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A SURVEY OF THE GRACEFULNESS OF DIGRAPHS

Wei Feng[1], Chunlei Xu[2], Jirimutu[3 §]

[1,3]College of Mathematics
Inner Mongolian University for Nationalities
Tongliao, 028043, P.R. CHINA
[3]e-mail: jrmt@sina.com

[2]College of Computer Science And Technology
Inner Mongolian University for Nationalities
Tongliao 028043, P.R. CHINA

Abstract: Graph labeling traces its origin to the famous conjecture that all trees are graceful presented by A.Rosa in 1966. In recent years, many research papers on graceful graph have been published. In this survey we have collected studies on the gracefulness of the directed circuit $\vec{C}_m$ and the union graph of mutually non-intersecting $\vec{C}_m$, the digraph $n \cdot \vec{C}_m$, the digraph $n - \vec{C}_m$, the union of the digraph $n \cdot \vec{C}_3$, and give some results.

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1. Introduction

Graph labeling traces its origin to the famous conjecture that all trees are graceful presented by A. Rosa [2] in 1966. A. Rosa developed β-valuations (which Golomb called graceful) as an attempt to answer Ringel's Conjecture [3] that if T is a tree of size q then there exists T-decomposition of K_{2q+1}. Another reason for the development of graceful labelings (as opposed to the more general β-valuations is that it can be shown that certain families of graphs do not obtain a graceful labelings. Vertex labeling is a mapping that maps the vertex

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§Correspondence author

set into an integer set(in general into an group). In recent years, many graph labelings have been evolved, many research papers on graceful graphs have been published, see [4]. Graceful labelings have been applied in many different fields in the modern world, such as coding theory, X-ray diffraction, crystallography, radar, radio astronomy, circuit design, and communication network addressing, see [5], [6], [7]. In 1979, Koh et al [8] conjectured that $C_n^{(t)}$ is graceful if and only if $nt \equiv 0, 3 \pmod 4$. Yang and Xu et al [9], [10], [11], [12] have proved the conjecture for the cases $n = 5, 7, 9$, and 11. Zhao et al [13] have proved the conjecture for the cases $n = 15$. A complete prove of the conjecture has not been given.

Definition 1. (see [14]) A graph $G(V, E)$ is said to be graceful if there exists an injection $f : V(G) \to \{0, 1, \cdots, |E|\}$, such that the induced function $f^{'} : E(G) \to \{1, 2, \cdots, |E|\}$ which is defined by $f^{'}(u, v) = |f(u) - f(v)|$ for every edge (u, v) is a bijection. Here, f is called a graceful labeling (graceful numbering) of G.

Definition 2. (see [15]) A digraph $D(V, E)$ is said to be graceful if there exists an injection $f : V(G) \to \{0, 1, \cdots, |E|\}$, such that the induced function $f^{'} : E(G) \to \{1, 2, \cdots, |E|\}$ which is defined by $f^{'}(u, v) = [f(v) - f(u)]$ (mod $(|E| + 1)$) for every directed edge (u, v) is a bijection. Here $[v]$ (mod n) denotes the least positive residue of v modulo n.

For any integers $a \leq b$, let $[a, b]$ denote the set of all consecutive integers from a to b. Let $\vec{C}_m$ denote the directed cycle on m vertices; $n\vec{C}_m$ denotes the graph obtained from any n copies of $\vec{C}_m$ which have no common vertex; $n \cdot C_m$ denotes the graph obtained from any n copies of C_m which have just one common vertex; $n \cdot \vec{C}_m$ denotes the graph obtained from any n copies of $\vec{C}_m$ which have just one common vertex; $n - \vec{C}_m$ denotes the graph obtained from any n copies of $\vec{C}_m$ which have just one common arc.

2. Some Results and Conjecture

2.1. Some Results on Directed Circuit $\vec{C}_m$ and the Union Digraph of Mutually Non-Intersecting $\vec{C}_m$

As to the gracefulness of directed circuit $\vec{C}_m$ and the union digraph of mutually non-intersecting $\vec{C}_m$, we have the following results.

In 1991, Ma gives some results and present some conjectures:

Theorem 1. (see [15]) *Let D be a digraph with p vertices and q arcs. If D is graceful, then* $p \leq q + 1$.

Theorem 2. (see [15]) *Any digraph has at least two different graceful labelings.*

Theorem 3. (see [15]) *The directed circuit $\vec{C}_m$ is graceful if and only if* $m \equiv 0 \ (mod\ 2)$.

Theorem 4. (see [15]) *The necessary condition of the digraph $n\vec{C}_3$ is graceful is* $n \equiv 0 \ (mod\ 2)$.

Theorem 5. (see [15]) *A necessary condition of the digraph $n\vec{C}_m$ is graceful is* $nm \equiv 0 \ (mod\ 2)$.

Conjecture 1. (see [15]) *A sufficient condition of the digraph $n\vec{C}_3$ is graceful is* $n \equiv 0 \ (mod\ 2)$.

Conjecture 2. (see [15]) *A sufficient condition of the digraph $n\vec{C}_m$ is graceful is* $nm \equiv 0 \ (mod\ 2)$.

2.2. Some Results on the Gracefulness of Digraph $n \cdot \vec{C}_m$

As to the gracefulness of digraph $n \cdot \vec{C}_m$, we have the following results.

In 1991, Ma gives a result and present a conjecture as follows,

Theorem 6. (see [15]) *A necessary condition of the digraph $n \cdot \vec{C}_3$ is graceful is* $n \equiv 0 \ (mod\ 2)$.

Conjecture 3. (see [15]) *A sufficient condition of the digraph $n \cdot \vec{C}_3$ is graceful is* $n \equiv 0 \ (mod\ 2)$.

In 2000, the third author of this paper has showed this conjecture is true.

Theorem 7. (see [16]) *A sufficient condition of the digraph $n \cdot \vec{C}_3$ is graceful is* $n \equiv 0 \ (mod\ 2)$.

In 1994, Du Zhiting et al. give some results and present a conjecture as follow,

Theorem 7. (see [17]) *For any integer $n \geq 1$ and any even integer $m \geq 3$, the digraph $n \cdot \vec{C}_m$ is graceful.*

Theorem 8. (see [17]) *For any positive integer n, and $m \geq 3$, a necessary condition of the digraph $n \cdot \vec{C}_m$ is graceful is that* $nm \equiv 0 \ (mod\ 2)$.

Conjecture 4. (see [17]) *For any even integer $n \geq 1$, odd integer $m \geq 3$, the digraph $n \cdot \vec{C}_m$ is graceful.*

Jirimutu et al [18] have proved Conjecture 4 for the cases $m = 5, 7$. Liu Xiaodong et al [19] proved Conjecture 4 for the case $m = 9$. Jirimutu et al. proved Conjecture 4 for the cases $m = 11, 13$, see [20]; $m = 15, 17$, see [21]; $m = 19$, see [22]; $m = 21$, see [23]; $m = 23$, see [24], respectively. In [25], Jirimutu et al. provided a complete and detailed proof of Conjecture 4 as following:

Theorem 9. (see [25]) *For any even integer $n \geq 2$, any odd $m \geq 3$, the digraph $n \cdot \vec{C}_m$ is graceful.*

Combining Theorem 7 with Theorem 9, we obtain a sharp corollary as follows:

Corollary 1. *For any integer $n \geq 1$ and $m \geq 3$, a sufficient condition of the digraph $n \cdot \vec{C}_m$ is graceful is that $nm \equiv 0 \pmod 2$.*

Combining Theorem 8 with Corollary 1, we have another sharp corollary as following:

Corollary 2. *For any integer $n \geq 1$ and $m \geq 3$, the necessary and sufficient condition of the digraph $n \cdot \vec{C}_m$ is graceful is that $nm \equiv 0 \pmod 2$.*

2.3. Some Results on the Gracefulness of Digraph $n - \vec{C}_m$

As to the gracefulness of digraph $n - \vec{C}_m$, we have the following results.

Theorem 10. (see [15]) *A necessary condition of the digraph $n - \vec{C}_3$ is graceful is $n \equiv 0 \pmod 2$.*

In [26], [27], Jirimutu et al. give a result, and put forward a conjecture and pose a problem.

Theorem 11. (see [26]) *For any positive even n, integer m and $4 \leq m \leq 13$, the digraph $n - \vec{C}_m$ is graceful.*

Conjecture 5. (see [26]) *For any positive even n, and any integer $m \geq 14$, the digraph $n - \vec{C}_m$ is graceful.*

Problem 1. (see [27]) *For any positive odd n and $m \geq 14$, determine whether the digraph $n - \vec{C}_m$ is graceful.*

Theorem 12. *For any positive integer n, and any integer $m \geq 3$, a necessary condition of the digraph $n - \vec{C}_m$ is graceful is $nm \equiv 0 \pmod 2$.*

Proof. Let $\vec{C}_m^1, \vec{C}_m^2, \cdots, \vec{C}_m^n$ denote n directed circuits of the digraph $n - \vec{C}_m$. Let v_0 and v_{m-1} denote two vertices with common edge of $\vec{C}_m^i$, respectively.

The other $m-2$ vertices of $\vec{C}^i_m$ are denoted by v^i_j ($j = 1, \ldots, m-2$; $i = 1, 2, \cdots, n$), respectively. For convenience, we put $v^1_0 = v^2_0 = \cdots = v^n_0 = v_0$, $v^1_m = v^2_m = \cdots = v^n_m = v_m$, and take subscripts j's modulo m. Obviously, the number of arcs of $n - \vec{C}_m$ is $q = (m-1)n + 1$.

Suppose that $n - \vec{C}_m$ is graceful, $f(v^i_j)$ and $[f(v^i_j) - f(v^i_{j-1})]$ are the graceful labeling and the induced edge's graceful labeling, respectively.

Then

$$\sum_{j=0}^{m-1} \left[f(v^i_j) - f(v^i_{j-1})\right] \equiv \sum_{j=0}^{m-1} f(v^i_j) - \sum_{j=0}^{m-1} f(v^i_{j-1}) = 0 \pmod{((m-1)n+2)},$$

This means there exists integer k_i, such that

$$\sum_{j=0}^{m-1} \left[f(v^i_j) - f(v^i_{j-1})\right] = k_i((m-1)n+2) \ (1 \le i \le n). \tag{1}$$

This implies that there exists integer $k = k_1 + \cdots + k_n$, such that

$$\sum_{i=1}^{n} \sum_{j=0}^{m-1} \left[f(v^i_j) - f(v^i_{j-1})\right] = k((m-1)n+2). \tag{2}$$

On the other hand, put $d = [f(v_0) - f(v_{m-1})]$, according to the definition, we have

$$\sum_{i=1}^{n} \sum_{j=0}^{m-1} \left[f(v^i_j) - f(v^i_{j-1})\right] = (n-1)d + \frac{1}{2}q(q+1) = k(q+1). \tag{3}$$

We obtain a necessary condition on $n - \vec{C}_m$ to be graceful as follows:

$$(n-1)d \equiv 0 \pmod{\frac{(q+1)}{2}}. \tag{4}$$

Without loss of generality, we assume that $f(v_0) = 0$. This implies that $f(v_{m-1}) = \frac{q+1}{2}$ by Equation 4. Since, $d = [f(v_0) - f(v_{m-1})] = [-\frac{q+1}{2}] \equiv \frac{q+1}{2} \pmod{(q+1)}$, it satisfies the condition of Equation 4. Put $d = \frac{q+1}{2}$ into Equation 3, $(n-1)(\frac{q+1}{2}) + (\frac{1}{2}q(q+1)) = k(q+1)$, and $\frac{n-1}{2} + \frac{n(m-1)}{2} = k$, namely, $nm \equiv 0 \pmod 2$. □

Problem 2. *For any odd n, and even $m \ge 14$, determine whether the digraph $n - \vec{C}_m$ is graceful?*

2.4. Some Results on Gracefulness of the Union of $n \cdot \vec{C}_3$

2.4.1. Some Results on Gracefulness of the Union of Non-Intersect Digraphs $n \cdot \vec{C}_3$

In [28], Siqingaowa et al. proved that the union of four graphs consisting mutually non-intersecting digraphs is graceful. They also proved the union of three mutually non-intersect digraph (namely, $n \cdot \vec{C}_3$, $n \cdot \vec{C}_3$, $2n \cdot \vec{C}_3$)is graceful. In [29], Siqinbate et al. proved the union of six graphs consisting mutually non-intersecting digraph $n \cdot \vec{C}_3$ is graceful, and pose a conjecture as follows,

Conjecture 6. (see [29]) *For positive integer n, the union of even mutually non-intersect digraphs $n \cdot \vec{C}_3$ is graceful.*

2.4.2. Other Results the Gracefulness of the Union of $n \cdot \vec{C}_3$

In [30], Siqinbate et al. further verified the gracefulness of the union of two non-intersecting digraphs $n \cdot \vec{C}_3$ and $n \cdot \vec{C'}_3$, and the common vertex of $n \cdot \vec{C}_3$ and the common vertex of $n \cdot \vec{C'}_3$ connected with two arcs in the opposite direction, where n is positive integer.

In [31], Siqinbate proved that a digraph in which the common vertices of two non-intersecting digraph $n \cdot \vec{C}_3$ connects with a new additional vertex through directional arcs where outdegree equals to 2 or indegree equals to 2 are graceful.

In [32], Siqinbate proved that the digraph which was obtained by sticking two non-intersecting digraphs $n \cdot \vec{C}_3$ together in opposite direction at two adjacent vertices of two degrees is graceful.

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