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# CONSTRUCTING EXACT SOLUTIONS TO DIFFERENCE-DIFFERENTIAL EQUATIONS VIA THE PROJECTIVE RICCATI EQUATION APPROACH

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**Abstract:** An algebraic algorithm is devised to derive exact traveling wave solutions of differential-difference equations by means of the projective Riccati equation approach. For illustration, we apply this method to solve relativistic Toda lattice system. Some explicit and exact traveling wave solutions including periodic solutions, solitary wave solutions are constructed.

**AMS Subject Classification:** 35R10
**Key Words:** difference-differential equation, soliton solution, exact solution, relativistic Toda lattice system

## 1. Introduction

Nonlinear differential-difference equations (DDEs) are important models describing nonlinear phenomena. Since the work of Fermi et al [1] in the 1950s, DDEs have been the focus of many nonlinear studies, see [2]-[5]. In order to find exact solutions of DDEs, some methods have been proposed, such as exp-function method (see [6]), hyperbolic function method (see [7], and so on [8]-[13]). Recently, Wang et al [14] improved the projective Ricatti equation expansion approach, which was introduced to solve PDEs in [15], to find exact traveling wave solutions of DDEs. To our knowledge, besides Wang et al.,

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there's no one else solves exact solutions of DDEs through this method.

In this paper, we would like to modify Wang's method to investigate the relativistic Toda lattice(RTL) system, see [3], [9], [16]. With the help of symbolic computation soft *Maple*, abundant explicit and exact traveling wave solutions are obtained.

## 2. Summary of the Proposed Method

Given a nonlinear system of $l$ polynomial DDE

$$H(U_{n+n_1}(X),\cdots,U_{n+n_k}(X),U'_{n+n_1}(X),\cdots,U'_{n+n_k}(X),\\ \cdots,U^{(r)}_{n+n_1}(X),\cdots,U^{(r)}_{n+n_k}(X))=0, \quad (1)$$

where dependent variable $U=(u_1,\cdots,u_l)$, independent variable $X=(x_1,\cdots,x_m)$, the discrete variable $n=(n_1,\cdots,n_q)$, and $U_i^r(X)$ denotes the set of all $r$th-order mixed derivative terms of $U_i$ with respect to $X$.

To seek the traveling wave solutions of (1), the first step is to introduce the wave transformation

$$U_n(x)=U(\xi_n), \quad \xi_n=\sum_{i=1}^{q}d_in_i+\sum_{j=1}^{m}c_jx_j+\delta, \quad (2)$$

where the coefficients $d_i$, $c_j$ and the phase $\delta$ are constants to be determined later. Then (1) becomes

$$H(U_{n+n_1}(\xi_{n+n_1}),\cdots,U_{n+n_k}(\xi_{n+n_k}),U'_{n+n_1}(\xi_{n+n_1}),\cdots,\\ U'_{n+n_k}(\xi_{n+n_k}),\cdots,U^{(r)}_{n+n_1}(\xi_{n+n_1}),\cdots,U^{(r)}_{n+n_k}(\xi_{n+n_k}))=0, \quad (3)$$

We suppose that the traveling wave solutions of (3) is in the following form:

$$U_n=A_0+\sum_{i=1}^{N_n}(A_if_n+B_ig_n)f_n^{i-1},\ U_{n+k}=A_0+\sum_{i=1}^{N_{n+k}}(A_if_{n+k}+B_ig_{n+k})f_{n+k}^{i-1}, \quad (4)$$

where $A_0$, $A_i$, $B_i$ are constants to be determined later, $N_n$ can be obtained by discrete Painlevé property, $f_n$ and $g_n$ satisfy the projective Ricatti equation

$$f_n{}'(\xi)=pf_n(\xi)g_n(\xi), \quad g_n{}'(\xi)=q+pg_n{}^2(\xi)-rf_n(\xi). \quad (5)$$

Equations (5) has three types solutions, while $f_{n+k},\ \ g_{n+k}$ can be presented by $f_n$ and $g_n$, see [14].

*Case A.* Hyperbolic function solutions

$$f_n(\xi_n) = \frac{1}{r + s\cosh(\xi_n) + h\sinh(\xi_n)},$$

$$g_n(\xi_n) = \frac{s\sinh(\xi_n) + h\cosh(\xi_n)}{r + s\cosh(\xi_n) + h\sinh(\xi_n)}. \tag{6}$$

a relationship between $f_n(\xi_n)$ and $g_n(\xi_n)$ as square relation small

$$ {g_n}^2 = 1 - 2rf_n + (r^2 + h^2 - s^2){f_n}^2, \tag{7}$$

and recursive formulas

$$\begin{aligned} f_{n+k}(\xi_{n+k}) &= f_{n+k}(\xi_n + \omega_k) \\ &= \frac{f_n}{rf_n[\cosh(\omega_k) - 1] - g_n\sinh(\omega_k) + \cosh(\omega_k)}, \\ g_{n+k}(\xi_{n+k}) &= g_{n+k}(\xi_n + \omega_k) \\ &= \frac{rf_n\sinh(\omega_k) - g_n\cosh(\omega_k) + \sinh(\omega_k)}{rf_n[\cosh(\omega_k) - 1] - g_n\sinh(\omega_k) + \cosh(\omega_k)}. \end{aligned} \tag{8}$$

Substitute (4) into (3), under conditions (5), (7) and (8), then set all coefficients of ${f_n}^i {g_n}^j$ (j=0,1, i,= 0, $1,\cdots$) to be zero, we obtain an over-determined nonlinear algebraic system with respect to $A_i,\ B_i,\ c_j$. Finally, with the help of *Maple*, solving the obtained algebraic system and making use of the explicit solutions of equations (5), we could find the traveling wave solutions to nonlinear DDE (1) with the form of (4).

The similar way could be deal with following two cases.

*Case B.* Periodic function solutions

$$f_n = \frac{1}{r + s\cos(\xi_n) + h\sin(\xi_n)}, \qquad g_n = \frac{s\sin(\xi_n) - h\cos(\xi_n)}{r + s\cos(\xi_n) + h\sin(\xi_n)}. \tag{9}$$

$$ {g_n}^2 = -1 + 2rf_n - (r^2 - h^2 - s^2){f_n}^2, \tag{10}$$

$$f_{n+k} = \frac{-f_n}{\sin(\omega_k)g_n - rf_n + \cos(\omega_k)f_n r - \cos(\omega_k)},$$

$$g_{n+k} = \frac{-\sin(\omega_k) - \cos(\omega_k) g_n + \sin(\omega_k) f_n r}{\sin(\omega_k) g_n - r f_n + \cos(\omega_k) f_n r - \cos(\omega_k)}. \tag{11}$$

*Case C.* Constant solutions

$$f_n = \frac{2}{pr{\xi_n}^2 + C_1 \xi_n - C_2}, \qquad g_n = -\frac{2\,pr\xi_n + C_1}{(pr{\xi_n}^2 + C_1\xi_n - C_2)p}. \tag{12}$$

$${g_n}^2 = \frac{2r}{p} f_n + \frac{{C_1}^2 + 4C_2 pr}{4p^2} {f_n}^2, \tag{13}$$

$$f_{n+k} = \frac{2 f_n}{2 + pr{\omega_k}^2 f_n - 2\,\omega_k p g_n}, \qquad g_{n+k} = \frac{2\,g_n - 2\,r\omega_k f_n}{2 + pr{\omega_k}^2 f_n - 2\omega_k p g_n}. \tag{14}$$

**Remark 1.** Now three sets basic solutions of (5), the corresponding square relationships and recursive formulas are presented. This ensures that the original equations and the system, which consists of all the coefficients about the items ${f_n}^i {g_n}^j$ in the reduced equations, are equivalent.

**Remark 2.** For simple, let $\omega_1 = -\omega_{-1} \equiv d$ here and throughout the text. In other words, $\xi_n$ is supposed to be an arithmetic progression with respect to the discrete variable $n$.

## 3. Application to Relativistic Toda Lattice System

We derive the solutions of relativistic Toda lattice (RTL) system (see [3], [9], [16]):

$$\begin{aligned} \frac{\partial U_n(t)}{\partial t} &= [1 + \alpha\, U_n(t)][V_n(t) - U_{n-1}(t)], \\ \frac{\partial V_n(t)}{\partial t} &= V_n(t)[U_{n+1}(t) - U_n(t) + \alpha(V_{n+1}(t) - V_{n-1}(t))]. \end{aligned} \tag{15}$$

Firstly, we make a transformation

$$U_n(t) = U_n(\xi_n), \quad V_n(t) = V_n(\xi_n), \quad \xi_n = dn + ct + \delta. \tag{16}$$

where $d, c, \delta$ are constants. Equation (15) reduces into

$$\begin{aligned} cU_n'(\xi_n) &= (1 + \alpha\, U_n(\xi_n))(V_n(\xi_n) - U_{n-1}(\xi_n)), \\ cV_n'(\xi_n) &= V_n(\xi_n)(U_{n+1}(\xi_{n+1}) - U_n(\xi_n) + \alpha V_{n+1}(\xi_{n+1}) - \alpha V_{n-1}(\xi_{n-1})). \end{aligned} \tag{17}$$

We suppose that (17) has the following formal solutions

$$U_n(\xi_n) = A_0 + A_1 f_n(\xi_n) + B_1 g_n(\xi_n),\ V_n(\xi_n) = a_0 + a_1 f_n(\xi_n) + b_1 g_n(\xi_n), \quad (18)$$

where $f_n$ and $g_n$ satisfy the projective Ricatti equation (5).

For Case A, substituting (18) with (5), (7), (8) into (17), we yield an algebraic equation for ${f_n}^i {g_n}^j (i = 0, 1, 2, 3, \cdots, j = 0, 1)$. Setting all coefficients of these terms to zero, we obtain a set of over-determined algebraic equations with respect to $A_0$,$A_1$, $B_1$, $a_0$, $a_1$, $b_1$,$c$, $d$. Solve this system and get

$$A_1 = B_1 = b_1 = 0,\ A_0 = -\frac{1}{\alpha},\ r = \pm\sqrt{s^2 - h^2}\cosh(\frac{d}{2}),$$

$$c = -2\alpha a_0 \sinh(d), \quad a_1 = \pm 2\sqrt{s^2 - h^2} a_0 \sinh(d)\sinh(\frac{d}{2})[1 + 2\cosh(d)], \quad (19)$$

with $h, s, d, a_0$ arbitrary constants.

From equations (6),(16) and (19),we get a family hyperbolic function solutions to RTL system.

*Family 1.*

$$U_n(t) = -\frac{1}{\alpha},$$

$$V_n(t) = a_0\left[1 + \frac{\pm 2\sqrt{s^2 - h^2} a_0 \sinh(d)\sinh(\frac{d}{2})[1 + 2\cosh(d)]}{\pm\sqrt{s^2 - h^2}\cosh(\frac{d}{2}) + s\cosh\xi + h\sinh\xi}\right], \quad (20)$$

where $\xi = dn \pm 2\alpha(1 - \cosh^2 d)t + c_0$.

*Family 2.*

$$U_n(t) = \frac{\alpha(1 - \cosh d)}{2a_0}\left[-1 + \frac{\pm\sqrt{r^2 - s^2 + h^2} + s\sinh\xi + h\cosh\xi}{r + s\cosh\xi + h\sinh\xi}\right],$$

$$V_n(t) = a_0\left[1 + \frac{\pm\sqrt{r^2 - s^2 + h^2} + s\sinh\xi + h\cosh\xi}{r + s\cosh\xi + h\sinh\xi}\right], \quad (21)$$

where $\xi = dn \pm 2\alpha(1 - \cosh d)t + c_0$.

In the same way, we could obtain its trigonometric function solutions

*Family 3.*

$$U_n(t) = \pm iB_1 + B_1\frac{\sin(dn + ct + c_0) \pm i\cos(dn + ct + c_0)}{\cos(dn + ct + c_0) \mp i\sin(dn + ct + c_0)},$$

$$V_n(t) = \pm i b_1 - b_1 \frac{\sin(dn+ct+c_0) \pm i\cos(dn+ct+c_0)}{\cos(dn+ct+c_0) \mp i\sin(dn+ct+c_0)}. \tag{22}$$

*Family 4.*

$$U_n(t) = \frac{i\alpha\,(\cos^2 d - 1)}{b_1} + \frac{\alpha\,(\cos^2 d - 1)(s\sin(\xi) - h\cos(\xi))}{b_1(s\cos(\xi) + h\sin(\xi))},$$
$$V_n(t) = -ib_1 + \frac{b_1(s\sin(\xi) - h\cos(\xi))}{s\cos(\xi) + h\sin(\xi)}, \tag{23}$$

with $\xi = dn \pm 2i\alpha(\cos^2 d - 1)t + c_0$.

*Family 5.*

$$U_n(t) = \frac{i\alpha\,(\cos(d) - 1)}{2b_1} \pm \frac{\sqrt{h^2+s^2}\alpha\,(\cos(d) - 1)}{2b_1(s\cos(\xi) + h\sin(\xi))} + \frac{\alpha\,(\cos(d) - 1)(s\sin(\xi) - h\cos(\xi))}{2b_1(s\cos(\xi) + h\sin(\xi))},$$

$$V_n(t) = -ib_1 \pm \frac{\sqrt{h^2+s^2}b_1}{s\cos(\xi) + h\sin(\xi)} + \frac{b_1(s\sin(\xi) - h\cos(\xi))}{s\cos(\xi) + h\sin(\xi)}, \tag{24}$$

with $\xi = dn \pm 2i\alpha(\cos d - 1)t + c_0$.

Now we analyze the solutions obtained in equation (21). Only one of them need to be considered. For example, if constants $r, s, h$ are fixed, the solution $V_n(t)$ has following cases.

$$V_n(t) = a_0(1 \pm i\text{sech}\xi + \tanh\xi) \qquad (r = h = 0, s = 1), \tag{25}$$

$$V_n(t) = a_0(1 \pm \text{csch}\xi + \coth\xi) \qquad (r = s = 0, h = 1), \tag{26}$$

$$V_n(t) = a_0(1 + \frac{\tanh\xi}{1 + \text{sech}\xi}) \qquad (r = s = 1, h = 0), \tag{27}$$

$$V_n(t) = a_0(1 + \frac{\coth\xi}{1 + \text{csch}\xi}) \qquad (r = h = 1, s = 0), \tag{28}$$

$$V_n(t) = \frac{2a_0(1 + \tanh\xi)}{1 + \tanh\xi + \text{sech}\xi} \qquad (r = s = h = 1), \tag{29}$$

The solution (25) is a kink-type soliton solution, while the solution (26) or (28) has singularity. Hence there exists the phenomenon of 'blow up' at the singular point, respective.

## 4. Conclusion

Introducing the projective Ricatti equations in the procedure of solving the difference-differential equation, we obtained many solutions to RTL system including soliton solutions presented by hyperbolic functions sinh and cosh, periodic solutions presented by sin and cos. This method can also be used to other nonlinear difference-differential equations.


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