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LINEAR PROJECTIONS OF SMOOTH CURVES OVER A FINITE FIELD

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Abstract: Let $C \subset \mathbb{P}^n$ be a smooth curve defined over $\mathbb{F}_q$ with degree d and genus g. Here we give conditions on d, g, q which assures the existence of an isomorphic linear projection into $\mathbb{P}^3$ (case $n \geq 4$) or a birational projection into $\mathbb{P}^2$ with only double points and defined over $\mathbb{F}_q$.

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1. Introduction

Fix a prime p and a p-power q. Here we look at the existence of plane curves defined over $\mathbb{F}_q$ with prescribed degree and geometric genus and with only double points as their singularities. We do not assume that all singular points of the curve are contained in $\mathbb{P}^2(\mathbb{F}_q)$.

Theorem 1. *Let $C \subset \mathbb{P}^n$, $n \geq 3$, be a smooth, geometrically connected and non-degenerate curve such that both C and the inclusion $C \hookrightarrow \mathbb{P}^n$ are defined over $\mathbb{F}_q$. Set $d := \deg(C)$ and $g := p_a(C)$. If $n > 3$, then assume $q \geq (d-1)(d-2)/2 - g + 4 - n$.*

(a) Assume $q > 2d + 2g - 2$ and $q^2(q - 2d - 2g + 2) + q^2 > (d-1)^3 - d$. Then there exists a linear subspace $V \subset \mathbb{P}^n$ such that $\dim(V) = n - 3$, $V \cap C = \emptyset$, V is defined over $\mathbb{F}_q$, and the linear projection from V induces an unramified morphism $f : C \to \mathbb{P}^2$ with $f(C)$ a plane curve of degree d birational to C. MThere are at least $q^2(q - 2d - 2g + 2) + q^2 - (d-1)^3 + d$ such linear subspaces V.

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(b) Assume $q \geq 2d+2g-2+(d-2)(d-1)(d-3)/3-g(d-2)$*. Then there exists a linear subspace* $V \subset \mathbb{P}^n$ *such that* $\dim(V) = n-3$, $V \cap C = \emptyset$, V *is defined over* $\mathbb{F}_q$*, and the linear projection from* V *induces an unramified morphism* $f : C \to \mathbb{P}^2$ *with* $f(C)$ *a plane curve of degree* d *birational to* C *and such that* $f(C)$ *has only singular points with multiplicity* 2 *and two formal branches. There are at least* $q^2(q-(2d+2g-2+(d-2)(d-1)(d-3)/3-g(d-2)))+q^2$ *such linear subspaces* V*.*

In the case of linear projections from $\mathbb{P}^n$ onto $\mathbb{P}^{n-1}$ we have a better lower bound for the number of admissible linear projections (see Corollary 1).

2. Proofs and Related Results

Proposition 1. *Let* $C \subset \mathbb{P}^n$, $n \geq 4$*, be a smooth, geometrically connected and non-degenerate curve. Assume that both* C *and the inclusion* $C \hookrightarrow \mathbb{P}^n$ *are defined over* $\mathbb{F}_q$*. Set* $d := \deg(C)$ *and* $g := p_a(C)$*. Fix an integer* r *such that* $3 \leq r \leq n-1$*. Assume* $q \geq (d-1)(d-2)/2 - g + r + 1 - n$*. Then then exists a linear subspace* $V \subset \mathbb{P}^n$ *such that* $\dim(V) = n-r-1$, $V \cap C = \emptyset$, V *is defined over* $\mathbb{F}_q$*, and the linear projection from* V *induces an embedding of* C *into* $\mathbb{P}^r$ *defined over* $\mathbb{F}_q$*.*

Proof. By induction on n we reduce to the case $r = n-1$. Let $S(C) \subset \mathbb{P}^n$ be the secant variety of C. $S(C)$ is an integral 3-dimensional variety (see [1], Remark 1.6). To prove the case $r = n-1$ it is sufficient to find $P \in \mathbb{P}^n(\mathbb{F}_q)$ such that $P \notin S(C)$. The genus formula for plane curves gives $\deg(S(C)) \leq (d-1)(d-2)/2 - g$. If $n = 4$, then we are done, because no hypersurface of degree q contains $\mathbb{P}^n(\mathbb{F}_q)$. Now assume $n > 4$. It is sufficient to prove that $S(C)$ is contained in a hypersurface of degree at most $(d-1)(d-2)/2-g+4-n$. Take a general $P \in S(C)$. Hence $S(C)$ is not a cone with vertex containing P. Thus the cone with vertex P and $S(C)$ as its basis is a 4-dimensional integral variety of degree at most $d-1)(d-2)/2-g-1$. Iterating this step we get a hypersurface F containing $S(C)$ and with $\deg(F) \leq (d-1)(d-2)/2 - g + 4 - n$. □

By the definition of J-embeddings in terms of secant varieties (see [8], 1.2 and Proposition 1.5, [14], Chapter II, page 37) the proof of Proposition 1 gives the following result.

Proposition 2. *Let* $Y \subset \mathbb{P}^n$, $n \geq 4$*, be a geometrically integral and non-degenerate curve. Assume that both* Y *and the inclusion* $Y \hookrightarrow \mathbb{P}^n$ *are defined over* $\mathbb{F}_q$*. Set* $d := \deg(Y)$*. Let* g *be the genus of the normalization of* Y*. Fix an* *integer r such that $3 \leq r \leq n-1$. Assume $q \geq (d-1)(d-2)/2 - g + r + 1 - n$. Then then exists a linear subspace $V \subset \mathbb{P}^n$ such that $\dim(V) = n - r - 1$, $V \cap C = \emptyset$, V is defined over $\mathbb{F}_q$, and the linear projection from V induces a J-embedding of Y into $\mathbb{P}^r$ defined over $\mathbb{F}_q$, i.e. $V \cap \mathrm{Sec}(Y) = \emptyset$.*

Lemma 1. *Let $C \subset \mathbb{P}^n$, $n \geq 3$, be a smooth and geometrically connected curve. Assume that both C and the inclusion $C \hookrightarrow \mathbb{P}^n$ are defined over $\mathbb{F}_q$. Set $d := \deg(G)$ and $g := p_a(C)$. Let $T(C) \subset \mathbb{P}^n$ be the tangent developable of C. Then $T(C)$ is defined over $\mathbb{F}_q$ and $\deg(T(C)) \leq 2d + 2g - 2$.*

Proof. Obviously $T(C)$ is an integral surface defined over $\mathbb{F}_q$. Take a general $(n-2)$-dimensional linear subspace $V \subset \mathbb{P}^n$. The linear projection from V induces a degree d morphism $f : C \to \mathbb{P}^1$. For general V at least one of the tangent lines of C is not contained in V. Thus for general V the morphism f is separable. Thus the ramification formula for g gives $\sharp(V \cap T(C)) \leq 2d + 2g - 2$. $\square$

Lemma 2. *Let $C \subset \mathbb{P}^3$ be a smooth and non-degenerate curve. Set $d := \deg(C)$ and $g := p_a(C)$. Let $\Lambda_C \subset \mathbb{P}^3$ be the closure of the union of all lines $D \subset \mathbb{P}^3$ such that $\deg(D \cap C) \geq 3$ and D is not tangent to C. Then Λ_C has pure dimension 2 and $\deg(T) \leq (d-2)(d-1)(d-3)/3 - g(d-2)$.*

Proof. For us $\emptyset$ has pure dimension x for any $x \in \mathbb{N}$. Since C is smooth and it is not a plane conic in characteristic 2, C is not a strange curve. Hence it is not very strange, i.e. its general hyperplane section is in linearly general position (see [11], Lemma 1.1). Hence the set B_1 of all lines $D \subset \mathbb{P}^3$ such that $\deg(D \cap C) \geq 3$ has dimension at most one. Let $B := \{D \in B_1 :$ the scheme $D \cap C$ is reduced $\}$. To prove that Λ_C has pure dimension 2 it is sufficient to prove that B has pure dimension 1. Fix any $D \in B$ and any $P \in (D \cap C)$. Let $\ell_P : \mathbb{P}^3 \setminus \{P\} \to \mathbb{P}^2$ be the linear projection from P. First assume that $\ell_P|(C \setminus \{P\})$ is not birational onto its image. In this case a one-dimensional component of B containing D is formed by a non-empty open subset of all lines $\{\langle\{P,Q\}\rangle\}_{Q \in C \setminus \{Q\}}$. Now assume that $\ell_P|(C \setminus \{P\})$ is birational onto its image. Let $E_P \subset \mathbb{P}^2$ be the closure of $\ell_P(C \setminus \{P\})$. Since C is smooth at P and $\ell_P|(C \setminus \{P\})$ is birational onto its image, E_P is an integral degree $d-1$ curve. Since C is smooth at P, the map $\ell_P|(C \setminus \{P\})$ extends to a morphism $u : C \to E_P$. By assumption u is the normalization map. Hence E has geometric genus g. Fix a neighborhood U of P in C such that $\ell_O|(C \setminus \{O\})$ is birational onto its image for all $O \in U$. Let $E_O \subset \mathbb{P}^2$ be the closure of $\ell_O(C \setminus \{O\})$. We get a family of $\{E_O\}_{O \in U}$ of degree $d-1$ integral plane curves with geometric genus g. Restricting if necessary U we get that for each $O \in U \setminus \{P\}$ there is a finite non-empty subset S_O of B. We get a finite morphism $f : C \times U \to Y \subset \mathbb{P}^2 \times U$ which is birational onto its image. The target Y is reduced, with pure dimension 2 and locally Cohen-Macaulay (the latter being true because it is a hypersurface of $\mathbb{P}^2 \times U$). Since Y is reduced, f is the normalization map. Since Y is locally Cohen-Macaulay, Serre's criterion of normality (see [5], Proposition II.8.23) gives that non-normal locus of Y has no isolated point. Hence D is not an isolated point of B. The upper bound for the degree of the surface Λ_C is classical (see [9], [10]). □

We stress that in the next lemma we get V such that $V \cap Y = \emptyset$, not only $Y(\mathbb{F}_q) \cap V = \emptyset$.

Lemma 3. *Let $Y \subset \mathbb{P}^n$ be an integral and non-degenerate subvariety. Assume that both Y and the inclusion $Y \hookrightarrow \mathbb{P}^n$ are defined over $\mathbb{F}_q$. Set $y := \dim(Y)$ and $z := \deg(Y)$. Assume $y < n$ and $q > z$. Then there exists an $(n-y-1)$-dimensional linear subspace $V \subset \mathbb{P}^n$ defined over $\mathbb{F}_q$ such that $V \cap Y = \emptyset$.*

Proof. First assume $y = n-1$. Write $Y = \{f = 0\}$ with f homogeneous degree z polynomial with coefficients in $\mathbb{F}_q$. Since $z < q$, it is obvious the existence of $V \in \mathbb{P}^n(\mathbb{F}_q)$ such that $V \notin Y$. Now assume $y \le n-2$. Since $\mathbb{P}^n(\mathbb{F}_q)$ spans $\mathbb{P}^n$ and $y < n$, there is at least one $P \in \mathbb{P}^n(\mathbb{F}_q)$ such that Y is not a cone with vertex containing P. We use induction on the codimension $n-y$ of Y. Fix any $P \in \mathbb{P}^n(\mathbb{F}_q)$ such that Y is not a cone with vertex containing P. Let Y' be the cone with vertex P and Y as its basis. Since Y is not a cone with vertex containing P, Y' is an integral variety of dimension $y+1$. Obviously $\deg(Y') \le \deg(Y) \le z$. By the inductive assumption there is an $(n-y-2)$-dimensional linear subspace W of $\mathbb{P}^n$ defined over $\mathbb{F}_q$. Such that $W \cap Y' = \emptyset$, i.e. $V \cup Y = \emptyset$, where V is the linear span of W and P. Since $P \in Y'$, we have $\dim(V) = n-y-1$. Since W and V are defined over $\mathbb{F}_q$, the linear space V is defined over $\mathbb{F}_q$. □

Notation 1. For any integral and non-degenerate subvariety $Y \subset \mathbb{P}^n$ let $\mathfrak{S}(Y)$ be the closure in $\mathbb{P}^n$ of the set of all $P \in \mathbb{P}^n \setminus Y$ such that the linear projecton from P does not map Y birationally onto its image (see [3]).

Take Y as in Notation 1. If Y is a curve, then $\mathfrak{S}(Y)$ is finite and $\sharp(\mathfrak{S}(Y)) \le (d-1)^3 - d$ (see [2]).

Proof of Theorem 1. First assume $n = 3$. For any $P \in \mathbb{P}^3 \setminus C$, let $u_P : C \to \mathbb{P}^2$ be the finite morphism induced by the linear projection from P. The morphism u_P is unramified if and only if $P \notin T(C)$. The morphism u_P is birational onto its image if and only if $P \in \mathfrak{S}(C)$. The surface $T(C)$ is defined over $\mathbb{F}_q$ and $\deg(T(C)) \le 2d + 2g - 2$ (Lemma 1). Hence $\sharp(T(C)(\mathbb{F}_q)) \le q^2(2d+2g-2)+q+1$ (see [13] or [12], Theorem 2). We have $\sharp(\mathfrak{S}(Y)) \le (d-1)^3-d$ (see [2]). Thus $\sharp(\mathbb{P}^3(\mathbb{F}_q)) = q^3+q^2+q+1 > \sharp(T(C)(\mathbb{F}_q))+(d-1)^3-d$, proving the case $n = 3$ for part (a). Assume $q \ge 2d+2g-2+(d-2)(d-1)(d-3)/3-g(d-2)$. By Lemma 1 there is $P \in \mathbb{P}^3(\mathbb{F}_q)$ such that $P \notin (T(C) \cup \Lambda_C)$ and the set of all such points P has cardinality at least $q^2(q - 2d + 2g - 2 + (d - 2)(d - 1)(d - 3)/3 - g(d - 2)) + q^2$. For any such P the morphism P is unramified. Since $P \notin \Lambda_C$, to get that every singular point of $u_P(C)$ has multiplicity 2 it is sufficient to prove $P \notin \mathfrak{S}(C)$. Assume $P \in \mathfrak{S}(C)$. Let $m_P : E_P \to u_P(C)$ be the normalization map. Since C is smooth there is a morphim $v_P : C \to E_P$ such that $u_P = m_P \circ v_P$. Since $P \notin T(C)$, u_P is unramified. Hence v_P is étale. Since $P \notin \Lambda_C$, u_P and v_P have degree two and no fiber of u_P has cardinality at least 2. Since each fiber of v_P has cardinality 2, the normalization map m_P is bijective. Since u_P is unramified, m_P must be an isomorphism. Thus u_P is a smooth plane curve of degree $d/2$. Hence d is even and $p_a(u_P(C)) = (d/2-1)(d/2-2)/2$. Since $u_P(C)$ is not algebraically simply connected, we get $d/2 \ge 3$. Since u_P is a degree 2 étale covering, Riemann-Hurwitz formula gives $g = (d/2-1)(d/2-2)-1 = d^2/4-3d/2+1$. Since $d^2/4-3d/2+1 < d(d-3)/6+1$, the space curve C is contained in a quadric surface S (see [4], Note A at page 313). Let Γ denote the cone with base $u_P(C)$ and P as its vertex. Since $C \subseteq S \cap \Gamma$, Bezout's Theorem gives that C is the complete intersection of Γ and S. Thus the adjunction formula gives $g = d^2/4 - d + 1$ (see [5], Remark IV.6.4.1 (a) and Example V.2.9), contradiction.

Now assume $n > 3$. We fix any $(n - 4)$-dimension linear subspace $W \subset \mathbb{P}^n$ defined over $\mathbb{F}_q$ such that $W \cap \mathrm{Sec}(C) = \emptyset$ (Proposition 1). Fix a 3-dimensional linear subspace $M \subset \mathbb{P}^n$ defined over $\mathbb{F}_q$ such that $W \cap M = \emptyset$. See the linear projection from W as a morphism $\ell_W : \mathbb{P}^n \setminus W \to M$. Since W and M are defined over $\mathbb{F}_q$, then ℓ_W is defined over $\mathbb{F}_q$. Let $C_W \subset M$ denote the curve obtained projecting C from W. Apply the case $n = 3$ to C_W. For each admissible center of projection $P \in M$ the linear space $V_P := \langle W \cup \{P\}\rangle$ is an admissible center of projection for C. If $P_1 \ne P_2$, then $V_{P_1} \ne V_{P_2}$. □

Proposition 3. *Fix positive integers n, m, x such that $n > m$ and $q > x$. Let $E \subset \mathbb{P}^n$ be an equidimensional m-dimensional subvariety defined over $\mathbb{F}_q$ such that* $\deg(E) = x$. *Then $\sharp(E(\mathbb{F}_q)) \le x(q^{m+1} - 1)/(q - 1)$.*

Proof. As in the proof of Proposition 1 we find a linear subspace $W \subset \mathbb{P}^n$ defined over $\mathbb{F}_q$ such that $W \cap E = \emptyset$ and $\dim(W) = n - m - 1$. Let $\ell_W : \mathbb{P}^n \setminus W \to \mathbb{P}^m$ denote the linear projection from W. Notice that $\ell_W|E$ is a finite morphism of degree x. Bezout's Theorem gives that each fiber of $\ell_W|X$

has cardinality at most x. Since W is defined over $\mathbb{F}_q$, ℓ_W is defined over $\mathbb{F}_q$. Hence $\ell_W(E(\mathbb{F}_q) \subseteq \mathbb{P}^m(\mathbb{F}_q)$. Since each fiber of $\ell_W|E$ has cardinality at most x, we get $\sharp(E(\mathbb{F}_q)) \le x(q^{m+1}-1)/(q-1)$. □

Corollary 1. *Let $C \subset \mathbb{P}^n$, $n \ge 4$, be a smooth and geometrically connected curve. Assume that both C and the inclusion $C \hookrightarrow \mathbb{P}^n$ are defined over $\mathbb{F}_q$. Set $d := \deg(G)$ and $g := p_a(C)$. Assume $q \ge (d-1)(d-2)/2 - g$. Then there at least $(q^{n+1}-1-((d-1)(d-2)/2-g)(q^4-1))/(q-1)$ points $P \in \mathbb{P}^n(\mathbb{F}_q)$ such that the linear projection from P induces an embedding of C into $\mathbb{P}^{n-1}$.*

Proof. Since $\deg(S(C)) \le (d-1)(d-2)/2-g$ and $\dim(S(C)) = 3$, we may apply Proposition 1. □

Question 1. If $n \ge 2m+1$, then Proposition 3 is sharp (take as E a disjoint union of x m-dimensional linear subspaces, each of them being defined over $\mathbb{F}_q$). Under what assumptions on q, n, m, x is this the only example and by how much it can be improved if we exclude that E has a linear subspace as one of its irreducible components?

For the case $m = n-1$, see [12], Theorem 3. A deep study of the case $(n, x) = (2, 1)$ is done in several papers by M. Homma and S.J. Kim (see [6], [7] and references therein).

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