International Journal of Pure and Applied Mathematics
Volume 69 No. 1 2011, 63-65

SINGULAR CURVES WITH LINE BUNDLES L DEFINED OVER $\mathbb{F}_q$ AND WITH GOOD COHOMOLOGY

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here (following a previous paper of mine) we prove the existence of several pairs (X, L), where X is a geometrically integral projective curve defined over $\mathbb{F}_q$ and L is a line bundle on X defined over $\mathbb{F}_q$ and with either $H^0(X, L) = 0$ or $H^1(X, L) = 0$. These examples are obtained using the existence of similar line bundles on the normalization of X, i.e. a case studied by C. Ballet, C. Ritzenthaler and R. Roland.

AMS Subject Classification: 14H20, 14H05, 12E20
Key Words: singular curve over a finite field, line bundle defined over a finite field

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Fix a prime p and a p-power q. In [1] the authors considered the following problem. Let X be a smooth and geometrically connected projective curve defined over $\mathbb{F}_q$. Is there a line bundle L on X defined over $\mathbb{F}_q$ such that $\deg(L) = g - 1$ and $h^0(X, L) = 0$? By Riemann-Roch we have $h^0(X, L) = h^1(X, L)$ for all $L \in \mathrm{Pic}^{g-1}(X)$. In [2] we considered the same problem for singular curves. Here we add a few results to [2] and recast some of the results of [2] in a more general way.

Let X be a geometrically integral projective curve of arithmetic genus g defined over $\mathbb{F}_q$. For any integer $t \leq g-1$ (resp. $t \geq g-1$) let $\mathcal{S}(X, q, t)$ denote the set of all isomorphism classes of degree t line bundles L defined over $\mathbb{F}_q$ and

Received: January 18, 2011

such that $h^0(X,L) = 0$ (resp. $h^1(X,L) = 0$). We recall that ω_X is defined over any field on which X is defined. Thus if X is Gorenstein, then Serre's duality gives $\sharp(\mathcal{S}(X,t)) = \sharp(\mathcal{S}(X,2g-2-t))$ for all t.

Theorem 1. *Let $u : Y \to X$ be a birational morphism defined over $\mathbb{F}_q$ between two geometrically integral projective curves defined over $\mathbb{F}_q$. Fix an integer $t \geq p_a(X) - 1$. If $\mathcal{S}(Y,q,t) = \emptyset$, then $\mathcal{S}(X,q,t) = \emptyset$.*

Remark 1. Let $u : Y \to X$ be a birational morphism defined over $\mathbb{F}_q$ between two geometrically integral projective curves defined over $\mathbb{F}_q$. Fix an integer $t \geq p_a(X) - 1$. Very often $t \geq p_a(Y)$. Assume $t \geq p_a(Y)$ and Y Gorenstein (e.g. smooth). We have $\sharp(\mathcal{S}(Y,q,t)) = \sharp(\mathcal{S}(Y,q,2p_a(Y)-2-t))$. For the nonemptiness of $\mathcal{S}(Y,q,y)$, $y < p_a(Y)$, when Y is smooth, see [1], Theorem 3.3 and Corollary 3.4.

Proof of Theorem 1. Assume the existence of $R \in \mathcal{S}(X,q,t)$. Since $t \geq p_a(X) - 1$ and $R \in \mathcal{S}(X,q,t)$, we have $h^1(X,R) = 0$. Since $u_*(\mathcal{O}_X)/\mathcal{O}_Y$ has finite support, the exact sequence

$$0 \to R \to R \otimes u_*(\mathcal{O}_Y) \to R \otimes (u_*(\mathcal{O}_X)/\mathcal{O}_Y) \to 0$$

gives $h^1(X, R \otimes u_*(\mathcal{O}_Y)) = 0$. The projection formula gives $u_ast(u^*(R)) \cong R \otimes u_*(\mathcal{O}_Y)$. Since u is finite, the Leray spectral sequence of u gives $h^1(Y,u^*(R)) = 0$. Since $\deg(u^*(R)) = 0$, we get $u^*(R) \in \mathcal{S}(Y,q,t)$, contradiction. □

We also give the following results which complements [2].

Proposition 1. *Let Y be a geometrically integral projective curve of genus $g \geq 0$. Let X be the curve obtained from Y creating an ordinary cusp at Q, i.e. $p_a(Y) = g+1$, there is a birational morphism $u : Y \to X$ which is injective, $u|(Y \setminus \{Q\})$ is an isomorphism and $u(Q)$ is an ordinary cusp of X. Both X and u are defined over $\mathbb{F}_q$. We have $\mathcal{S}(Y,q,g) \neq \emptyset$ if and only if $\mathcal{S}(X,q,g) \neq \emptyset$. If $\mathcal{S}(Y,q,g) \neq \emptyset$, then $\sharp(\mathcal{S}(X,q,g)) \geq (q-1) \cdot \sharp(\mathcal{S}(Y,q,g))$.*

Proposition 2. *Assume $q \neq 2$. Let Y be a geometrically integral projective curve of genus $g \geq 0$ defined over $\mathbb{F}_q$. Assume $\sharp(Y_{reg}(\mathbb{F}_q)) \geq 2$ and fix $P, Q \in Y_{reg}(\mathbb{F}_q)$ such that $P \neq Q$. Let X be the curve obtained from Y gluing together P and Q to get an ordinary node, i.e. $p_a(Y) = g+1$, there is an unramified birational morphism $u : Y \to X$ such that $u|Y \setminus \{P\}$ is injective, $u|(Y \setminus \{P,Q\})$ is an isomorphism and $u(Q)$ is an ordinary node of X. Both X and u are defined over $\mathbb{F}_q$. We have $\mathcal{S}(Y,q,g) \neq \emptyset$ if and only if $\mathcal{S}(X,q,g) \neq \emptyset$. If $\mathcal{S}(Y,q,g) \neq \emptyset$, then $\sharp(\mathcal{S}(X,q,g)) \geq (q-2) \cdot \sharp(\mathcal{S}(Y,q,g))$.*

Proposition 3. *Let Y be a geometrically integral projective curve defined over $\mathbb{F}_q$ such that $N(Y,2) > 0$. Fix $P \in Y_{reg}(\mathbb{F}_{q^2})$ such that $P \notin Y_{reg}(\mathbb{F}_q)$. Set*

$Q := F_q(P)$. Since $P \in Y_{reg}(\mathbb{F}_{q^2}) \setminus Y_{reg}(\mathbb{F}_q)$, we have $Q \in Y_{reg}(\mathbb{F}_{q^2}) \setminus Y_{reg}(\mathbb{F}_q)$ and $Q \neq P$. Let X be the curve obtained from Y gluing together P and Q, i.e. $p_a(X) = g+1$ and there is a birational morphism $u : Y \to X$ such that $u|(Y \setminus \{Q\})$ is an isomorphism onto $Y \setminus \{u(Q\}$), $u(P) = u(Q)$ and $u(Q)$ is an ordinary node of X. Both X and u are defined over $\mathbb{F}_q$. set $g := p_a(Y)$. We have $\mathcal{S}(Y,q,g) \neq \emptyset$ if and only if $\mathcal{S}(X,q,g) \neq \emptyset$. If $\mathcal{S}(Y,q,g) \neq \emptyset$, then $\sharp(\mathcal{S}(X,q,g)) \geq q \cdot \sharp(\mathcal{S}(Y,q,g))$.

Proof of Propositions 1, 2 and 3. If $\mathcal{S}(X,q,g) \neq \emptyset$, then $\mathcal{S}(X,q,g) \neq \emptyset$ by Theorem 1. Assume $\mathcal{S}(Y,q,g) \neq \emptyset$. We modify the proofs in [2] in the following way. Instead of starting with $R \in \mathcal{S}(Y,q-g-1)$ and playing with the line bundle $R(Q) \in \mathcal{S}(Y,q,g)$ we play with an arbitrary $R' \in \mathcal{S}(Y,q,g)$. □

Acknowledgments

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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