. *For each* $m \geq 1$

$$\operatorname*{Limit}_{k\to\infty} \operatorname*{Limit}_{T\to\infty} M^*(k,T,m) = 0.$$

Proof. This follows from Lemma 3.6 and Sterling's formula. □

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