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NUMERICAL SOLUTION OF HYPERSINGULAR INTEGRAL EQUATIONS

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Abstract: In this article, an accurate numerical solution for solving hypersingular integral equation is presented. Chebyshev orthogonal polynomials of the second kind are used to approximate the unknown function. The regular kernel is interpolated using Chebyshev interpolation formula of the first kind. Numerical examples are solved using the proposed numerical technique. Numerical results show the accuracy of the present numerical solution.

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1. Introduction

Consider the hypersingular integral equation (HSIE) of the form

$$\oint_{-1}^{1} \frac{\varphi(t)}{(t-x)^2}\,dt + \int_{-1}^{1} K(x,t)\varphi(t)\,dt = f(x), \quad -1 < x < 1, \tag{1}$$

with the conditions

$$\varphi(\pm 1) = 0, \tag{2}$$

where K and f' are assumed to be Hölder-continues functions and φ is unknown function to be determined. The first integral in the left-hand side of equation

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(1) is understood as a Hadamard finite-part integral, which is defined by (see Lifanov et al [1] and Martin [2])

$$\oint_{-1}^{1} \frac{\varphi(t)}{(t-x)^2}\,dt = \lim_{\epsilon \longrightarrow 0} \left\{ \int_{-1}^{x-\epsilon} \frac{\varphi(t)}{(t-x)^2}\,dt + \int_{x+\epsilon}^{1} \frac{\varphi(t)}{(t-x)^2}\,dt - \frac{2\varphi(x)}{\epsilon} \right\}. \quad (3)$$

The Hadamard finite-part integral (3) is also defined by the Cauchy principal-value integral as

$$\oint_{-1}^{1} \frac{\varphi(t)}{(t-x)^2}\,dt = -\frac{d}{dx} \int_{-1}^{1} \frac{\varphi(t)}{t-x}\,dt. \quad (4)$$

where the Cauchy principal-value integral is defined as

$$\int_{-1}^{1} \frac{\varphi(t)}{t-x}\,dt = \lim_{\epsilon \longrightarrow 0} \left\{ \int_{-1}^{x-\epsilon} \frac{\varphi(t)}{t-x}\,dt + \int_{x+\epsilon}^{1} \frac{\varphi(t)}{t-x}\,dt \right\}. \quad (5)$$

Equation (1) arises in a variety of mixed boundary value problems in mathematical physics such as water wave scatting and radiation problems involving thin submerged, fluid mechanics, fracture mechanics, acoustics and elasticity (see Chan et al [3], Mandal et al [4] and Parsons et al [5]).

In this paper, we present a numerical solution for solving the HSIE(1). In the approximation, we use the Chebyshev polynomials of the second kind, U_i, to approximate the unknown function $\varphi(t)$ and Chebyshev polynomials of the first kind , T_i, to interpolate the regular kernel $K(x,t)$ with respect to t in the zeros of T_{M+1}.

2. Numerical Technique

Using (4), equation (1) can be converted into

$$-\frac{d}{dx} \int_{-1}^{1} \frac{\varphi(t)}{t-x}\,dt + \int_{-1}^{1} K(x,t)\varphi(t)\,dt = f(x), \quad -1 < x < 1. \quad (6)$$

Rewriting the unknown function $\varphi(t)$ in equation (6) which satisfies the conditions in (2) as

$$\varphi(t) = \sqrt{1-t^2}\,\psi(t), \quad (7)$$

where the function $\psi(t)$ is well behavior on the interval $[-1,1]$.
Now, approximating $\psi(t)$ by using Chebyshev polynomials of the second kind as follows

$$\psi(t) \approx \sum_{i=0}^{N} a_i\, U_i(t), \quad (8)$$

where a_i, $i = 0, 1, \ldots, N$ are the unknown coefficients to be determined. Due to (7-8), the numerical solution of equation (1) is of the form

$$\varphi(t) \approx \sqrt{1-t^2} \sum_{i=0}^{N} a_i \, U_i(t). \tag{9}$$

Substituting (9) into equation (6) yields

$$\sum_{i=0}^{N} a_i \left[P_i(x) + Q_i(x) \right] = f(x), \tag{10}$$

where

$$P_i(x) = -\frac{d}{dx} \fint_{-1}^{1} \frac{\sqrt{1-t^2}\, U_i(t)}{t-x} \, dt \tag{11}$$

and

$$Q_i(x) = \int_{-1}^{1} \sqrt{1-t^2}\, K(x,t)\, U_i(t)\, dt. \tag{12}$$

It is known that (see Mason et al [6])

$$\left. \begin{aligned} &\fint_{-1}^{1} \frac{\sqrt{1-t^2}\, U_i(t)}{t-x} \, dt = -\pi \, T_{i+1}(x), \quad -1 < x < 1, \\ &\frac{d}{dx}(T_{i+1}(x)) = (i+1)\, U_i(x), \quad -1 < x < 1. \end{aligned} \right\} \tag{13}$$

Taking into account (13) we obtain

$$P_i(x) = -\pi \,(i+1)\, U_i(x), \quad i = 0, 1, \ldots, N. \tag{14}$$

In order to evaluate $Q_i(x)$, the integral in (12) may be calculated analytically. If we can not evaluate the integral in (12) analytically, we may do so approximately, for instant, by interpolating the regular kernel $K(x,t)$, with respect to t, in the zeros of T_{M+1} as a sum of Chebyshev polynomials in the form (see Mason et al [6])

$$K(x,t) \approx {\sum_{k=0}^{M}}' c_k(x)\, T_k(t), \tag{15}$$

where

$$c_k(x) = \frac{2}{M+1} \sum_{\ell=1}^{M+1} K(x, t_\ell)\, T_k(t_\ell), \tag{16}$$

and

$$t_\ell = \frac{(2\,\ell - 1)\pi}{2(M+1)}, \quad \ell = 1, 2, \ldots, M+1. \tag{17}$$

Then

$$Q_i(x) = {\sum_{k=0}^{M}}{}' \, c_k(x) \int_{-1}^{1} \sqrt{1-t^2}\, T_k(t)\, U_i(t)\, dt. \tag{18}$$

It is easy to verify that (see Mason et al [6] and Abdulkawi et al [7])

$$T_k(t)\, U_i(t) = \frac{1}{2}\big[U_{k+i}(t) + U_{i-k}(t)\big], \quad i, k = 0, 1, 2, \ldots. \tag{19}$$

Using (19) into (18) yields

$$Q_i(x) = \frac{1}{2} {\sum_{k=0}^{M}}{}' \, c_k(x) \big[b_{i,k} + d_{i,k}\big], \tag{20}$$

where

$$b_{i,k} = \int_{-1}^{1} \sqrt{1-t^2}\, U_{k+i}(t)\, dt = \begin{cases} \frac{\pi}{2}, & k+i = 0, \\ 0, & k+i \neq 0. \end{cases} \tag{21}$$

and

$$d_{i,k} = \int_{-1}^{1} \sqrt{1-t^2}\, U_{i-k}(t)\, dt = \begin{cases} \frac{\pi}{2}, & i = k, \\ -\frac{\pi}{2}, & k - i = 2, \\ 0, & Oherwise. \end{cases} \tag{22}$$

Then, choosing the suitable collocation points x_j, $j = 0, 1, \ldots, N$ for equation (10) such as the zeros of U_{N+1}. These lead to a system of linear equations

$$\sum_{i=0}^{N} a_i \big[P_i(x_j) + Q_i(x_j)\big] = f(x_j), \quad j = 0, 1, \ldots, N. \tag{23}$$

Solving the system (23) for the unknown coefficients a_i, $i = 0, 1, \ldots, N$ and substituting the values of a_i into (9) we obtain the numerical solution of equation (1). Note that the dash in $\sum'$ denotes that the first term in the sum is to be halved.

3. Numerical Results

Example 1. Consider the following HSIE

$$\fint_{-1}^{1} \frac{\varphi(t)}{(t-x)^2}\,dt + 16\int_{-1}^{1} x^3\, t^3 \varphi(t)\,dt = -\pi[31x^3 - 16x], \tag{24}$$

with the conditions

$$\varphi(\pm 1) = 0. \tag{25}$$

It is not difficult to verify that the exact solution of equation (24) is

$$\varphi(t) = \sqrt{1-t^2}\big[8t^3 - 4t\big]. \tag{26}$$

Due to (12) we have

$$Q_i(x) = 16\int_{-1}^{1} \sqrt{1-t^2}\, x^3\, t^3\, U_i(t)\,dt. \tag{27}$$

With the help of the recurrence relation of Chebyshev polynomials of the second kind

$$\left.\begin{aligned} &U_0(x) = 1, \quad U_1(x) = 2x, \\ &U_n(x) = 2x\,U_{n-1}(x) - U_{n-2}(x), \quad n \geq 2. \end{aligned}\right\}, \tag{28}$$

we have

$$t^3 = \frac{1}{8}\big[U_3(t) + 2U_1(t)\big]. \tag{29}$$

Using (29) into (27) yields

$$Q_i(x) = 2\,x^3 \int_{-1}^{1} \sqrt{1-t^2}\,\big[U_3(t) + 2U_1(t)\big]\,U_i(t)\,dt. \tag{30}$$

It is known that

$$\int_{-1}^{1} \sqrt{1-t^2}\, U_i(t)\, U_j(t)\,dt = \begin{cases} 0, & i \neq j, \\ \dfrac{\pi}{2}, & i = j. \end{cases} \tag{31}$$

From (30-31) we obtain

$$\left.\begin{aligned} &Q_0(x) = 0,\ Q_1(x) = 2\,\pi\,x^3,\ Q_2(x) = 0, \\ &Q_3(x) = \pi\,x^3,\ Q_i(x) = 0,\ i \geq 4. \end{aligned}\right\} \tag{32}$$

Thus, the system of linear equations (23) for $N = 3$ becomes

$$\left.\begin{aligned}\sum_{i=0}^{3} a_i\big[-\pi(i+1)U_i(x_j)\big] + 2\,\pi a_1 x_j^3 + \pi a_3 x_j^3 \\ = -\pi[31x_j^3 - 16x_j],\quad j = 0,1,2,3.\end{aligned}\right\} \tag{33}$$

which is equivalent to

$$\left.\begin{aligned}&[2\pi a_1 - 31\pi a_3]x_j^3 - 12\pi a_2\, x_j^2 + [-4\pi a_1 + 16\pi a_3]x_j \\ &+3\pi a_2 - \pi a_0 = -31\pi x_j^3 + 16\pi x_j,\quad j = 0,1,2,3.\end{aligned}\right\} \tag{34}$$

Comparing the coefficients of various powers of x_j from the both sides of equation (34) yields the following system

$$\left.\begin{aligned}3\,\pi\, a_2 - \pi\, a_0 &= 0, \\ -4\,\pi\, a_1 + 16\pi\, a_3 &= 16\,\pi, \\ -12\,\pi\, a_2 &= 0, \\ 2\,\pi\, a_1 - 31\pi\, a_3 &= -31\,\pi.\end{aligned}\right\} \tag{35}$$

It is easy to see that the solution of the above system (35) is

$$a_0 = a_1 = a_2 = 0,\quad a_3 = 1. \tag{36}$$

Substituting the values of the coefficients a_i, $i = 0,1,2,3$, into (9) we obtain the numerical solution of equation (24) which is identical to the exact solution given by (26).

Example 2. Consider the following HSIE

$$\ne\!\!\!\!\!\!\int_{-1}^{1} \frac{\varphi(t)}{(t-x)^2}dt + \int_{-1}^{1} \sin(x)\, t^4 \varphi(t)dt = f(x) \tag{37}$$

with the conditions

$$\varphi(\pm 1) = 0, \tag{38}$$

where

$$f(x) = -\pi\left[5(16x^4 - 12x^2 + 1) - \frac{\sin(x)}{32}\right].$$

The exact solution of equation (37) is

$$\varphi(x) = \sqrt{1-x^2}\,(16x^4 - 12x^2 + 1). \tag{39}$$

By using (20-22) with $M = 5$, we obtain

$$\left.\begin{aligned} &Q_0(x) = 0.06249999990\,\pi\ \sin(x),\ Q_1(x) = 0, \\ &Q_2(x) = 0.09374999990\,\pi\ \sin(x),\ Q_3(x) = 0, \\ &Q_4(x) = 0.03125000012\,\pi\ \sin(x),\ Q_5(x) = 0. \end{aligned}\right\} \tag{40}$$

Choosing the zeros of U_{N+1} as the collocation points x_j for equation (23) where $N = 5$, i.e.

$$x_j = \cos\left(\frac{(j+1)\,\pi}{N+2}\right), \quad j = 0, 1, \ldots, 5, \tag{41}$$

and solving the obtained system for the unknown coefficients a_i, $i = 0, 1, \ldots, 5$. Substituting the values of a_i into (9) we obtain the numerical solution of equation (37).The errors of numerical solution of equation (37) are given by Table 1.

Example 3. Consider the following HSIE

$$\neq\!\!\!\!\!\!\int_{-1}^{1} \frac{\varphi(t)}{(t-x)^2}\,dt + \int_{-1}^{1} e^x\, t^2 \varphi(t)\,dt = -\pi\left[12\,x^2 - \frac{1}{8}\,e^x - 3\right]. \tag{42}$$

with the conditions

$$\varphi(\pm 1) = 0. \tag{43}$$

It is easy to see that the exact solution of equation (37) is

$$\varphi(x) = \sqrt{1-x^2}\,(4x^2 - 1). \tag{44}$$

Using (20-22) with $M = 5$, we obtain

$$\left.\begin{aligned} &Q_0(x) = 0.125\,\pi\, e^x,\ Q_1(x) = 0, \\ &Q_2(x) = 0.1249999999\,\pi\, e^x,\ Q_3(x) = 0, \\ &Q_4(x) = 1.75 \times 10^{-10}\,\pi\, e^x,\ Q_5(x) = 0. \end{aligned}\right\} \tag{45}$$

Using x_j in (41) as the collocation points for equation (23) where $N = 5$ and solving the obtained system for the unknown coefficients a_i, $i = 0, 1, \ldots, 5$. Substituting the values of a_i into (9) we obtain the numerical solution of equation (42).The errors of numerical solution of equation (42) are given by Table 2.

x	$N = M = 5$	Error
-1.0		0
-0.9		5.00×10^{-10}
-0.8		2.10×10^{-09}
-0.7		3.60×10^{-09}
-0.6		4.00×10^{-09}
-0.5		2.90×10^{-09}
-0.4		3.40×10^{-09}
-0.3		2.81×10^{-09}
-0.2		2.30×10^{-09}
-0.1		2.10×10^{-09}
0.0		2.00×10^{-09}
0.1		2.10×10^{-09}
0.2		2.30×10^{-09}
0.3		2.72×10^{-09}
0.4		3.30×10^{-09}
0.5		2.90×10^{-09}
0.6		4.00×10^{-09}
0.7		3.60×10^{-09}
0.8		2.70×10^{-09}
0.9		4.00×10^{-10}
1.0		0

Table 1: Errors of numerical solution of equation (37)

4. Conclusion

We have developed collocation method based on the Chebyshev orthogonal polynomials for solving the hypersingular integral equations. Chebyshev interpolation formula helped us to approximate the regular kernel. The collocation points in examples $2-3$ are chosen to be the zeros of Chebyshev polynomials of the second kind $U_N + 1$. The errors of the collocation method shown in the Tables 2-3 are computed as the absolute value of the difference between the exact and numerical solutions. We used Maple 13 to carry the computations. The developed collocation method gives a very accurate numerical results for any singular point $x \in [-1, 1]$ with only small number of collocation points $N = 5$ (Tables 1-2). Moreover, the developed collocation method gives the exact solution for some hypersingular integral equations (Example 1).

x	$N = M = 5$	Error
-1.0		0
-0.9		5.0×10^{-10}
-0.8		6.0×10^{-10}
-0.7		5.0×10^{-10}
-0.6		6.0×10^{-10}
-0.5		9.6×10^{-10}
-0.4		1.1×10^{-09}
-0.3		1.0×10^{-09}
-0.2		9.0×10^{-10}
-0.1		1.0×10^{-09}
0.0		1.0×10^{-09}
0.1		1.0×10^{-9}
0.2		9.0×10^{-10}
0.3		1.0×10^{-09}
0.4		1.1×10^{-9}
0.5		9.3×10^{-10}
0.6		6.0×10^{-10}
0.7		4.0×10^{-10}
0.8		6.0×10^{-10}
0.9		5.0×10^{-10}
1.0		0

Table 2: Errors of numerical solution of equation (42)

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