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NORM, NORM RATIO CALCULATIONS AND ANISOTROPY DEGREE

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Abstract: In this paper, for elastic constant tensor, the norm concept, norm ratio and anisotropy degree are described. The norm of a tensor is used as a criterion for comparing the overall effect of the properties of anisotropic materials and norm ratios are used as a criterion to represent the anisotropy degree of the properties of these materials. Norm and norm ratios as well as the measure of "nearness" to the nearest isotropic tensor are computed for several examples from various anisotropic materials possessing elastic symmetries such as cubic, transversely isotropic, tetragonal, trigonal and orthorhombic. These computations are used to compare and assess the anisotropy in various anisotropic materials by means of strength or magnitude and also determine the "nearness" of the nearest isotropic tensor for the materials with lower symmetry types.

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1. Introduction

Most of the elastic materials in engineering are anisotropic; metal crystals, fiber-reinforced composites, polycrystalline textured materials, biological tissues, rock structures. In order to understand the physical properties of the anisotropic materials, use of tensors by decomposing them is inevitable.

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The constitutive relation for linear anisotropic elasticity, defined by using stress and strain tensors, is the generalized Hooke's law

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}. \tag{1}$$

This formula demonstrates the well known general linear relation between the stress tensor (symmetric second order tensor) whose components are σ_{ij} and the strain tensor (symmetric second order tensor) whose components are ε_{kl}. The coefficients of linearity, namely C_{ijkl} are the components of elastic constant tensor (elasticity tensor) and satisfies three important symmetry restrictions. These are

$$C_{ijkl=}C_{jikl}, \quad C_{ijkl=}C_{ijlk}, \quad C_{ijkl=}C_{klij}, \tag{2}$$

which follow from the symmetry of the stress tensor, the symmetry of the strain tensor and the elastic strain energy. These restrictions reduce the number of independent elastic constants C_{ijkl} from 81 to 21.

The purpose of the work is to elaborate on the subjects like norm, norm ratio and anisotropy degree and emphasize their importance on decomposition of elastic constant tensor for any symmetry type of anisotropic materials.

In the present paper, norm concept is defined in section 2. A proposed relation between the norm ratio and the anisotropy degree is given and examples from different symmetry types materials are illustrated in section 3 and 4 respectively. Finally, in the last section, the results of computations are discussed and conclusions pertinent to this work are stated.

2. The Norm Concept

Norm is an invariant of the material. For instance, the magnitude of length is a norm for a vector. If it is a unit vector, its norm will be equal to 1.There are many types of norm in literature. Those norms are Euclidean, Riemannian, log-Euclidean, Taxicab, infinity, uniform, zero and so on. These norms are used in different fields of science and engineering. For instance, zero norm is related with machine learning and optimization. Log-Euclidean norm is a measure for tensors such as symmetric positive-definite matrices in medical imaging, modeling of anatomical variability i.e. human brain variability and Riemannian and log-Euclidean norms are used to find the shortest distance between an elasticity tensor of arbitrary symmetry and an elasticity tensor of lower symmetry. These two norms are effective when elastic compliance tensor is considered. Since the most appropriate and reliable norm for elastic constant tensor is Euclidean norm in literature, Euclidean norm is used for computations as a measure in this paper. Comparison of magnitudes of the Euclidean norm give a valuable information about the origin of the physical property under examination. Euclidean norm also represents the stiffness effect in the material like fiber-reinforced composites.

Euclidean norm of a Cartesian tensor is defined as the square root of the contracted product over all the indices with itself, which is given as follows (see [1])

$$N = \|C\| = \{C_{ijkl\ldots}C_{ijkl\ldots}\}^{\frac{1}{2}}. \tag{3}$$

Since the basis constructed in this thesis is orthonormal and $C_{ijkl\ldots}$ is in the space spanned by that orthonormal basis $\{A^K\}$, it is straightforward to see that, now the norm [2]

$$N = \|C\| = \{\textstyle\sum_K (C, A^K_{ijkl})^2\}^{\frac{1}{2}}. \tag{4}$$

The norm of nearest isotropic tensor, denoted by C^o_{iikl}, of C_{ijkl} is therefore

$$N_i = \|C^o\| = \{\textstyle\sum_{K=I} (C^o, A^K_{ijkl})^2\}^{\frac{1}{2}}, \quad (K = I, II). \tag{5}$$

In similar way, with respect to the tensor C_{ijkl}, the nearest tensors of other symmetry classes within the class spanned by the basis $\{A^K\}$ can be read off from the representation and their norms may be computed according to equation (2).

By using the norms, the nearest isotropic tensors of lower symmetries such as cubic, transversely isotropic, tetragonal, trigonal and orthorhombic can be found via the following formula

$$\varepsilon^o = \frac{\|C\| - \|C^o\|}{\|C\|}, \tag{6}$$

where ε^o is a scalar constant independent of the rotation of the axes. It is a measure of "nearness" of the nearest isotropic tensor.

3. A Proposed Relation Between the Norm Ratio and the Anisotropy Degree

It is obvious that the anisotropy of the material, for instance, the symmetry group of the material and the anisotropy of the measured property depicted in the same materials may be quite different. Clearly, the property tensor must show, at least, the symmetry of the material. For instance, a property which is measured in a material can almost be isotropic but the material symmetry group itself may have very few symmetry elements. It is known that, for isotropic materials, the elastic constant tensor has two scalar (isotropic) parts, so the norm of the elastic constant tensor for isotropic materials depends only on the norm of the scalar parts, i.e., $N = N_i$. So, the ratio $\frac{N_i}{N} = 1$ for isotropic materials. For cubic materials, the elastic constant tensor has the same two parts that consisting the isotropic symmetry and a third which is designated as the anisotropic part, hence we define two ratios: $\frac{N_i}{N}$ for the isotropic parts and $\frac{N_a}{N}$ for the anisotropic part. For lower symmetry type materials such as transversely isotropic, tetragonal, trigonal and orthorhombic, the elastic constant tensor additionally contains more anisotropic parts, so $\frac{N_a}{N}$ can be defined for all the anisotropic parts.

Although the norm ratios of different parts represent the anisotropy of that particular part, they can also be used to asses and compare the anisotropy degree of a material property as a whole.

4. Examples From Various Anisotropic Elastic Symmetries

The concept of norm, norm ratio and anisotropy degree are implemented in this section for different anisotropic materials possessing symmetries such as cubic, transversely isotropic, tetragonal, trigonal, orthorhombic [3-5] and computed for several examples. These computations are compared in strength or magnitude of any property in different materials. The numerical data for elastic constants are mainly taken from Landolt-Börnstein [6].

4.1. Examples From Cubic Symmetry

Elastic constants of cubic materials are given in Table 4.1. The units are in GPa.

The norm and norm ratios (the anisotropy degrees) for cubic materials are calculated in order to determine which one is close to isotropy or anisotropy. The results for norm, norm ratios and the measure of "nearness" of the nearest isotropic tensor are presented in Table 4.2.

According to the calculated results in Table 4.2, the most isotropic material

Cubic Media	C_{11}	C_{12}	C_{44}
AlSb [7]	87.7	43.4	40.8
Indium Phosphide(InP) [12]	102	58	46
Gallium Arsenide(GaAs) [13]	118	53.5	59.4
Gallium Antimonide(GaSb) [14]	88.4	40.3	43.4
Indium Arsenide(InAs) [15]	84.4	46.4	39.6
Gallium Phosphide(GaP) [16]	142	63	71.6

Table 4.1: Elastic constants of cubic materials

Cubic Media	N_i	N_a	N	$\frac{N_i}{N}$	$\frac{N_a}{N}$	ε^o
AlSb	229.524	40.8601	233.1328	0.9845	0.1753	0.0155
InP	272.072	52.5814	277.1065	0.9818	0.1898	0.0182
GaAs	312.646	59.4827	318.2543	0.9824	0.1869	0.0176
GaSb	232.3655	42.3937	236.2011	0.9838	0.1795	0.0162
InAs	225.9841	45.1323	230.4469	0.9806	0.1958	0.0194
GaP	375.3382	70.3276	381.87	0.9829	0.1842	0.0171

Table 4.2: The norm and norm ratios (the anisotropy degrees)for cubic materials

among the other six materials is Aluminium Antimonide (AlSb). Since mathematically, $\frac{N_i}{N}$ for AlSb is so close to 1 that implies the closeness to the isotropic behaviour of the cubic materials which agrees with the physical understanding of the materials with cubic symmetry. This case is also verified by taking into account the results of ε^o which is closer to 0 than those of other five materials which indicates that AlSb is nearest to isotropy among the other materials. The most anisotropic material is selected as Indium Arsenide (InAs). Since the value of $\frac{N_i}{N}$ for InAs is the smallest and in reverse manner, the value of $\frac{N_a}{N}$ for InAs is the largest among the cubic materials. This case shows that the property of Indium Arsenide is the most anisotropic.

4.2. Examples From Transversely Isotropic Symmetry

Elastic constants of transversely isotropic materials are given in Table 4.3. The units are in GPa.

For transversely isotropic materials, the norm and norm ratios, ε^o (the anisotropy degrees) are computed in order to determine which one is close to isotropy or anisotropy. The results for norm, norm ratios and the measure of “nearness” of the nearest isotropic tensor are presented in Table 4.4.

Transversely isotropic Media	C_{11}	C_{12}	C_{13}	C_{33}	C_{44}
Polystyrene [8]	5.20	2.75	2.75	5.70	1.30
Hardened tool steel [17]	277	113	112	272	80.3
Zinc(Zn) [18]	165	31.1	50	61.8	39.6
Cadmium [19]	116	42	41	50.9	19.6
Normal tool steel [20]	289	116	117	284	84.5

Table 4.3: Elastic constants of transversely isotropic materials

Transversely isotropic Media	N_i	N_a	N	$\frac{N_i}{N}$	$\frac{N_a}{N}$	ε^o
Polystyrene	12.2996	0.432	12.3100	0.9994	0.0351	0.000617
Hardened tool steel	617.745	5.257	617.768	1.000	0.0085	0.000036
Zinc(Zn)	301.619	98.510	317.298	0.9506	0.3105	0.049400
Cadmium	211.340	60.330	219.7819	0.9616	0.2745	0.038400
Normal tool steel	645.282	5.367	645.3038	1.000	0.0083	0.0000346

Table 4.4: The norm and norm ratios (the anisotropy degrees) for transversely isotropic materials

From Table 4.4, it is seen that the ratio $\frac{N_i}{N}$ gives the same result for hardened and normal tool steel which is equal to 1 and the results for ε^o is close to each other. But ε^o of normal tool steel is smaller than ε^o of hardened tool steel which shows that normal tool steel is more isotropic than the hardened one. The same case is also proved by comparing the $\frac{N_a}{N}$ for both tool steel type. The larger ratio $\frac{N_a}{N}$ and ε^o, the more anisotropic property exists for a transversely isotropic material and in reverse manner, the smaller ratio $\frac{N_i}{N}$, a transversely isotropic material possesses the more anisotropic property. So Zinc is the most anisotropic material whereas normal tool steel is the elastically strongest material among the other transversely isotropic materials.

4.3. Examples From Tetragonal Symmetry

Elastic constants of tetragonal materials are presented in Table 4.5. The units are in GPa.

The norm and norm ratios, ε^o (the anisotropy degrees) for tetragonal materials are calculated in order to determine the effect of anisotropy in other words which one is more anisotropic or isotropic. The results for norm, norm ratios and the measure of "nearness" of the nearest isotropic tensor are summarized in Table 4.6.

Tetragonal Media	C_{11}	C_{12}	C_{13}	C_{33}	C_{44}	C_{66}
Zircon, $ZrSiO_4$ (metamict) [9]	284	73	119	309	77.5	47.7
Indium-cadmium alloy, In-3.42 at %Cd [21]	44.8	41	40.5	44.1	6.86	11.25
Ammonium dihydrogen arsenate (piezoel.), $NH_4H_2ASO_4$[22]	62.2	8.6	18.4	29.6	6.69	6.22
Rolled steel [23]	284	96	112	269	82.1	68.9
Indium bismuth(InBi) [24]	51.1	37	32	34.6	19.8	15.9

Table 4.5: Elastic constants of tetragonal materials

Tetragonal Media	N_i	N_a	N	$\frac{N_i}{N}$	$\frac{N_a}{N}$	ε^o
Zircon, $ZrSiO_4$ (metamict)	610.083	95.11	617.45	0.9881	0.1540	0.0119
Indium-cadmium alloy, In-3.42 at %Cd	128.521	15.717	129.478	0.9926	0.1214	0.0074
Ammonium dihydrogen arsenate (piezoel.), $NH_4H_2ASO_4$	95.653	38.462	103.097	0.9278	0.3731	0.0722
Rolled steel	611.468	36.06	612.53	0.9983	0.0589	0.0017
Indium bismuth(InBi)	128.03	31.77	131.91	0.9706	0.2408	0.0294

Table 4.6: The norm and norm ratios (the anisotropy degrees) for tetragonal materials

Trigonal Media	C_{11}	C_{12}	C_{13}	C_{14}	C_{33}	C_{44}
Haematite,Fe_2O_3 [10]	242	54.9	15.7	-12.7	228	85.3
Antimony [25]	99.4	30.9	26.4	21.6	44.5	39.5
Magnesite,$MgCO_3$ [26]	259	75.6	58.8	-19	156	54.8
As-Sb at % As 25.5 [27]	106.7	48.4	28.5	18.8	48	40.8
Arsenic [28]	130.2	30.3	64.3	-3.71	58.7	22.5

Table 4.7: Elastic constants of trigonal materials

According to Table 4.6, by comparing the ratio $\frac{N_i}{N}$ and ε^o, rolled steel exhibits the most isotropic property among the others. On the other hand, by taking into account the ratio $\frac{N_a}{N}$, Ammonium dihydrogen arsenate (piezoel.) shows the most anisotropic property and zircon is the elastically strongest material among the other tetragonal materials.

4.4. Examples From Trigonal Symmetry

Elastic constants of trigonal materials are given in Table 4.7. The units are in GPa.

For trigonal materials, the norm and norm ratios, ε^o (the anisotropy degrees) are computed in order to determine which one is close to isotropy or

Trigonal Media	N_i	N_a	N	$\frac{N_i}{N}$	$\frac{N_a}{N}$	ε^o
Haematite,Fe_2O_3	515.476	78.782	521.46	0.9885	0.1511	0.0115
Antimony	202.253	100.854	226.004	0.8949	0.4462	0.1051
Magnesite,$MgCO_3$	457.455	116.54	472.068	0.969	0.247	0.031
As-Sb at % As 25.5	214.39	97.027	235.327	0.911	0.4123	0.089
Arsenic	250.355	85.421	264.53	0.9464	0.3229	0.0536

Table 4.8: The norm and norm ratios (the anisotropy degress) for trigonal materials

Orthorhombic Media	C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}	C_{44}	C_{55}	C_{66}
Olivine [11]	192	66	60	160	56	272	60	62	49
Pine(Softwood) [29]	1.24	0.74	0.76	17.1	0.94	1.79	1.18	0.079	0.91
Olivinite [30]	232	93	92	210	82	199	73.3	70.9	68.6
Marble [30]	119	51	52	110	47	104	29.7	30.7	32.6
Canine femora [31]	19	9.73	11.9	22.2	11.9	29.7	6.67	5.67	4.67

Table 4.9: Elastic constants of orthorhombic materials

anisotropy. The results for norm, norm ratios and the measure of "nearness" of the nearest isotropic tensor are shown in Table 4.8.

From Table 4.8, it is understood that Haematite is the most isotropic and elastically strongest material among the others by comparing the ratio $\frac{N_i}{N}$ and ε^o. Besides among trigonal materials, Antimony is the most anisotropic material by investigating the effect of the ratio $\frac{N_a}{N}$.

4.5. Examples From Orthorhombic Symmetry

Elastic constants of orthorhombic materials are presented in Table 4.9. The units are in GPa.

The norm and norm ratios, ε^o (the anisotropy degrees) for orthorhombic materials are calculated in order to determine the effect of anisotropy in other words which one is more anisotropic or isotropic. The results for norm, norm ratios and the measure of "nearness" of the nearest isotropic tensor are summarized in Table 4.10.

In Table 4.10, by taking into account the effect of the norm ratios; $\frac{N_i}{N}$, $\frac{N_a}{N}$ and ε^o, it is obvious that marble is an orthorhombic material that possesses the

Orthorhombic Media	N_i	N_a	N	$\frac{N_i}{N}$	$\frac{N_a}{N}$	ε^o
Olivine	435.35	93.267	445.228	0.9778	0.2095	0.0222
Pine(Softwood)	11.0247	15.9396	19.381	0.5688	0.8224	0.4312
Olivinite	494.479	35.1969	495.73	0.9975	0.0710	0.0025
Marble	251.9798	12.8411	252.3067	0.9987	0.0509	0.0013
Canine femora	53.0038	8.9226	53.7495	0.9861	0.1660	0.0139

Table 4.10: The norm and norm ratios (the anisotropy degress) for orthorhombic materials

most isotropic effect among the other orthorhombic materials with the largest value for ratio $\frac{N_i}{N}$ and the smallest value for ε^o. Pine, which is a softwood, exhibits the most anisotropic property among the others with the largest value for ratio $\frac{N_a}{N}$ and ε^o. Olivinite is the elastically strongest among these five materials.

5. Discussion and Conclusion

In conclusion, the following significant notes should be taken into account when the computed results are evaluated in above tables. These notes are:

1. Norm can be used as a parameter representing and comparing the overall effect of a certain property of anisotropic materials of the same or different symmetry. If the norm value of a material is large, it has more effective property than the other materials of the same symmetry type.

2. When N_i is the largest among norms of the decomposed parts, if the norm ratio $\frac{N_i}{N}$ is closer to one, the material property is closer to isotropic.

3. When N_i is not the largest or not present, norms of the other parts can be used as a criterion. But in this case the situation is reverse; if the norm ratio value is larger than the others, the material property is more anisotropic.

Finally, I hope this paper prepares interested readers to appreciate a deep understanding of norm, norm ratio calculations and anisotropy degree as well as the significant effects on many applications in different fields such as examining the material symmetry types in detail and determination of materials possessing same crystal symmetry type which are highly anisotropic or close to isotropy.

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