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ON A SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY CONVOLUTION AND INTEGRAL CONVOLUTION

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Abstract: In this paper, we introduce and study a subclass of harmonic univalent functions defined by convolution and integral convolution. Coefficient bounds, extreme points, distortion bounds, convolution conditions and convex combination are determined for functions in this family. Consequently, many of our results are either extensions or new approaches to those corresponding to previously known results.

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1. Introduction

A continuous complex valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D, if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of

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f. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D.

Let S_H denote the family of function $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \ |b_1| < 1. \tag{1.1}$$

The harmonic function $f = h + \overline{g}$ for $g \equiv 0$ reduces to an analytic univalent function $f \equiv h$. In 1984, Clunie and Sheil-Small [3] investigated the class S_H and as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses. For more basic results one may refer to the following standard introductory text book by Duren [5], see also Ahuja [1], Jahangiri et al. [8] and Ponnusamy and Rasila ([9], [10]).

The convolution of two function of form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad F(z) = z + \sum_{k=2}^{\infty} A_k z^k \quad \text{is defined as}$$

$$(f * F)(z) = f(z) * F(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k \tag{1.2}$$

while the integral convolution is defined by

$$f \diamond F(z) = z + \sum_{k=2}^{\infty} \frac{a_k A_k z^k}{k}. \tag{1.3}$$

In 1999, Jahangiri [7] defined the class $S_H^*(\alpha)$ consisting of functions $f = h + \overline{g}$ such that h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \ g(z) = \sum_{k=1}^{\infty} |b_k| z^k \tag{1.4}$$

which satisfy the condition

$$\frac{\partial}{\partial \theta}(\arg f(re^{i\theta})) \geq \alpha, \ 0 \leq \alpha < 1, \ |z| = r < 1.$$

Recently, these results were generalized by Frasin [6] by using convolution techniques. It is worthy to note that, results of [6] were improved and generalized by Dixit and Porwal [4].

Motivated with the work of Dixit and Porwal [4], Frasin [6] and Porwal et al. [11] we consider the subclass $HS(\phi,\psi;\alpha)$ of functions of the form (1.1) satisfying the condition

$$Re\left\{\frac{h(z)*\phi(z)-\overline{g(z)*\psi(z)}}{h(z)\diamond\phi(z)+\overline{g(z)\diamond\psi(z)}}\right\}>\alpha, \tag{1.5}$$

where $0\le\alpha<1,\ \phi(z)=z+\sum_{k=2}^{\infty}\lambda_k z^k$ and $\psi(z)=z+\sum_{k=2}^{\infty}\mu_k z^k$ are analytic in U with the condition $\lambda_k\ge 0,\ \mu_k\ge 0$.

We further let $\overline{HS}(\phi,\psi,\alpha)$ denote the subclass of $HS(\phi,\psi,\alpha)$ consisting of functions $f=h+\overline{g}\in S_H$ such that h and g are of the form (1.4).

We note that by specializing the ϕ and ψ we obtain the following known subclasses studied earlier by various authors

$$\overline{HS}\left(\frac{z}{(1-z)^2},\frac{z}{(1-z)^2};\alpha\right)=S_H^*(\alpha),$$

studied by Jahangiri [7]

and

$$\overline{HS}\left(\frac{z+z^2}{(1-z)^3},\frac{z+z^2}{(1-z)^3};\alpha\right)\equiv K(\alpha),$$

studied by Jahangiri [7]. For case $\alpha=0$ these subclass studied by Silverman and Silvia [13] and for $\alpha=b_1=0$ these subclass studied by Avci and Zlotkiewicz [2] and Silverman [12].

In the present paper, we extend the above results to the families $HS(\phi,\psi,\alpha)$ and $\overline{HS}(\phi,\psi,\alpha)$. We also obtain extreme points, distortion bounds, convolution conditions and convex combinations for the class $\overline{HS}(\phi,\psi,\alpha)$.

2. Coefficient Bounds

We begin with a sufficient condition for function in $HS(\phi,\psi;\alpha)$.

Theorem 2.1. *Let the function $f=h+\overline{g}$ be so that h and g are given by* (1.1)*. Furthermore, let*

$$\sum_{k=2}^{\infty}\frac{\lambda_k}{k}\left(\frac{k-\alpha}{1-\alpha}\right)|a_k|+\sum_{k=1}^{\infty}\frac{\mu_k}{k}\left(\frac{k+\alpha}{1-\alpha}\right)|b_k|\le 1, \tag{2.1}$$

where $0\le\alpha<1,\ k^2(1-\alpha)\le\lambda_k(k-\alpha)$ and $k^2(1-\alpha)\le\mu_k(k+\alpha)$ for $k\ge 2$.

Then f is sense-preserving, harmonic univalent in U and $f\in HS(\phi,\psi,\alpha)$.

Proof. First we note that f is locally unvialent and sense-preserving in U. This is because

$$\begin{aligned} |h'(z)| &\geq 1 - \sum_{k=2}^{\infty} k|a_k|r^{k-1} > 1 - \sum_{k=2}^{\infty} k|a_k| \\ &\geq 1 - \sum_{k=2}^{\infty} \frac{\lambda_k}{k}\left(\frac{k-\alpha}{1-\alpha}\right)|a_k| \\ &\geq \sum_{k=1}^{\infty} \frac{\mu_k}{k}\left(\frac{k+\alpha}{1-\alpha}\right)|b_k| \geq \sum_{k=1}^{\infty} k|b_k| \\ &> \sum_{k=1}^{\infty} k|b_k|r^{k-1} \geq |g'(z)|. \end{aligned}$$

To show that f is univalent in U, suppose $z_1, z_2 \in U$ so that $z_1 \neq z_2$, then

$$\begin{aligned} &\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\ &= 1 - \left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k - z_2^k)}{(z_1-z_2) + \sum_{k=2}^{\infty} a_k(z_1^k - z_2^k)}\right| > 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=2}^{\infty} k|a_k|} \\ &> 1 - \frac{\sum_{k=1}^{\infty} \frac{\mu_k}{k}\left(\frac{k+\alpha}{1-\alpha}\right)|b_k|}{1 - \sum_{k=2}^{\infty} \frac{\lambda_k}{k}\left(\frac{k-\alpha}{1-\alpha}\right)|a_k|} \\ &\geq 0. \end{aligned}$$

Now, we show that $f \in HS(\phi, \psi, \alpha)$ by using the fact that $\mathrm{Re}\, w \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$ it suffices to show that

$$|(1-\alpha)B(z) + A(z)| \ - \ |(1+\alpha)B(z) - A(z)| \geq 0 \tag{2.2}$$

where $A(z) = h(z) * \phi(z) - \overline{g(z) * \psi(z)}$ and $B(z) = h(z) \diamond \phi(z) + \overline{g(z) \diamond \psi(z)}$.

Substituting for $A(z)$ and $B(z)$ in L.H.S. of (2.2) and making use of (2.1), we obtain

$$|A(z) + (1-\alpha)B(z)| \ - \ |A(z) - (1+\alpha)B(z)| \geq 0,$$

$$
\begin{aligned}
&= |(1-\alpha)\{h(z)\diamond\phi(z)+\overline{g(z)\diamond\psi(z)}\}+\{h(z)*\phi(z)-\overline{g(z)*\psi(z)}\}| \\
&- |(1+\alpha)\{h(z)\diamond\phi(z)+\overline{g(z)\diamond\psi(z)}\}-\{h(z)*\phi(z)-\overline{g(z)*\psi(z)}\}| \\
&= \left|(1-\alpha)\left\{z+\sum_{k=2}^{\infty}\frac{a_k\lambda_k}{k}z^k+\overline{\sum_{k=1}^{\infty}\frac{b_k\mu_k}{k}z^k}\right\}+\left\{z+\sum_{k=2}^{\infty}a_k\lambda_k z^k\right.\right. \\
&\qquad \left.\left.-\overline{\sum_{k=1}^{\infty}b_k\mu_k z^k}\right\}\right| - \left|(1+\alpha)\left\{z+\sum_{k=2}^{\infty}\frac{a_k\lambda_k}{k}z^k+\overline{\sum_{k=1}^{\infty}\frac{b_k\mu_k}{k}z^k}\right\}\right. \\
&\qquad\qquad \left.-\left\{z+\sum_{k=2}^{\infty}a_k\lambda_k z^k-\overline{\sum_{k=1}^{\infty}b_k\mu_k z^k}\right\}\right| \\
&= \left|(2-\alpha)z+\sum_{k=2}^{\infty}a_k\lambda_k\left(\frac{1-\alpha}{k}+1\right)z^k+\overline{\sum_{k=1}^{\infty}b_k\mu_k\left(\frac{1-\alpha}{k}-1\right)z^k}\right| \\
&- \left|\alpha z+\sum_{k=2}^{\infty}a_k\lambda_k\left(\frac{1+\alpha}{k}-1\right)z^k+\overline{\sum_{k=1}^{\infty}b_k\mu_k\left(\frac{1+\alpha}{k}+1\right)z^k}\right| \\
&\geq (2-\alpha)|z|-\sum_{k=2}^{\infty}\lambda_k\left(\frac{k+1-\alpha}{k}\right)|a_k|\;|z^k|-\sum_{k=1}^{\infty}\mu_k\left(\frac{k-1+\alpha}{k}\right)|b_k|\;|z^k| \\
&- \alpha|z|-\sum_{k=2}^{\infty}\lambda_k\left(\frac{k-1-\alpha}{k}\right)\;|a_k|\;|z|^k-\sum_{k=1}^{\infty}\mu_k\left(\frac{k+1+\alpha}{k}\right)\;|b_k|\;|z|^k \\
&= 2(1-\alpha)|z|\left\{1-\sum_{k=2}^{\infty}\frac{\lambda_k(k-\alpha)}{k(1-\alpha)}|a_k|\;|z|^{k-1}-\sum_{k=1}^{\infty}\frac{\mu_k(k+\alpha)}{k(1-\alpha)}|b_k|\;|z|^{k-1}\right\} \\
&> 2(1-\alpha)|z|\left\{1-\sum_{k=2}^{\infty}\frac{\lambda_k(k-\alpha)}{k(1-\alpha)}|a_k|\;|z|^{k-1}-\sum_{k=1}^{\infty}\frac{\mu_k(k+\alpha)}{k(1-\alpha)}|b_k|\;|z|^{k-1}\right\} \\
&\geq 0.
\end{aligned}
$$

The coefficient bound (2.1) is sharp for the function

$$
f(z)=z+\sum_{k=2}^{\infty}\frac{k(1-\alpha)}{\lambda_k(k-\alpha)}x_k z^k+\sum_{k=1}^{\infty}\frac{k(1-\alpha)}{\mu_k(k+\alpha)}\overline{y_k z^k} \tag{2.3}
$$

where $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$. □

Next, we show that the above sufficient condition is also necessary for functions in $\overline{HS}(\phi,\psi,\alpha)$.

Theorem 2.2. *Let the function $f = h + \overline{g}$ be so that h and g are given by* (1.4)*. Then $f \in \overline{HS}(\phi, \psi, \alpha)$ if and only if*

$$\sum_{k=2}^{\infty} \frac{\lambda_k}{k} \left(\frac{k-\alpha}{1-\alpha} \right) |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k}{k} \left(\frac{k+\alpha}{1-\alpha} \right) |b_k| \leq 1, \tag{2.4}$$

where $0 \leq \alpha < 1$ and $k^2(1-\alpha) \leq \lambda_k(k-\alpha), k^2(1-\alpha) \leq \mu_k(k+\alpha)$ for $k \geq 2$.

Proof. The if part, follows from Theorem 2.1. To prove the only if part, let $f \in \overline{HS}(\phi, \psi, \alpha)$ then from (1.5), we have

$$Re \left\{ \frac{h(z) * \phi(z) - \overline{g(z) * \psi(z)}}{h(z) \diamond \phi(z) + \overline{g(z) \diamond \psi(z)}} - \alpha \right\} \geq 0,$$

which is equivalent to

$$Re \left\{ \frac{(1-\alpha)z - \sum_{k=2}^{\infty} \frac{\lambda_k(k-\alpha)}{k} |a_k| z^k - \sum_{k=1}^{\infty} \frac{\mu_k(k+\alpha)}{k} |b_k| \overline{z}^k}{z - \sum_{k=2}^{\infty} \frac{\lambda_k}{k} |a_k| z^k + \sum_{k=1}^{\infty} \frac{\mu_k}{k} |b_k| \overline{z}^k} \right\} \geq 0. \tag{2.5}$$

The above required condition (2.5) must hold for all values of z in U. Upon choosing the value of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$\frac{(1-\alpha) - \sum_{k=2}^{\infty} \frac{\lambda_k(k-\alpha)}{k} |a_k| r^{k-1} - \sum_{k=1}^{\infty} \frac{\mu_k(k+\alpha)}{k} |b_k| r^{k-1}}{1 - \sum_{k=2}^{\infty} \frac{\lambda_k}{k} |a_k| r^{k-1} + \sum_{k=1}^{\infty} \frac{\mu_k |b_k| r^{k-1}}{k}} \geq 0 \tag{2.6}$$

If the condition (2.4) does not hold, then the numerator in (2.6) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in (0,1) for which the quotient in (2.6) is negative. This contradicts the required condtion for $f \in \overline{HS}(\phi, \psi, \alpha)$ and so the proof is complete. □

Corollary 2.3. *Let the function $f = h + \overline{g}$ be so that h and g are given by* (1.4) *then $f \in \overline{HS}\left(\frac{z+z^2}{(1-z)^3}, \frac{z+z^2}{(1-z)^3}, \alpha \right)$ if and only if*

$$\sum_{k=2}^{\infty} \frac{k(k-\alpha)}{1-\alpha} |a_k| + \sum_{k=1}^{\infty} \frac{k(k+\alpha)}{1-\alpha} |b_k| \leq 1, \quad \text{where } 0 \leq \alpha < 1.$$

3. Distortion Theorem

In next theorem, we determine distortion bounds for function in $\overline{HS}(\phi,\psi,\alpha)$ which yields a covering result for this family.

Theorem 3.1. *let* $f \in \overline{HS}(\phi,\psi,\alpha)$ *and*

$$A \leq \frac{\lambda_k}{k}(k-\alpha),\; A \leq \frac{\mu_k}{k}(k+\alpha) \quad \text{for } k \geq 2, A = \min\left\{\lambda_2\frac{(2-\alpha)}{2},\; \mu_2\frac{(2+\alpha)}{2}\right\}.$$

Then for $|z| = r < 1$, *we have*

$$|f(z)| \leq (1+|b_1|)r + \left(\frac{1-\alpha}{A} - \frac{1+\alpha}{A}|b_1|\right) r^2$$

and

$$|f(z)| \geq (1-|b_1|)r - \left(\frac{1-\alpha}{A} - \frac{1+\alpha}{A}|b_1|\right) r^2.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f \in \overline{HS}(\phi,\psi,\alpha)$. Taking the absolute value of f, we have

$$\begin{aligned}
&|f(z)| < (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \\
&\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2 \\
&= (1+|b_1|)r + \frac{1-\alpha}{A}\sum_{k=2}^{\infty}\left(\frac{A}{1-\alpha}|a_k| + \frac{A}{1-\alpha}|b_k|\right) r^2 \\
&\leq (1+|b_1|)r + \frac{1-\alpha}{A}\sum_{k=2}^{\infty}\left(\frac{\lambda_k(k-\alpha)}{k(1-\alpha)}|a_k| + \frac{\mu_k(k+\alpha)}{k(1-\alpha)}|b_k|\right) r^2 \\
&\leq (1+|b_1|)r + \frac{1-\alpha}{A}\left(1 - \frac{(1+\alpha)|b_1|}{(1-\alpha)}\right) r^2 \\
&= (1+|b_1|)r + r^2\left(\frac{1-\alpha}{A} - \frac{1+\alpha}{A}|b_1|\right).
\end{aligned}$$

□

The following covering result follows from the left hand inequality in Theorem 3.1.

Corollary 3.2. *Let f of the form* (1.4) *be so that $f \in \overline{HS}(\phi, \psi, \alpha)$ and*

$$A \leq \frac{\lambda_k}{k}(k-\alpha),\ A \leq \frac{\mu_k}{k}(k+\alpha) \quad \text{for } k \geq 2, A = \min\left\{\lambda_2 \frac{(2-\alpha)}{2},\ \mu_2 \frac{(2+\alpha)}{2}\right\}.$$

Then for $|z| = r < 1$, we have

$$\left\{w : |w| < \frac{A+(1-\alpha)}{A} + \frac{A-(1+\alpha)}{A}|b_1|\right\} \subset f(U).$$

For our next theorem, we need the following definition of the convolution of two harmonic functions, harmonic functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \overline{z}^k$$

and

$$F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| \overline{z}^k$$

the convolution of two functions $f(z)$ and $F(z)$ as

$$\begin{aligned}(f * F)(z) &= f(z) * F(z) \\ &= z - \sum_{k=2}^{\infty} |a_k A_k| z^k + \sum_{k=1}^{\infty} |b_k B_k| \overline{z}^k. \end{aligned} \tag{3.1}$$

Following theorems are an easy consequences of this definition.

Theorem 3.3. *For $0 \leq \beta \leq \alpha < 1$ let $f(z) \in \overline{HS}(\phi, \psi, \alpha)$ and $F(z) \in \overline{HS}(\phi, \psi, \beta)$ then $f(z) * F(z) \in \overline{HS}(\phi, \psi, \alpha) \subseteq \overline{HS}(\phi, \psi, \beta)$, i.e. $\overline{HS}(\phi, \psi, \alpha)$ is closed under convolution.*

Next, we discuss the convex combinations of the class $\overline{HS}(\phi, \psi, \alpha)$.

Theorem 3.4. *The family $\overline{HS}(\phi, \psi, \alpha)$ is closed under convex combination.*

4. Extreme Points

In this section we determine the extreme points of $\overline{HS}(\phi, \psi, \alpha)$.

Theorem 4.1. *Let* $h_i(z) = z,\ h_k(z) = z - \frac{(1-\alpha)k}{\lambda_k(k-\alpha)}\ z^k (k \geq 2)$ *and* $g_k(z) = z + \frac{k(1-\alpha)}{\mu_k(k+\alpha)}\overline{z}^k (k \geq 1)$. *Then* $f \in \overline{HS}(\phi, \psi, \alpha)$ *if and only if it can be expressed as*

$$f(z) = \sum_{k=1}^{\infty}(x_k h_k + y_k g_k)$$

where $x_k \geq 0,\ y_k \geq 0,\ \sum_{k=1}^{\infty}(x_k + y_k) = 1$. *In particular, the extreme points of* $\overline{HS}(\phi, \psi, \alpha)$ *are* $\{h_k\}$ *and* $\{g_k\}$.

Proof. The proof of Theorem 4.1 is similar to that of Theorem 3.1 of [6] therefore we omit details involved. □

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